Journal of Informatics and Mathematical Sciences
Vol. 9, No. 3, pp. 721–732, 2017
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

Proceedings of the Conference
Current Scenario in Pure and Applied Mathematics
December 22-23, 2016
Kongunadu Arts and Science College (Autonomous)
Coimbatore, Tamil Nadu, India

Research Article

On Detour Distance Laplacian Energy

V. Kaladevi and A. Abinayaa*

PG and Research Department of Mathematics, Bishop Heber College, Tiruchirappalli 620017, Tamil Nadu, India
***Corresponding author:** abinayaa.am@gmail.com

Abstract. The Detour distance laplacian energy of a simple connected graph G is defined as the sum of the absolute values of the Eigen values of the detour distance laplacian matrix of G. In this paper, the bounds for detour distance laplacian energy is obtain and also the detour distance laplacian energy of standard graphs and the Cartesian product of certain graphs with P_2 are computed.

Keywords. Detour distance Laplacian matrix; Detour distance Laplacian Eigen value; Detour distance Laplacian energy

MSC. 05CXX; 05C12; 05C50

Received: January 10, 2017 **Accepted:** March 4, 2017

1. Introduction

Let G be a connected graph of order n, with vertex set $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$. Let $A = [a_{ij}]_{n\times n}$ be the adjacency matrix of G. The eigen values $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ of A, assumed to be in non-increasing order, are the eigen values of the G. The Energy $E(G)$ of G is defined to be the sum of the absolute values of the eigen values of G. That is

$$E(G)=\sum_{i=1}^{n}|\lambda_i|. \quad [5,10,11]$$

Let $L=\mathrm{Diag}(\mathrm{Deg})-A$, where Diag(Deg) is the diagonal matrix whose diagonal entries are the degrees of v_i, $i=1,2,3,\dots,n$ in G is called the Laplacian matrix of G. The Laplacian eigen values of L are denoted by $\mu_1,\mu_2,\mu_3,\dots,\mu_n$ assumed to be in non-increasing order, are the laplacian eigen values of G. The laplacian eigen values of G is defined as

$$LE(G)=\sum_{i=1}^{n}|\mu_i|.$$

The distance laplacian matrix of G is defined as $D^L=\mathrm{Diag}(D_i)-D(G)$, where $\mathrm{Diag}(D_i)$ denotes the diagonal matrix of the distance degrees. The eigen values $\delta_1^L(G),\delta_2^L(G),\delta_3^L(G),\dots,\delta_n^L(G)$ of D^L are assumed in non-increasing order, are the distance laplacian eigen values of G. The distance laplacian energy of G is defined as

$$E_{DL}(G)=\sum_{i=1}^{n}|\delta_i^L(G)|. \quad [3,16]$$

Analogous to the distance laplacian, we define the detour distance laplacian matrix of a connected graph G as $DDL(G)=\mathrm{Diag}(T_r)-DD$, where $\mathrm{Diag}(T_r)$ denotes the diagonal matrix of the vertex transmissions in G. Let $\phi_{DDL}(\gamma)$ denotes the characteristic polynomial of $DDL(G)$. The eigen values of $DDL(G)$ are such that $\gamma_1^L(G),\gamma_2^L(G),\gamma_3^L(G),\dots,\gamma_n^L(G)$ are the detour distance laplacian eigen values of G and form the DDL spectrum of G denoted by $spec_{DDL}(G)$. The detour distance laplacian energy is defined as $E_{DDL}(G)=\sum_{i=1}^{n}|\gamma_i^L(G)|$ [1,2,4,6–9,12–15,17,20,22].

Two Graphs with equal DDL energy are said to be DDL-equienergetic.

The detour index of the graph G is defined as $DI(G)=\frac{1}{2}\sum_{u,v\in v(G)}D(u,v)$. The use of the detour index in quantitative structure activity relationship (QSAR) is studied by Lukovits [18]. Further, Trinajstic [21] analysed and compared, the wiener index and detour index in structure boiling point modelling. In [19] the authors proved the detour index as a descriptor for boiling points of acyclic and cyclic alkanes and saturated hydrocarbons.

Definition 1.1. An $n\times n$ circulant matrix C is defined as

$$C=\begin{bmatrix} C_0 & C_{n-1} & \cdots & \cdots & C_2 & C_1\\ C_1 & C_0 & C_{n-1} & \cdots & \cdots & C_2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ C_{n-2} & \cdots & \cdots & \cdots & C_{n-1} & \cdots\\ C_{n-1} & C_{n-2} & \cdots & \cdots & C_1 & C_0\end{bmatrix}.$$

The eigen values of circulant matrix are

$$\lambda_j=C_0+C_{n-1}\omega^j+C_{n-2}\omega^{2j}+\dots+C_1\omega^{(n-1)j},\quad j=0,1,2,\dots,n-1$$

where $\omega^j=\exp\left(\frac{2\pi ij}{n}\right)$ are the nth roots of unity.

Definition 1.2. The crown graph S_n^0 for an integer $n \geq 2$ is the graph with vertex set $\{u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n\}$ and edge set $\{u_i v_j : 1 \leq i, j \leq n, i \neq j\}$.

Definition 1.3. The cocktail party graph is denoted by $K_{n\times 2}$, is a graph having the vertex set $V = \bigcup_{i=1}^{n} \{u_i, v_i\}$ and the edge set $E = \{u_i u_j, v_i v_j : i \neq j\} \cup \{u_i u_j, v_i v_j : 1 \leq i < j \leq n\}$.

Result 1.4. *Let G be a connected graph with $n \geq 3$ vertices. Then $(n-1)^2 \leq DI(G) \leq \frac{n(n-1)^2}{2}$* [22].

In this paper, we establish some properties of the detour distance laplacian energy via detour index. We give bounds for the detour distance laplacian energy. Further the detour distance laplacian energy of standard graphs and the Cartesian product of certain graphs with P_2 are computed.

2. Bound-I for Detour Distance Laplacian Energy

Lemma 2.1. *Let G be a connected (n,m) graph and let $\gamma_1^L(G)$, $\gamma_2^L(G)$, $\gamma_3^L(G), \ldots, \gamma_n^L(G)$ are the detour distance laplacian eigen values. Then*

$$\sum_{i=1}^{n} \gamma_i^L = \sum_{i=1}^{n} DDL_{ii} \text{ and } \sum_{i=1}^{n} (\gamma_i^L)^2 = \sum_{i=1}^{n} DDL_{ii}^2 + 2 \sum_{1\leq i<j\leq n}^{n} DDL_{ij}^2.$$

Proof. $trace(DDL(G)) = \sum_{i=1}^{n} DDL_{ii}$. Since $\sum_{i=1}^{n} \gamma_i^L = trace(DDL(G))$.

It follows that $\sum_{i=1}^{n} \gamma_i^L = \sum_{i=1}^{n} DDL_{ii}(G)$.

For $i = 1, 2, 3 \ldots, n$, the (i,i) entry of $(DDL(G))^2$ is equal to sum of square of (i,i)-entry of $DDL(G)$ and $\sum_{i=1}^{n} \left(DDL_{ij}\right)^2$. Hence

$$\begin{aligned}
\sum_{i=1}^{n} (\gamma_i^L)^2 &= trace(DDL(G))^2 \\
&= \sum_{i=1}^{n} DDL_{ii}^2 + 2 \sum_{1\leq i<j\leq n}^{n} DDL_{ij}^2. \qquad \square
\end{aligned}$$

Theorem 2.2. *If G is a connected (n,m) graph, then*

$$\sqrt{\sum_{i=1}^{n} DDL_{ii}^2 + 2 \sum_{1\leq i<j\leq n}^{n} DDL_{ij}^2} \leq E_{DDL}(G) \leq \sqrt{n(\sum_{i=1}^{n} DDL_{ii}^2 + 2 \sum_{1\leq i<j\leq n}^{n} DDL_{ij}^2)}\ .$$

Proof. Consider the Cauchy-Schwartz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Let us choose $a_i = 1$ and $b_i = |\gamma_i^l|$, we get

$$\left(\sum_{i=1}^{n}\left|\gamma_i^L\right|\right)^2 \le n\left(\sum_{i=1}^{n}(\gamma_i^L)^2\right),$$

$$E_{DDL}(G)^2 \le n\left(\sum_{i=1}^{n} DDL_{ii}^2 + 2\sum_{1\le i<j\le n}^{n} DDL_{ij}^2\right).$$

This is the upper bound for $E_{DDL}(G)$. Now,

$$E_{DDL}(G)^2 = \left(\sum_{i=1}^{n}|\gamma_i^L(G)|\right)^2$$

$$E_{DDL}(G)^2 \ge \sum_{i=1}^{n}|\gamma_i^L(G)|^2$$

$$\ge \sum_{i=1}^{n} DDL_{ii}^2 + 2\sum_{1\le i<j\le n}^{n} DDL_{ij}^2.$$

This is the lower bound for $E_{DDL}(G)$. □

Lemma 2.3. *If G is a connected (n,m) graph, then $E_{DDL}(G) \le n(n-1)\sqrt{n(n-1)}$.*

Proof. For any connected Graph $DDL_{ij} \le n-1$, $i \ne j$. There are $n(n-1)/2$ pairs of vertices in G.

$$\sum_{1\le i<j\le n}^{n} DD_{ij}^2 \le \frac{(n-1)^2\, n(n-1)}{2},$$

$$\sum_{1=n}^{n} DD_{ii}^2 = n\,(n-1)^4.$$

From the upper bound of Theorem 2.2,

$$\le \sqrt{n\left(n(n-1)^4 + 2\left(\frac{n\,(n-1)^3}{2}\right)\right)}$$

$$\le \sqrt{n^2(n-1)^4 + n^2(n-1)^3}$$

$$E_{DDL}(G) \le n(n-1)\sqrt{n(n-1)}.$$ □

3. Bound-II for Detour Distance Laplacian Energy

Lemma 3.1. *For any connected graph $G(n,m), 2DI = \sum_{i=1}^{n}\gamma_i^L$.*

Proof. The Detour distance matrix is a lower or upper triangular square matrix of order $n\times n$, whose entries are $D(u_i,u_j)$ for $i \ne j$ and $D(u_i,u_i) = 0$.

The detour index DI is defined as

$$DI(G)=\sum_{1\le i<j\le n}^{n} D(u_i,u_j),$$

$$2DI(G)=2\sum_{1\le i<j\le n}^{n} D(u_i,u_j)$$

$$=trace(DDL(G)),$$

$$2DI=\sum_{i=1}^{n}\gamma_i^L.$$

Remark 3.2. By the result [22], $2(n-1)^2 \le E_{DDL}(G)\le n(n-1)^2$. The left inequality holds if and only if $G=S_n$ and the right inequality holds if and only if $G=K_n$.

4. Detour Distance Laplacian Energy of Standard Graphs

Theorem 4.1. *If G is a complete graph of order n, then the detour distance laplacian energy of G is $E_{DDL}(G)=n(n-1)^2$.*

Proof. In G, the detour distance between two adjacency vertices is $(n-1)$. The transmission $T_r(V)$ of a vertex v is defined as the sum of the detour distances from v to all other vertices in G. It follows that, $\mathrm{Diag}(T_r)=(n-1)(n-1)$.

Then the detour distance laplacian matrix $DDL(G)=(n-1)(n-1)-DD$.

The characteristic polynomial of $DDL(G)$ is

$$\phi_{DDL}(\gamma^L)=(\gamma^L)(\gamma^L-n(n-1))^{n-1}.$$

Detour distance laplacian spectra is $spec_{DDL}(G)=\begin{pmatrix}0 & n(n-1)\\ 1 & n-1\end{pmatrix}$.

Hence $E_{DDL}(G)=n(n-1)^2$. □

Corollary 4.2. *The detour distance laplacian energy of circulant graph $C_n\left(\pm\left\{1,2,3,\dots,\left\lfloor\frac{n}{2}\right\rfloor\right\}\right)$, Cocktail party graph and wheel graph is same as complete graph.*

Theorem 4.3. *If G is a complete bipartite graph K_{n_1,n_2} $(n_1+n_2=n)$ then the detour distance laplacian energy of K_{n_1,n_2} is $E_{DDL}(G)=n/2(2n^2-5n+4)$, when $n_1=n_2$ and*

$$E_{DDL}(G)=2n_1(n-1)^2,\quad when\ n_1<n_2.$$

Proof. Let $V(G)=V_1\cup V_2$.

Case (i): $n_1=n_2$
In K_{n_1,n_2} the detour distance between the vertices of V_1 to V_2 is $(2n-1)$ and the detour distance between the vertices of V_1 to itself is $2(n-1)$ and vice-versa.

It follows that $\text{Diag}(Tr) = \frac{(2n^2-5n+4)}{2}$.

Then $DDL(G)$ is $\frac{(2n^2-5n+4)}{2} - DD$.

The characteristic polynomial of G is $\phi_{DDL}(\gamma^L) = (\gamma^L)(\gamma^L - n(n-1))\left(\gamma^2 - \frac{n}{2}(4n-3)\right)^{(n-2)}$. Detour distance laplacian spectra is $spec_{DDL}(G) = \begin{pmatrix} 0 & n(n-1) & \frac{n}{2}(2n-3) \\ 1 & 1 & n-2 \end{pmatrix}$.

Hence $E_{DDL}(G) = n/2(2n^2 - 5n + 4)$.

Case (ii): $n_1 < n_2$

In K_{n_1,n_2} with $n_1 = |V_1| < |V_2| = n_2$, $n_1 + n_2 = n$. The detour distance between the distinct vertices $u, v \in V_1$ is $2n_1 - 2$ and the detour distance between the distinct vertices $u, v \in V_2$ is $2n_1$ and the detour distance between the vertices $u \in V_1$ and $v \in V_2$ is $2n_1 - 1$ then

$$DI(G) = (2n_1 - 2)[n_1(n_1-1)]/2 + 2n_1[n_{2(}n_2 - 1)]/2 + (2n_1 - 1)n_1n_2 = n_1(n-1)^2,$$

$$E_{DDL}(G) = 2n_1(n-1)^2. \qquad \square$$

Corollary 4.4. *If G is a crown graph, $n \geq 4$, then $E_{DDL}(G)$ is same as $K_{n,n}$.*

Theorem 4.5. *If G is a cycle of length n, then the detour distance laplacian energy of G is*

$$E_{DDL}(G) = \begin{cases} \frac{3n^3-4n^2+n}{4}, & \text{if } n \text{ is odd} \\ \frac{3n^3-4n^2}{4}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $G = C_n$ be the cycle graph of order n. Let $V(C_n) = \{u_1, u_2, u_3, \ldots, u_n\}$ and $E(Cn) = \{u_iu_{i+1}; 1 \leq i \leq n-1\} \cup \{u_nu_1\}$ be respectively the vertex set and edge set of G. Assume the vertices of G are arranged in clockwise direction. The detour distance matrix $DDM = (DD_{ij})$ is a $n \times n$ matrix, where DD_{ij} are the detour distances from u_i to u_j $(i \neq j)$ and $DD_{ii} = 0$.

Case (i): when n is odd, the detour distance values in the first row of detour distance matrix are $0, n-1, n-2, \ldots, \left(\frac{n+1}{2}\right), \left(\frac{n+1}{2}\right), \left(\frac{n+1}{2}+1\right), \ldots, n-2, n-1$.

All the entries of other rows are formed cyclically.

$$\text{Diag}(Tr) = 2(n-1) + 2(n-2) + 2(n-3) + \ldots + 2\left(\frac{n+1}{2} + 1\right) + 2\left(\frac{n+1}{2}\right),$$

$$\text{Diag}(Tr) = \frac{3n^2 - 4n + 1}{4},$$

$$DI(G) = \left(\frac{n(3n^2 - 4n + 1)}{8}\right),$$

$$E_{DDL}(G) = \left(\frac{3n^3 - 4n^2 + n}{4}\right).$$

Case (ii): when n is even, the detour distance values in the first row of detour distance matrix are $0, n-1, n-2, \ldots, \left(\frac{n}{2}+1\right), \left(\frac{n}{2}\right), \left(\frac{n}{2}+1\right), \ldots, n-2, n-1$.

All the entries of other rows are formed cyclically.

$$\mathrm{Diag}(Tr) = 2(n-1) + 2(n-2) + 2(n-3) + \ldots + 2\left(\frac{n}{2}+1\right) + \frac{n}{2},$$

$$\mathrm{Diag}(Tr) = \frac{3n^2-4n}{4},$$

$$DI(G) = \left(\frac{n(3n^2-4n)}{8}\right),$$

$$E_{DDL}(G) = \left(\frac{3n^3-4n^2}{4}\right).$$

Aliter. If G is a cycle of length n, then the detour distance laplacian energy of G is

$$E_{DDL}(G) = \sum_{j=0}^{n-1}\left|\sum_{k=1}^{\frac{n-1}{2}}\left[2(n-k)-2(n-k)\cos\left(\frac{2\pi jk}{n}\right)\right]\right|, \text{ when } n \text{ is odd}$$

$$= \sum_{j=0}^{n-1}\left|\sum_{k=1}^{\frac{n-2}{2}}\left[2(n-k)-2(n-k)\cos\left(\frac{2\pi jk}{n}\right)\right] + \frac{n}{2}\left[\cos\left(\frac{n\pi j}{n}\right)+i\sin\left(\frac{n\pi j}{n}\right)\right]\right|,$$

when n is even.

Proof. Let n be odd, the detour distance values in the first row are

$$0,(n-1),(n-2),\ldots,\left(\frac{n+1}{2}\right),\left(\frac{n+1}{2}\right),\left(\frac{n+1}{2}+1\right),\ldots,(n-2),(n-1).$$

It follows that $\mathrm{Diag}(T_r) = \frac{(3n^2-4n+1)}{4}$.

Let n be even,the detour distance values in the first row are

$$0,(n-1),(n-2),\ldots,\left(\frac{n}{2}+1\right),\left(\frac{n}{2}\right),\left(\frac{n}{2}+1\right),\ldots,(n-2),(n-1).$$

It follows that $\mathrm{Diag}(T_r) = \frac{(3n^2-4n)}{4}$.

All the entries of other rows are formed cyclically, it provides a circulant matrix.

The eigen values of G is

$$(\gamma_j^L) = \sum_{k=1}^{\frac{n-1}{2}}\left[2(n-k)-2(n-k)\cos\left(\frac{2\pi jk}{n}\right)\right],\ \ j=0,1,2,\ldots,n-1, \text{ when } n \text{ is odd}$$

$$= \sum_{k=1}^{\frac{n-2}{2}}\left[2(n-k)-2(n-k)\cos\left(\frac{2\pi jk}{n}\right)\right] + \frac{n}{2}\left[\cos\left(\frac{n\pi j}{n}\right)+i\sin\left(\frac{n\pi j}{n}\right)\right],$$

$j=0,1,2,\ldots,n-1$, when n is even.

Hence

$$E_{DDL}(G)=\sum_{j=0}^{n-1}\left|\sum_{k=1}^{\frac{n-1}{2}}\left[2(n-k)-2(n-k)\cos\left(\frac{2\pi jk}{n}\right)\right]\right|, \text{ when } n \text{ is odd}$$

$$=\sum_{j=0}^{n-1}\left|\sum_{k=1}^{\frac{n-2}{2}}\left[2(n-k)-2(n-k)\cos\left(\frac{2\pi jk}{n}\right)\right]+\frac{n}{2}\left[\cos\left(\frac{n\pi j}{n}\right)+i\sin\left(\frac{n\pi j}{n}\right)\right]\right|,$$

when n is even. □

Theorem 4.6. *If G is a ladder graph,then the detour distance laplacian energy of G is*

$$E_{DDL}(G)\begin{cases}8n^3-115n^2+830n-1537, & \textit{when } n \textit{ is odd.}\\ 8n^3-13n^2+14n-8, & \textit{when } n \textit{ is even.}\end{cases}$$

Proof. Let G be a ladder graph. Let $V(G)=\{u_1,u_2,u_3,\dots,u_n\}$ and $E(G)=\{u_iu_{i+1};i=1,3,5,\dots,2n-1\}\cup\{u_iu_{i+2};i=1,3,5,\dots,2n-3\}\cup\{u_iu_{i+2};i=2,4,6,\dots,2n-2\}$ be the vertex set and edge set, respectively.

Case (i): when n is odd and $n\ge 3$.
The following table shows that the number of vertices and its detour distances of G.

Number of vertices	Detour distance
n^2-n+2	$2n-1$
n^2-3n+6	$2n-2$
2	$2n-3$
4	$2n-4$
2	$2n-5$
4	$2n-6$
$\vdots$	$\vdots$
2	$n+2$
4	$n+1$
1	n

$$=(2n-1)n^2-n+2+(2n-2)n^2-3n+6+\dots+2\{2n-3+2n-5+\dots+n+2\}$$
$$+\ 4\{2n-4+2n-6+\dots+n+1\}+n\,,$$

$$DI(G)=\left(\frac{8n^3-115n^2+830n-1537}{2}\right), \text{ when } n \text{ is odd},$$

$$E_{DDL}(G)=8n^3-115n^2+830n-1537, \text{ when } n \text{ is odd}.$$

Case (ii): when n is even.
The following table shows that the number of vertices and its detour distances of G.

Number of vertices	Detour distance
n^2-n+2	$2n-1$
n^2-3n+6	$2n-2$
2	$2n-3$
4	$2n-4$
2	$2n-5$
4	$2n-6$
$\vdots$	$\vdots$
2	$n+3$
4	$n+2$
1	$n+1$
2	n

$$= (2n-1)n^2-n+2+(2n-2)n^2-3n+6+\ldots+2\{2n-3+2n-5+\ldots+n+1+n\}$$

$$+\ 4\{2n-4+2n-6+\ldots+n+2\},$$

$$DI(G)=\left(\frac{8n^3-13n^2+14n-8}{2}\right),\ \text{when } n \text{ is even},$$

$$E_{DDL}(G)=8n^3-13n^2+14n-8,\ \text{when } n \text{ is even}.$$

5. The Detour Distance Laplacian Energy of Cartesian product of certain Graphs with P_2

Theorem 5.1. *If G is a complete graph then the detour distance laplacian energy of $P_2\times G$ is $E_{DDL}(P_2\times G)=2n(2n-1)^2$.*

Proof. Let $V(P_2)=\{u_1,u_2\}$ and $V(G)=\{v_1,v_2,v_3,\ldots,v_n\}$ be the vertex set of P_2 and G, respectively.

Let $V(P_2\times G)=\{(u_1,v_j),(u_2,v_j);1\le j\le n\}$ be the vertex set of $P_2\times G$.

In $P_2\times G$ the detour distance between any two vertices (u_i,v_j), $i=1$ to 2, $j=1,2,3,\ldots,n$ is $2n-1$.

$\mathrm{Diag}(T_r)=(2n-1)^2$ then $DDL(G)=(2n-1)^2-DD$.

The characteristic polynomial of $P_2\times G$ is $\phi_{DDL}(\gamma^L)=\left(\gamma^L\right)\left(\gamma^L-2n(2n-1)\right)^{2n-1}$.

Detour distance laplacian spectra is $spec_{DDL}(P_2\times G)=\begin{pmatrix}0 & 2n(2n-1)\\ 1 & 2n-1\end{pmatrix}$.

Hence $E_{DDL}(P_2\times G)=2n(2n-1)^2$. □

Corollary 5.2. *If G is a cocktail party graph then $E_{DDL}(P_2\times G)=2n(2n-1)^2$.*

Theorem 5.3. *If G is a complete bipartite graph $K_{n1,n2}$ when $n_1+n_2=n$ and $n_1=n_2$, then the detour distance laplacian energy of $P_2\times G$ is $E_{DDL}(P_2\times G)=2n(4n^2-5n+2)$.*

Proof. Let $V(P_2) = \{u_1, u_2\}$ be the vertex set of P_2.

Let $V(G) = X \cup Y$ be the partition of $V(G)$ with $V(X) = \{x_1, x_2, x_3, \dots, x_{n1}\}$ and $V(Y) = \{y_1, y_2, y_3, \dots, y_n\}$.

Let $V(P_2 \times G) = \{(u_1, x_j), (u_1, y_j), (u_2, x_j), (u_2, y_j)\}$, $j = 1, 2, 3, \dots, n/2$ be the vertex set of $P_2 \times G$.

The detour distance between the vertices (u_1, x_j) and (u_2, x_j), (u_1, y_j) and (u_2, y_j), $j = 1, 2, 3, \dots, n/2$ is $2n-1$. The detour distance between any two vertices in (u_i, x_j) and the detour distance between any two vertices in (u_i, y_j), $i = 1$ to 2 and $j = 1, 2, 3, \dots, n/2$, $i \neq j$ is $2n-2$ and 0 if $i = j$.

It follows that $\text{Diag}(T_r) = 4n^2 - 5n + 2$.

Then $DDL(G) = 4n^2 - 5n + 2 - DD$.

The characteristic polynomial of $P_2 \times G$ is

$$\phi_{DDL}(\gamma^L) = (\gamma^L)(\gamma^L - n(4n-3))^{2n-2}(\gamma^L - n(4n-2)).$$

Detour distance laplacian spectra is $spec_{DDL}(P_2 \times G) = \begin{pmatrix} 0 & n(4n-3) & n(4n-2) \\ 1 & 2n-2 & 1 \end{pmatrix}$.

Hence $E_{DDL}(P_2 \times G) = 2n(4n^2 - 5n + 2)$. □

Corollary 5.4. *If G is a crown graph, $n \geq 4$, then $E_{DDL}(P_2 \times G) = 2n(4n^2 - 5n + 2)$.*

Theorem 5.5. *If G is a cycle C_n, then the detour distance laplacian energy of $P_2 \times G$ is $E_{DDL}(P_2 \times G) = 2n(2n-1)^2$, when n is odd, $E_{DDL}(P_2 \times G) = 2n(4n^2 - 5n + 2)$, when n is even.*

Proof. Let $V(P_2) = \{u_1, u_2\}$ and $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$ be the vertex set of P_2 and G, respectively.

Let $V(P_2 \times G) = \{(u_1, v_j), (u_2, v_j); 1 \leq j \leq n\}$ be the vertex set of $P_2 \times G$.

Case (i): when n is odd and $n \geq 3$.
In $P_2 \times G$ the detour distance between any two vertices (u_i, v_j), $i = 1$ to 2, $j = 1, 2, 3, \dots, n$ is $2n-1$.

$$\text{Diag}(T_r) = (2n-1)^2.$$

Then $DDL(G) = (2n-1)^2 - DD$.

The characteristic polynomial of $P_2 \times G$ is $\phi_{DDL}(\gamma^L) = (\gamma^L)(\gamma^L - 2n(2n-1))^{2n-1}$.

Detour distance laplacian spectra is $spec_{DDL}(P_2 \times G) = \begin{pmatrix} 0 & 2n(2n-1) \\ 1 & 2n-1 \end{pmatrix}$.

Hence $E_{DDL}(P_2 \times G) = 2n(2n-1)^2$.

Case (ii): when n is even and $n > 4$.

In $P_2 \times G$, the detour distance between (u_1, v_1) and (u_1, v_j), $j = 1, 2, 3, \dots, n$ are $0, 2n-2, 2n-1, 2n-2, \dots, 2n-2, 2n-1$. All the other entries between (u_1, v_2) and (u_1, v_j), (u_1, v_3) and

$(u_1,v_j),\ldots,(u_1,v_n)$ and (u_1,v_j) are formed cyclically. The detour distance between (u_1,v_1) and (u_2,v_j), $j=1,2,3,\ldots,n$ are $2n-1,2n-2,2n-1,2n-2,\ldots,2n-1,2n-2$. All the other entries between (u_1,v_2) and (u_2,v_j), (u_1,v_3) and $(u_2,v_j),\ldots,(u_1,v_n)$ and (u_2,v_j) are formed cyclically. The detour distance between (u_2,v_1) and (u_1,v_j), $j=1,2,3,\ldots,n$ are $2n-1,2n-2,2n-1,2n-2,\ldots,2n-1,2n-2$. All the other entries between (u_2,v_2) and (u_1,v_j), (u_2,v_3) and $(u_1,v_j),\ldots,(u_2,v_n)$ and (u_1,v_j) are formed cyclically. The detour distance between (u_2,v_1) and (u_2,v_j), $j=1,2,3,\ldots,n$ are $0,2n-1,2n-2,2n-1,2n-2,\ldots,2n-1$. All the other entries between (u_2,v_2) and (u_2,v_j), (u_2,v_3) and $(u_2,v_j),\ldots,(u_2,v_n)$ and (u_2,v_j) are formed cyclically.

It follows that $\mathrm{Diag}(T_r)=4n^2-5n+2$.

Then $DDL(G)=4n^2-5n+2-DD$.

The characteristic polynomial of $P_2\times G$ is

$$\phi_{DDL}(\gamma^L)=(\gamma^L)(\gamma^L-n(4n-3))^{2n-2}(\gamma^L-n(4n-2)).$$

Detour distance laplacian spectra is

$$spec_{DDL}(P_2\times G)=\begin{pmatrix}0 & n(4n-3) & n(4n-2)\\ 1 & 2n-2 & 1\end{pmatrix}.$$

Hence $E_{DDL}(P_2\times G)=2n(4n^2-5n+2)$. □

6. Conclusion

From the results of this paper it is concluded that the complete graph, wheel graph, cocktail party graph and circulant graph are detour distance laplacian equienergetic graphs. Further if the detour index is known the detour distance laplacian energy of a graph can be obtained.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] C. Adiga, A. Bayad, I. Gutman and S.A. Srinivas, The minimum coveringenergy of a graph, *Kragujevac J. Sci.* **34** (2012), 39 – 56.

[2] C. Adiga and M. Smitha, On maximum degree energy of a graph, *Int. J. Contemp. Math. Sciences* **4** (8) (2009), 385 – 396.

[3] M. Aouchiche and P. Hansen, Two Laplacians for the distance matrix of agraph, *Linear Algebra Appl.* **439** (2013), 21 – 33.

[4] S.K. Ayyasamy and S. Balachandran, On detour spectra of some graphs, *World Academy of Science, Engineering and Technology* **4**, 2010-07-21.

[5] R. Balakrishnan, The energy of a graph, *Lin. Algebra Appl.* **387** (2004), 287 – 295.

[6] S.B. Bozkunt, A.D. Gungor and B. Zhou, Note on the distance energy of graphs, *MATCH Commun. Math Comput. Chem.* **64** (2010), 129 – 134.

[7] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, Redwood (1990).

[8] M. Edelberg, M.R. Garey and R.L. Graham, On the distance matrix of a tree, *Discr. Math.* **14** (1976), 23 – 29.

[9] A.D. Gunger and S.B. Bozkurt, On the distance spectral radius and distance energy of graphs, *Lin. Multilin. Algebra* **59** (2011), 365 – 370.

[10] I. Gutman, The energy of a graph, *Ber. Math. Statist. Sekt. for Schungsz. Ghaz* **103** (1978), 1 – 22.

[11] I. Gutman, The energy of a graph: old and new results, in *Algebraic Combinatorics and Applications*, A. Betten, A. Kobnert, R. Lave and A. Wassermann (eds.), Springer, Berlin (2001), 196 – 211.

[12] R.L. Graham and L. Lovasz, Distance matrix polynomials of trees, *Adv. Math.* **29** (1978), 60 – 88.

[13] H. Hua, On minimal energy of unicycle graphs with prescribed girth and pendent vertices, *MATCH Commun. Math. Comput. Chem.* **57** (2007), 351 – 361.

[14] G. Indulal, Sharp bounds on the distance spectral radius and the distance energy of graphs, *Lin. Algebra Appl.* **430** (2009), 106 – 113.

[15] G. Indulal, I. Gutman and A. Vijayakumar, On distance energy of graphs, *MATCH Commun. Math. Comput. Chem.* **60** (2008), 461 – 472.

[16] J. Yang, L. You and I. Gutman, Bounds on the distance laplacian energy of graphs, *Kragujevac Journal of Mathematics* **37** (2) (2013), 245 – 255.

[17] J.H. Koolen and V. Moulton, Maximal Energy Graphs, *Adv. Appl. Math.* **26** (2001), 47 – 52.

[18] I. Lukovits, The detour index, *Croat. Chem. Acta* **69** (1996), 873 – 882.

[19] I. Lukovits and M. Razinger, On calculation of the detour index, *J. Chem. Inf. Comput. Sci.* **37** (1997), 283 – 286.

[20] H.S. Ramane, D.S. Revankar, I. Gutman, S.B. Rao, B.D. Acharya and H.B. Walikar, Bounds for the distance energy of a graph, *Kragujevac J. Math.* **31** (2008), 59 – 68.

[21] N. Trinajstic, S. Nikolic, B. Lucic, D. Amic and Z. Mihalic, The detour matrix in chemistry, *J. Chem. Inf. Comput. Sci.* **37** (1997), 631 – 638.

[22] B. Zohu and X. Cai, On detour index, *MATCH Commun. Math. Comput. Chem.* **63** (2010), 199 – 210.