Intuitionistic Fuzzy D^*-Metric Spaceand Some Topological Properties

Masoumeh Vali

Abstract. The notion of intuitionistic fuzzy metric space was introduced by Park [*Chaos Solitons and Fractals* **22**(2004), 1039–1046] and the concept of D^*-metric space by Sedghi, Shobe and Zhou [*Fixed Point Theory and Applications* 2007, **Article ID 27906**, 13 pages]. In this paper, we define intuitionistic fuzzy D^*-metric space and study precompactness and metrizability in this new setup.

1. Introduction and Preliminaries

Among various developments of the fuzzy set theory [29], a progressive development has been made to find the fuzzy analogues of the classical set theory. Infact the fuzzy theory has become an area of active research for the last forty years. It has a wide range of applications in the field of science and engineering, e.g. population dynamics [5], chaos control [19], computer programming [21], nonlinear dynamical systems [5], medicine [4] etc.

Fuzzy topology is one of the most important and useful tool studied by various authors, e.g. [16, 17, 18, 20, 23, 28]. The most fascinating application of the fuzzy topology in quantum physics arises in $e^{(\infty)}$-theory due to El Naschie ([9]-[15]) who presented the relation of fuzzy Kähler interpolation of $e^{(\infty)}$ to the recent work on cosmo-topology and the Poincaré dodecahedral conjecture and gave various applications and results of $e^{(\infty)}$-theory from nano technology to brain research.

Atanassov [2, 3] introduced the concept of intuitionistic fuzzy sets. For intuitionistic fuzzy topological spaces, we refer [1], [6], [23], [24] and [25]. Recently, Sedghi, Shobe and Zhou [27] studied the notion of D^*-metric space. $D^\star$-metric spaces is a probable modification of the definition of D-metric introduced by Dhage [7, 8].

In this paper we introduce the concept of intuitionistic fuzzy $D^\star$-metric space which would provide a more suitable funtional tool to deal with the inexactness

of the metric or $D^\star$-metric in some situations. We present here analogues of precompactness and metrizability and establish some interesting results in this new setup.

We recall some notations and basic definitions used in this paper.

Definition 1.1 ([26]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if satisfies the following conditions:

(a) $*$ is associative and commutative,
(b) $*$ is continuous,
(c) $a * 1 = a$ for all $a \in [0,1]$,
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Definition 1.2 ([26]). A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-conorm if satisfies the following conditions:

(a) $\Diamond$ is associative and commutative,
(b) $\Diamond$ is continuous,
(c) $a \Diamond 0 = a$ for all $a \in [0,1]$,
(d) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Definition 1.3 ([25]). The five-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space (for short, IFMS) if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm, and M, N fuzzy sets on $X \times X \times (0, \infty)$ satisfying the following conditions. For every $x, y, z \in X$ and $s, t > 0$,

(a) $M(x, y, t) + N(x, y, t) \leq 1$,
(b) $M(x, y, t) > 0$,
(c) $M(x, y, t) = 1$ if and only if $x = y$,
(d) $M(x, y, t) = M(y, x, t)$,
(e) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(f) $M(x, y, \cdot) : (0, \infty) \to [0,1]$ is continuous,
(g) $N(x, y, t) < 1$,
(h) $N(x, y, t) = 0$ if and only if $x = y$,
(i) $N(x, y, t) = N(y, x, t)$,
(j) $N(x, y, t) \Diamond N(y, z, s) \geq N(x, z, t + s)$,
(k) $N(x, y, \cdot) : (0, \infty) \to [0,1]$ is continuous,

Then (M, N) is called an intuitionistic fuzzy metric on X.

Definition 1.4 ([27]). Let X be a nonempty set. A generalized metric (or $D^\star$-metric) on X is a function, $D^\star : X^3 \to [0, \infty)$, that satisfies the following conditions for each $x, y, z, a \in X$:

(1) $D^\star(x, y, z) \geq 0$,
(2) $D^\star(x, y, z) = D^\star(p\{x, y, z\})$, (symmetry) where p is a permutation function,
(3) $D^\star(x, y, z) = 0$ if and only if $x = y = z$,

(4) $D^\star(x, y, z) \leq D^\star(x, y, a) + D^\star(a, z, z)$.

The pair $(X, D^\star)$ is called a generalized metric (or $D^\star$-metric)space.

 Immediate examples of such a function are

 (a) $D^\star(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,
 (b) $D^\star(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Here, d is the ordinary metric space on X.

2. **Intuitionistic Fuzzy $D^\star$-Metric Space**

 Recently, the concept of $D^\star$-metric spaces has been introduced by Sedghi, Shobe and Zhou [27]. In this section we introduce the concept of intuitionistic fuzzy $D^\star$-metric space and we also define here some topological concepts in intuitionistic fuzzy $D^\star$-metric space.

Definition 2.1. The 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy $D^\star$-metric space (for short, IF-$D^\star$-MS) if X is a $D^\star$-metric space, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm, and M, N fuzzy sets on $X \times X \times X \times (0, \infty)$, satisfying the following conditions for each $x, y, z, w \in X$ and $s, t > 0$,

 (a) $M(x, y, z; t) + N(x, y, z; t) \leq 1$,
 (b) $M(x, y, z; t) > 0$,
 (c) $M(x, y, z; t) = 1$ if and only if $x = y = z$,
 (d) $M(x, y, z; t) = M(p\{x, y, z\}; t)$, where p is a permutation function,
 (e) $M(x, y, w; t) * M(w, z, z; s) \leq M(x, y, z; t + s)$ for all $x, y, z, w \in X$,
 (f) $M(x, y, z; \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous,
 (g) $N(x, y, z; t) < 1$,
 (h) $N(x, y, z; t) = 0$ if and only if $x = y = z$,
 (i) $N(x, y, z; t) = N(p\{x, y, z\}; t)$, where p is a permutation function,
 (j) $N(x, y, w; t) \Diamond N(w, z, z; s) \geq N(x, y, z; t + s)$,
 (k) $N(x, y, z; \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous.

 In this case (M, N) is called an intuitionistic fuzzy $D^\star$-metric on X and we denote it by $(M, N)_{D^*}$. The functions $M(x, y, z; t)$ and $N(x, y, z; t)$ denote the degree of nearness and the degree of non nearness between x, y and z with respect to t, respectively.

Remark 2.1. In an intuitionistic fuzzy $D^\star$-metric space $(X, M, N, *, \Diamond)$, $M(x, y, z; \cdot)$ is non-decreasing and $N(x, y, z; \cdot)$ is non-increasing for all $x, y, z \in X$.

 Let $(X, D^\star)$ be a $D^\star$-metric space. Denote $a * b = ab$ and $a \Diamond b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let M_{D^*} and N_{D^*} be fuzzy sets on $X^3 \times (0, \infty)$ defined by:

$$M_{D^*}(x, y, z; t) = \frac{ht^n}{ht^n + mD^\star(x, y, z)}, \quad N_{D^*}(x, y, z; t) = \frac{D^\star(x, y, z)}{kt^n + mD^\star(x, y, z)}$$

for all $h, k, m, n \in \mathbb{R}^+$. Then $(X, M_{D^*}, N_{D^*}, *, \Diamond)$ is an intuitionistic fuzzy D^*-metric space.

Definition 2.2. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy D^*-metric space, and let $r \in (0, 1)$, $t > 0$ and $x \in X$. The set $\mathbb{B}_{D^*}(x, r, t) = \{y \in X : M(x, y, y; t) > 1 - r, \ N(x, y, y; t) < r\}$ is called the open ball with center x and radius r with respect to t.

Definition 2.3. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy D^*-metric space, then a set $U \subset X$ is said to an open set if each of its points is the center of some open ball contained in U. The open set in an intuitionistic fuzzy D^*-metric space $(X, M, N, *, \Diamond)$ is denoted by $\mathbb{U}$.

Definition 2.4. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy D^*-metric space. A subset A of X is said to be IF_{D^*}-bounded if there exists $t > 0$ and let $\epsilon \in (0, 1)$ such that $M(x, y, y; t) > 1 - \epsilon$ and $N(x, y, y; t) < \epsilon$ for all $x, y \in A$.

Definition 2.5. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy D^*-metric space. A sequence (x_n) in X is said to be Cauchy if for each $\epsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, x_m; t) > 1 - \epsilon$ and $N(x_n, x_m, x_m; t) < \epsilon$ for all $n, m \geq n_0$.

Definition 2.6. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy D^*-metric space. A sequence (x_n) in X is said to be convergence to $L \in X$, with respect to the intuitionistic fuzzy D^*-metric space $(M, N)_{D^*}$ if for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $M(x_k, L, L; t) > 1 - \epsilon$ and $N(x_k, L, L; t) < \epsilon$ for all $k \geq k_0$. In this case we write $(M, N)_{D^*} - \lim x_k = L$ or $x_k \xrightarrow{(M,N)_{D^*}} L$ as $k \to \infty$.

Definition 2.7. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy D^*-metric space. Define $\tau_{(M,N)_{D^*}} = \{A \subset X : \text{for each } x \in A, \text{ there exists } t > 0 \text{ and } r \in (0, 1) \text{ such that } \mathbb{B}_{D^*}(x, r, t) \subset A\}$. Then $\tau_{(M,N)_{D^*}}$ is a topology on $(X, M, N, *, \Diamond)$.

Definition 2.8. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy D^*-metric space. Then it is said to be complete if every Cauchy sequence is convergent with respect to $\tau_{(M,N)_{D^*}}$.

3. **Precompactness in Intuitionistic Fuzzy D^*-Metric Spaces**

In this section, we define and establish some results precompactness in intuitionistic fuzzy D^*-metric space.

Definition 3.1. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy D^*-metric space and $A \subset X$. Then A is precompact if for each $r \in (0, 1)$ and $t > 0$, there exists a finite subset S of A such that

$$A \subseteq \bigcup_{x \in S} \mathbb{B}_{D^*}(x, r, t).$$

Lemma 3.1. *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy $D^\star$-metric space and $A \subset X$. Then A is precompact if and only if for every $r \in (0, 1)$ and $t > 0$, there exists a finite subset S of A such that*

$$A \subseteq \bigcup_{x \in S} \mathbb{B}_{D^*}(x, r, t). \tag{3.1}$$

Proof. Let $r \in (0, 1)$ and $t > 0$ and condition (3.1) holds. By the continuity of $*, \Diamond$, there exists $s \in (0, 1)$ such that $(1 - s) * (1 - s) > 1 - r$ and $s \Diamond s < r$. Now we apply condition (3.1) for s and $\frac{t}{2}$, there exists a subset $S' = \{x_1, \cdots, x_n\}$ of X such that $A \subseteq \bigcup_{x_i \in S'} \mathbb{B}_{D^*}(x_i, s, \frac{t}{2})$. We assume that $\mathbb{B}_{D^*}(x_j, s, \frac{t}{2}) \bigcap A \neq \emptyset$, otherwise we omit x_j from S' and so we have $A \subseteq \bigcup_{x_i \in S' - \{x_j\}} \mathbb{B}_{D^*}(x_i, s, \frac{t}{2})$. For every $i = 1, \cdots, n$ we select y_i in $\mathbb{B}_{D^*}(x_i, s, \frac{t}{2}) \bigcap A$, therefore $M(x_i, y_i, y_i; \frac{t}{2}) > 1 - s$ and $N(x_i, y_i, y_i; \frac{t}{2}) < s$ and we put $S = \{y_1, \cdots, y_n\}$. Now for every $y \in A$, there exists $i \in \{1, \cdots, n\}$ such that $M(x_i, y, y; \frac{t}{2}) > 1 - s$ and $N(x_i, y, y; \frac{t}{2}) < s$. Therefore we have

$$M(y, y_i, y_i; t) > M(x_i, y_i, y_i; \frac{t}{2}) * M(x_i, y, y; \frac{t}{2}) > (1 - s) * (1 - s) > 1 - r$$

and

$$N(y, y_i, y_i; t) < N(x_i, y_i, y_i; \frac{t}{2}) \Diamond N(x_i, y, y; \frac{t}{2}) < s \Diamond s < r$$

which implies that $A \subseteq \bigcup_{x \in S} \mathbb{B}_{D^*}(x, r, t)$. The converse is trivial. $\square$

Lemma 3.2. *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy $D^\star$-metric space and $A \subset X$. If A is a precompact set then its closure $\overline{A}$ is also precompact.*

Proof. Let $r \in (0, 1)$ and $t > 0$. Then by the continuity of $*, \Diamond$, there exists $s \in (0, 1)$ such that $(1 - s) * (1 - s) > 1 - r$ and $s \Diamond s < r$, also there exists a finite subset $S' = \{x_1, \cdots, x_n\}$ of X such that $A \subseteq \bigcup_{x_i \in S'} \mathbb{B}_{D^*}(x_i, s, \frac{t}{2})$. But for every $y \in \overline{A}$ there exists $x \in A$ such that $M(x, y, y; \frac{t}{2}) > 1 - s$ and $N(x, y, y; \frac{t}{2}) < s$ and there exists $1 \leq i \leq n$, such that $M(x, x_i, x_i; \frac{t}{2}) > 1 - s$ and $N(x, x_i, x_i; \frac{t}{2}) < s$. Therefore we have

$$M(y, x_i, x_i; t) > M\left(x, x_i, x_i; \frac{t}{2}\right) * M\left(x, y, y; \frac{t}{2}\right) > (1 - s) * (1 - s) > 1 - r$$

and

$$N(y, x_i, x_i; t) < N\left(x, x_i, x_i; \frac{t}{2}\right) \Diamond N\left(x, y, y; \frac{t}{2}\right) < s \Diamond s < r.$$

Hence $\overline{A} \subseteq \bigcup_{x_i \in S} \mathbb{B}_{D^*}(x_i, r, t)$, i.e. $\overline{A}$ is precompact set. $\square$

Lemma 3.3. *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy $D^\star$-metric space and $A \subset X$. Then A is a precompact set if and only if every sequence has a Cauchy subsequence.*

Proof. Let A be a precompact set. Let (p_n) be a sequence in A. For every $k \in \mathbb{N}$, there exists a finite subset S_k of X such that $A \subseteq \bigcup_{x \in S_k} \mathbb{B}_{D^*}(x, \frac{1}{k}, \frac{1}{k})$. Hence, for $k = 1$, there exists $x_1 \in S_1$ and a sequence $(p_{1,n})$ of (p_n) such that $p_{1,n} \in \mathbb{B}_{D^*}(x_1, 1, 1)$ for

every $n \in \mathbb{N}$. Similarly, there exists $x_2 \in S_2$ and a subsequence $(p_{2,n})$ of $(p_{1,n})$ such that $p_{2,n} \in \mathbb{B}_{D^*}\left(x_2, \frac{1}{2}, \frac{1}{2}\right)$ for every $n \in \mathbb{N}$. Continuing this process, we get $x_k \in S_k$ and a subsequence $(p_{k,n})$ of $(p_{k-1,n})$ such that $p_{k,n} \in \mathbb{B}_{D^*}\left(x_k, \frac{1}{k}, \frac{1}{k}\right)$ for every $n \in \mathbb{N}$. Now we consider the subsequence $(p_{n,n})$ of (p_n). For every $r \in (0,1)$ and $t > 0$, by the continuity of $*, \Diamond$, there exists a $n_0 \in \mathbb{N}$ such that $\left(1 - \frac{1}{n_0}\right) * \left(1 - \frac{1}{n_0}\right) > 1 - r$, $\frac{1}{n_0} \Diamond \frac{1}{n_0} < r$ and $\frac{2}{n_0} < t$. Therefore for every $l, m \geq n_0$, we have

$$M(p_{l,l}, p_{m,m}, p_{m,m}; t) \geq M\left(p_{l,l}, p_{m,m}, p_{m,m}; \frac{2}{n_0}\right)$$

$$M\left(p_{l,l}, p_{m,m}, p_{m,m}; \frac{2}{n_0}\right) \geq M\left(x_{n_0}, p_{m,m}, p_{m,m}; \frac{1}{n_0}\right) * M\left(x_{n_0}, p_{l,l}, p_{l,l}; \frac{1}{n_0}\right)$$

$$> \left(1 - \frac{1}{n_0}\right) * \left(1 - \frac{1}{n_0}\right)$$

$$> 1 - r$$

and

$$N(p_{l,l}, p_{m,m}, p_{m,m}; t) \leq N\left(p_{l,l}, p_{m,m}, p_{m,m}; \frac{2}{n_0}\right)$$

$$N\left(p_{l,l}, p_{m,m}, p_{m,m}; \frac{2}{n_0}\right) \leq N\left(x_{n_0}, p_{m,m}, p_{m,m}; \frac{1}{n_0}\right) \Diamond M\left(x_{n_0}, p_{l,l}, p_{l,l}; \frac{1}{n_0}\right)$$

$$< \frac{1}{n_0} \Diamond \frac{1}{n_0}$$

$$< r.$$

Hence $(p_{n,n})$ is a Cauchy sequence in $(X, M, N, *, \Diamond)$.

Conversely, suppose that A is not a precompact set. Then there exists $r \in (0,1)$ and $t > 0$ such that for every finite subset S of X, A is not a subset of $A \subseteq \bigcup_{x \in S} \mathbb{B}_{D^*}(x, r, t)$. Fix $p_1 \in A$. Since A is not a subset of $A \subseteq \bigcup_{x \in \{p_1\}} \mathbb{B}_{D^*}(x, r, t)$, there exists $p_2 \in A$ such that $M(p_1, p_2, p_2; t) \leq 1 - r$ and $N(p_1, p_2, p_2; t) \geq r$. Since A is not a subset of $A \subseteq \bigcup_{x \in \{p_1, p_2\}} \mathbb{B}_{D^*}(x, r, t)$, there exists $p_3 \in A$ such that $M(p_1, p_3, p_3; t) \leq 1 - r$ and $N(p_1, p_3, p_3; t) \geq r$. Continuing this process, we construct a sequence (p_n) of distinct points in A such that $M(P_i, P_j, P_j; t) \leq 1 - r$ and $N(P_i, P_j, P_j; t) \geq r$ for every $i \neq j$. Therefore (p_n) has not Cauchy subsequence. $\square$

Lemma 3.4. *Let (x_n) be a Cauchy sequence in an intuitionistic fuzzy D^*-metric space $(X, M, N, *, \Diamond)$ having a cluster point $x \in X$. Then (x_n) is convergent to x.*

Proof. Since (x_n) is a Cauchy sequence in $(X, M, N, *, \Diamond)$ having a cluster point $x \in X$. Then, there is a subsequence (x_{n_k}) of (x_n) that converges to x with respect to $\tau_{(M,N)_{D^*}}$. Thus, given $r \in (0,1)$ and $t > 0$, there is a $l \in \mathbb{N}$ such that for each $k \geq l$, $M\left(x, x_{n_k}, x_{n_k}; \frac{t}{2}\right) > 1 - s$ and $N\left(x, x_{n_k}, x_{n_k}; \frac{t}{2}\right) < s$, where $s \in (0,1)$ and satisfies $(1-s)*(1-s) > 1-r$ and $s \Diamond s < r$. On the other hand, there is $n_t > n_l$ such

that for each $n, m \geq n_t$, we have $M\left(x_m, x_n, x_n; \frac{t}{2}\right) > 1 - s$ and $N\left(x_m, x_n, x_n; \frac{t}{2}\right) < s$. Therefore for each $n, n_k \geq n_t$, we have

$$M(x, x_n, x_n; t) \geq M\left(x_{n_k}, x_n, x_n; \frac{t}{2}\right) * M\left(x, x_{n_k}, x_{n_k}; \frac{t}{2}\right)$$
$$> (1-s) * (1-s)$$
$$> 1 - r$$

and

$$N(x, x_n, x_n; t) \leq M\left(x_{n_k}, x_n, x_n; \frac{t}{2}\right) \Diamond M\left(x, x_{n_k}, x_{n_k}; \frac{t}{2}\right)$$
$$< (1-s) \Diamond (1-s)$$
$$< r.$$

We conclude that the Cauchy sequence converges to x. □

4. Intuitionistic Fuzzy $D^\star$-Metrizability.

In this section, we define and study intuitionistic fuzzy $D^\star$-metrizability.

Lemma 4.1. *Let* $(X, M, N, *, \Diamond)$ *be an intuitionistic fuzzy $D^\star$-metric space. Then* $(X, \tau_{(M,N)_{D^*}})$ *is a metrizable topological space.*

Proof. For each $n \in \mathbb{N}$, let

$$U_n = \left\{ (x, y, y) \in X \times X \times X : M\left(x, y, y, \frac{1}{n}\right) > 1 - \frac{1}{n}, N\left(x, y, y, \frac{1}{n}\right) < \frac{1}{n} \right\}.$$

We shall prove that $(U_n : n \in \mathbb{N})$ is a base for a uniformity $\mathscr{U}$ on X whose induced topology coincides with $\tau_{(M,N)_{D^*}}$. We first note that for each $n \in \mathbb{N}$, $\{(x, x, x) : x \in X\} \subseteq U_n$, $U_{n+1} \subseteq U_n$ and $U_n = U_n^{-1}$ under the operation $\circ$ defined as

$$U_1 \circ U_2 = (U_1 - U_2) \cup (U_2 - U_1).$$

On the other hand, for each $n \in \mathbb{N}$, there is by the continuity of $*$, $\Diamond$, a $m \in \mathbb{N}$ such that $m > 2n$, $(1 - \frac{1}{m}) * (1 - \frac{1}{m}) > 1 - \frac{1}{n}$ and $\frac{1}{m} \Diamond \frac{1}{m} < \frac{1}{n}$. Then, $U_m \circ U_m \subseteq U_n$. Indeed, let $(x, y, w) \in U_m$, $(w, y, y) \in U_m$. Since $M(x, y, z, \cdot)$ is non-decreasing and $N(x, y, z, \cdot)$ is non-increasing for all $x, y, z \in X$, respectively, $M(x, y, y, \frac{1}{n}) \geq M(x, y, y, \frac{2}{m})$ and $N(x, y, y, \frac{1}{n}) \leq N(x, y, y, \frac{2}{m})$. So

$$M\left(x, y, y, \frac{1}{n}\right) \geq M\left(x, y, w, \frac{1}{m}\right) * M\left(w, y, y, \frac{1}{m}\right)$$
$$> \left(1 - \frac{1}{m}\right) * \left(1 - \frac{1}{m}\right)$$
$$> 1 - \frac{1}{n}$$

and

$$N(x, y, y, \frac{1}{n}) \le N\left(x, y, w, \frac{1}{m}\right) \Diamond N\left(w, y, y, \frac{1}{m}\right)$$
$$< \frac{1}{m} \Diamond \frac{1}{m}$$
$$< \frac{1}{n}.$$

Therefore $(x, y, y) \in U_n$. Thus $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity $\mathscr{U}$ on X. Since for each $x \in X$ and each $n \in \mathbb{N}$,

$$U_n(x) = \left\{ y \in X : M\left(x, y, y, \frac{1}{n}\right) > 1 - \frac{1}{n}, \ N\left(x, y, y, \frac{1}{n}\right) < \frac{1}{n} \right\}$$
$$= \mathbb{B}_{D^*}\left(x, \frac{1}{n}, \frac{1}{n}\right),$$

we deduced that the topology induced by $\mathscr{U}$ coincides with $\tau_{(M,N)_{D^*}}$. Then $(X, \tau_{(M,N)_{D^*}})$ is a metrizable topological space. $\qquad \square$

Note that, in every metrizable space every sequentially compact set is compact.

Corollary 4.2. *A subset A of an intuitionistic fuzzy D^*-metric space $(X, M, N, *, \Diamond)$ is compact if and only if it is precompact and complete.*

Lemma 4.3. *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy D^*-metric space and let $\lambda, \eta \in (0, 1)$ such that $\lambda + \eta \le 1$. Then there exists an intuitionistic fuzzy D^*-metric $(m, n)_{D^*}$ on X such that $m(x, y, z, t) \ge \lambda$ and $n(x, y, z, t) \le \eta$ for each $x, y, z \in X$ and $t > 0$ and $(m, n)_{D^*}$ and $(M, N)_{D^*}$ induce the same topology on X.*

Proof. We define $m(x, y, z, t) = \max\{\lambda, M(x, y, z, t)\}$ and $n(x, y, z, t) = \min\{\eta, N(x, y, z, t)\}$. We claim that $(m, n)_{D^*}$ is an intuitionistic fuzzy D^*-metric on X. The properties of $(a), (b), (c), (d), (f), (g), (h), (i), (k)$ are immediate from the definition. For the inequalities (e) and (j), suppose that $x, y, z, w \in X$ and $t, s > 0$. Then $m(x, y, z, t + s) \ge \lambda$ and so $m(x, y, z, t + s) \ge \lambda * 1$.

Since $0 < m(x, y, w, t)$ and $m(w, z, z, s) < 1$. Now if either of $m(x, y, w, t) = \lambda$ or $m(w, z, z, s) = \lambda$. Then $m(x, y, z, t + s) \ge m(x, y, w, t) * m(w, z, z, s)$. The only remaining case is when $m(x, y, w, t) = M(x, y, w, t) > \lambda$ and $m(w, z, z, s) = M(w, z, z, s) > \lambda$. But $M(x, y, z, t + s) \ge M(x, y, w, t) * M(w, z, z, s)$ and $m(x, y, z, t + s) \ge M(x, y, z, t + s)$ and so $m(x, y, z, t + s) \ge m(x, y, w, t) * m(w, z, z, s)$. Also, then $n(x, y, z, t + s) \le \eta$ and so $n(x, y, z, t + s) \le n(x, y, w, t) \Diamond n(w, z, z, s)$ when either $n(x, y, w, t) = \eta$ or $n(w, z, z, s) = \eta$. The only remaining case is when $n(x, y, w, t) = N(x, y, w, t) < \eta$ and $n(w, z, z, s) = N(w, z, z, s) < \eta$. But $N(x, y, z, t + s) \le N(x, y, w, t) \Diamond N(w, z, z, s)$ and so $n(x, y, z, t + s) \le n(x, y, w, t) \Diamond n(w, z, z, s)$. Thus $(m, n)_{D^*}$ is an intuitionistic fuzzy D^*-metric on X. It only remains to show that the topology induced by $(m, n)_{D^*}$ is same as that induced

by $(M, N)_{D^*}$. But we have $M(x_n, x, z, t) \to 1$ and $N(x_n, x, z, t) \to 0$ as $n \to \infty$ for all $z \in X$ and $t > 0$ if and only if $\{\lambda, M(x, y, z, t)\} \to 1$ and $\{\eta, N(x, y, z, t)\} \to 0$, and we are done. $\qquad\square$

The intuitionistic fuzzy $D^\star$-metric $(m, n)_{D^*}$ in above lemma is said to $D^\star$-bounded by (λ, η).

Definition 4.1. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy $D^\star$-metric space, $x \in X$ and $\emptyset \neq A \subseteq X$. We define

$$D(x, A, t) = \sup\{M(x, y, y, t) : y \in A\} \quad (t > 0)$$

and

$$C(x, A, t) = \inf\{N(x, y, y, t) : y \in A\} \quad (t > 0).$$

Note that $D(x, A, t)$ and $C(x, A, t)$ are a degree of closeness and a degree of non-closeness of x to A at t, respectively.

Definition 4.2. A topological space is called a topologically complete intuitionistic fuzzy $D^\star$-metric space if there exists a complete intuitionistic fuzzy $D^\star$-metric inducing the given topology on it.

Example 4.3. Let $X = (0, 1]$. The intuitionistic fuzzy $D^\star$-metric space $(X, M, N, \min, \max)$, where $M(x, y, z, t) = \frac{t}{t + D^*(x, y, z)}$ and $N(x, y, z, t) = \frac{D^*(x, y, z)}{t + D^*(x, y, z)}$; where $D^*(x, y, z) = \min\{|x - y|, |y - z|, |z - x|\}$ is not complete, because the Cauchy sequence $\{\frac{1}{n}\}$ in this space is not convergent. Now consider the 5-tuple $(X, m, n, \min, \max)$, where $m(x, y, z, t) = \frac{t}{t + D^*(x, y, z) + D^*(\frac{1}{x}, \frac{1}{y}, \frac{1}{z})}$ and $n(x, y, z, t) = \frac{D^*(x, y, z) + D^*(\frac{1}{x}, \frac{1}{y}, \frac{1}{z})}{t + D^*(x, y, z) + D^*(\frac{1}{x}, \frac{1}{y}, \frac{1}{z})}$. It is straightforward to show that $(X, m, n, \min, \max)$ is an intuitionistic fuzzy $D^\star$-metric space which is complete. Since x_n tends to x with respect to intuitionistic fuzzy $D^\star$-metric space $(M, N)_{D^*}$, if and only if $|x_n - x| \to 0$, if and only if x_n tends to x with respect to intuitionistic fuzzy $D^\star$-metric space $(m, n)_{D^*}$, hence $(M, N)_{D^*}$ and $(m, n)_{D^*}$ are equivalent intuitionistic fuzzy $D^\star$-metric spaces. Therefore the intuitionistic fuzzy $D^\star$-metric space $(X, m, n, \min, \max)$ is topologically complete intuitionistic fuzzy $D^\star$-metrizable.

Lemma 4.4. *Intuitionistic fuzzy $D^\star$-metrizability is preserved under countable Cartesian product.*

Proof. Without loss of generality we may assume that the index set is $\mathbb{N}$. Let $((X_n, m_n, n_n, *, \Diamond) : n \in \mathbb{N})$ be a collection of intuitionistic fuzzy $D^\star$-metrizable spaces. Let τ_n be the topology induced by $(m_n, n_n)_{D^*}$ on $(X_n, m_n, n_n, *, \Diamond)$ for $n \in \mathbb{N}$ and let (X, τ) be the cartesian product of $((X_n, \tau_n) : n \in \mathbb{N})$ with product topology. We have to prove there is an intuitionistic fuzzy $D^\star$-metric $(m, n)_{D^*}$ on X which induces the topology τ. By Lemma(4.1), we may suppose that $(m, n)_{D^*}$ is

D^*-bounded by $(1 - \epsilon^{(n)}, \epsilon^{[n]})$ where $\epsilon^{(n)} = \overbrace{\epsilon * \epsilon * \cdots * \epsilon}^{n}$, $\epsilon^{[n]} = \overbrace{\epsilon \Diamond \epsilon \Diamond \cdots \Diamond \epsilon}^{n}$ and $\epsilon \in (0,1)$ (see[2]), i.e. $m_n(x_n, y_n, y_n, t) = \max\{1 - \epsilon^{(n)}, M_n(x_n, y_n, y_n, t)\}$ and $n_n(x_n, y_n, y_n, t) = \min\{\epsilon^{[n]}, N_n(x_n, y_n, y_n, t)\}$. Points of $X = \prod_{n \in \mathbb{N}} X_n$ are denoted as sequences as $x = (x_n)$ with $x_n \in X_n$ for $n \in \mathbb{N}$. Define $m(x, y, y, t) = \prod_{n=1}^{\infty} m_n(x_n, y_n, y_n, t)$ and $n(x, y, y, t) = \coprod_{n=1}^{\infty} n_n(x_n, y_n, y_n, t)$, for each $x, y \in X$ and $t > 0$ where $\prod_{n=1}^{m} a_n = a_1 * a_2 * \cdots * a_m$, $\coprod_{n=1}^{m} a_n = a_1 \Diamond a_2 \Diamond \cdots \Diamond a_m$. First note that $(m, n)_{D^*}$ is well defined since $a_j = \prod_{n=1}^{j} (1 - \epsilon^{(n)})$ is decreasing and D^*-bounded then converges to $\alpha \in (0,1)$ also $b_i = \coprod_{n=1}^{i} \epsilon^{[n]}$ is increasing and D^*-bounded then converges to $\beta \in (0,1)$.

Also $(m, n)_{D^*}$ is an intuitionistic fuzzy D^*-metric on X because each $(m_n, n_n)_{D^*}$ is an intuitionistic fuzzy D^*-metric. Let $\mathcal{U}$ be the topology induced by an intuitionistic fuzzy D^*-metric $(m_n, n_n)_{D^*}$. We claim that $\mathcal{U}$ coincides with τ. If $G \in \mathcal{U}$ and $x = (x_n) \in G$, then there exists $r \in (0,1)$ and $t > 0$ such that $\mathbb{B}_{D^*}(x, r, t) \subset G$. For each $r \in (0,1)$, we can find a sequence (δ_n) in $(0,1)$ and a positive integer n_0 such that

$$\prod_{n=1}^{n_0} (1 - \delta_n) * \prod_{n=n_0+1}^{\infty} (1 - \epsilon^{(n)}) > 1 - r$$

and

$$\coprod_{n=1}^{n_0} \delta_n \Diamond \coprod_{n=n_0+1}^{\infty} \epsilon^{[n]} < r .$$

For each $n = 1, 2, \cdots, n_0$, let $V_n = \mathbb{B}_{D^*}(x_n, \delta_n, t)$, where the ball is with respect to intuitionistic fuzzy D^*-metric $(m_n, n_n)_{D^*}$. Let $V_n = X_n$ for $n > n_0$. Put $V = \prod_{n \in \mathbb{N}} V_n$, then $x \in V$ and V is an open set in the product topology τ on X denoted by $\mathbb{V}$. Furthermore $\mathbb{V} \subset \mathbb{B}_{D^*}(x, r, t)$, since for each $y \in \mathbb{V}$

$$
\begin{aligned}
m(x, y, y, t) &= \prod_{n=1}^{\infty} m_n(x_n, y_n, y_n, t) \\
&= \prod_{n=1}^{n_0} m_n(x_n, y_n, y_n, t) * \prod_{n=n_0+1}^{\infty} m_n(x_n, y_n, y_n, t) \\
&\geq \prod_{n=1}^{n_0} (1 - \delta_n) * \prod_{n=n_0+1}^{\infty} (1 - \epsilon^{(n)}) > 1 - r
\end{aligned}
$$

$$
\begin{aligned}
n(x, y, y, t) &= \coprod_{n=1}^{\infty} n_n(x_n, y_n, y_n, t) \\
\\
&= \coprod_{n=1}^{n_0} n_n(x_n, y_n, y_n, t) \Diamond \coprod_{n=n_0+1}^{\infty} n_n(x_n, y_n, y_n, t) \\
&\leq \coprod_{n=1}^{n_0} \delta_n \Diamond \coprod_{n=n_0+1}^{\infty} \epsilon^{[n]} < r.
\end{aligned}
$$

Hence $\mathbb{V} \subset \mathbb{B}_{D^*}(x, r, t) \subset G$. Therefore G is open in the product topology.

Conversely, suppose $\mathbb{G}$ is open in the product topology and let $x = (x_n) \in \mathbb{G}$. Choose a standard basic open set $\mathbb{V}$ such that $x \in \mathbb{V}$ and $\mathbb{V} \subset \mathbb{G}$. Let $\mathbb{V} = \prod_{n \in \mathbb{N}} \mathbb{V}_n$, where each $\mathbb{V}_n$ is open in X_n and $\mathbb{V}_n = X_n$ for all $n > n_0$. For $n = 1, 2, \cdots, n_0$, let $1 - r_n = D_n(x_n, X_n - \mathbb{V}_n, t)$ and $q_n = C_n(x_n, X_n - \mathbb{V}_n, t)$, if $X_n \neq \mathbb{V}_n$, and $r_n = \epsilon^{(n)}$ and $q_n = \epsilon^{[n]}$, otherwise. Let $r = \min\{r_1, r_2, \cdots, r_0\}$, $q = \min\{q_1, q_2, \cdots, q_0\}$ and

$p = \min\{r, q\}$. We claim that $\mathbb{B}_{D^*}(x, p, t) \subset \mathbb{V}$. If $y = (y_n) \in \mathbb{B}_{D^*}(x, p, t)$, then $m(x, y, y, t) = \prod_{n=1}^{\infty} m_n(x_n, y_n, y_n, t) > 1 - p$ and so $m_n(x_n, y_n, y_n) > 1 - p \geq 1 - r \geq 1 - r_n$ and $n(x, y, y, t) = \coprod_{n=1}^{\infty} n_n(x_n, y_n, y_n, t) < p \leq q \leq q_n$ for each $n = 1, 2, \cdots, n_0$. Then $y_n \in \mathbb{V}_n$, for $n = 1, 2, \cdots, n_0$. Also for $n > n_0$, $y_n \in \mathbb{V}_n = X_n$. Hence $y \in \mathbb{V}$ and so $\mathbb{B}_{D^*}(x, p, t) \subset \mathbb{V} \subset \mathbb{G}$. Therefore $\mathbb{G}$ is open with respect to intuitionistic fuzzy D^*-metric topology and $\tau \subset \mathscr{U}$. Hence τ and $\mathscr{U}$ coincide. $\square$

Theorem 4.5. *An open subspace of a complete intuitionistic fuzzy D^*-metrizable space is a topologically complete intuitionistic fuzzy D^*-metrizable space.*

Proof. Let $(X, M, N, *, \Diamond)$ be a complete intuitionistic fuzzy D^*-metric space and $\mathbb{G}$ an open subspace of X. If the restriction of $(M, N)_{D^*}$ to $\mathbb{G}$ is not complete we can replace $(M, N)_{D^*}$ on $\mathbb{G}$ by other intuitionistic fuzzy D^*-metric as follows. Define $f : \mathbb{G} \times (0, \infty) \to \mathbb{R}^+$ by $f(x, t) = \frac{1}{1 - D(x, X - \mathbb{G}, t)}$ (f is undefined if $X - \mathbb{G}$ is empty, but then there is nothing to prove.) Fix an arbitrary $s > 0$ and for $x, y \in \mathbb{G}$ and X, define

$$m(x, y, z, t) = \begin{cases} M(x, y, z, t) * M(f(x, s), f(y, s), f(z, s), t) & \text{for } z \in \mathbb{G} \\ M(x, y, z, t) & \text{for } z \in X - \mathbb{G}. \end{cases}$$

and

$$n(x, y, z, t) = N(x, y, z, t)$$

for each $t > 0$. We claim that $(m, n)_{D^*}$ is an intuitionistic fuzzy D^*-metric on $\mathbb{G}$. The properties of (a), (b), (c), (d), (f), (g), (h), (i), (j) and (k) are immediate from the definition. For inequality (e), suppose that $w, x, y, z \in \mathbb{G}$ and $t, s, u > 0$, then

$$\begin{aligned} & m(w, y, z, t) * m(w, x, x, u) \\ &= (M(w, y, z, t) * M(f(w, s), f(y, s), f(z, s), t) * (M(w, x, x, u) \\ &\quad * M(f(w, s), f(x, s), f(x, s), u)) \\ &= (M(w, y, z, t) * M(w, x, x, u)) * (M(f(w, s), f(y, s), f(z, s), t) \\ &\quad * M(f(w, s), f(x, s), f(x, s), u)) \\ &\leq M(x, y, z, t + u) * M(f(x, s), f(y, s), f(z, s), t + u) \\ &= m(x, y, z, t + u) \quad \text{for } z \in \mathbb{G} \end{aligned}$$

Similarly

$$m(w, y, z, t) * m(w, x, x, u) \leq m(x, y, z, t + u) \quad \text{for } z \in X - \mathbb{G}.$$

Now we show that $(m, n)_{D^*}$ and $(M, N)_{D^*}$ are equivalent intuitionistic fuzzy D^*-metric on $\mathbb{G}$. We do this by showing that $m(x_n, x, x, t) \to 1$ if and only if $M(x_n, x, x, t) \to 1$ and $n(x_n, x, x, t) \to 0$ if and only if $N(x_n, x, x, t) \to 0$ of course the second part is trivial. Since $m(x, y, y, t) \leq M(x, y, y, t)$ for all $x, y \in \mathbb{G}$ and $t > 0$, $M(x_n, x, x, t) \to 1$ whenever $m(x_n, x, x, t) \to 1$. To prove the converse, let

$M(x_n, x, x, t) \to 1$, we know from Proposition of [19], M is continuous function on $X \times X \times X \times (0, \infty)$, then since

$$\lim_n D(x_n, X - \mathbb{G}, s) = \lim_n (\sup\{M(x_n, y, y, s) : y \in \mathbb{G}\})$$
$$\geq \lim_n M(x_n, y, y, s)$$
$$= M(x, y, y, s),$$

we have $\lim_n D(x_n, X - \mathbb{G}, s) \geq \lim_n D(x, X - \mathbb{G}, s)$. On the other hand, there exist a $y_0 \in X - \mathbb{G}$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have

$$D(x_n, X - \mathbb{G}, s) * \left(1 - \frac{1}{n}\right) \leq M(x_n, y_0, y_0, s).$$

Then $\lim_n D(x_n, X - \mathbb{G}, s) \leq M(x, y_0, y_0, s) \leq \sup\{M(x, y, y, s) : y \in X - \mathbb{G}\} = D(x, X - \mathbb{G}, s)$. Therefore $\lim_n D(x_n, X - \mathbb{G}, s) = \lim_n D(x, X - \mathbb{G}, s)$. This implies $M(f(x_n, s), f(x, s), f(x, s), t) \to 1$. Hence $m(x_n, x, x, t) \to 1$. Therefore $(m, n)_{D^*}$ and $(M, N)_{D^*}$ are equivalent. Next we show that $(m, n)_{D^*}$ is a complete intuitionistic fuzzy $D^\star$-metric. Suppose that (x_n) is a Cauchy sequence in $\mathbb{G}$ with respect to $(m, n)_{D^*}$. Since, for each $m, n \in \mathbb{N}$ and $t > 0$, $m(x_m, x_n, x_n, t) \leq M(x_m, x_n, x_n, t)$ and $n(x, y, y, t) = N(x, y, y, t)$, the sequence (x_n) is also a Cauchy sequence with respect to $(M, N)_{D^*}$. By the completeness of $(X, M, N, *, \Diamond)$, (x_n) converges to point p in X. We claim that $p \in \mathbb{G}$. Assume otherwise, then for each $n \in \mathbb{N}$, if $p \in X - \mathbb{G}$ and $M(x_n, p, p, t) \leq D(x_n, X - \mathbb{G}, t)$, then

$$1 - M(x_n, p, p, t) \geq 1 - D(x_n, X - \mathbb{G}, t) > 0.$$

Therefore

$$\frac{1}{1 - D(x_n, X - \mathbb{G}, t)} \geq \frac{1}{1 - M(x_n, p, p, t)},$$

that is

$$f(x_n, t) \geq \frac{1}{1 - M(x_n, p, p, t)},$$

for each $t > 0$. Therefore as $n \to \infty$, for every $t > 0$ we get $f(x_n, t) \to \infty$. In particular, $f(x_n, s) \to \infty$. On the other hand, $M(f(x_n, s), f(x_m, s), f(x_m, s), t) \geq m(x_n, x_m, x_m, t)$ for every $m, n \in \mathbb{N}$, that is $(f(x_n, s))$ is an $F - D^*$- bounded sequence ([9]). This contradiction shows that $p \in \mathbb{G}$. Hence (x_n) converges to p with respect to $(m, n)_{D^*}$ and thus $(G, m, n, *, \Diamond)$ is a complete intuitionistic fuzzy $D^\star$-metric space. $\square$

References

[1] S.E. Abbas, On intuitionistic fuzzy compactness, *Information Sciences* **173** (2005), 75–91.

[2] K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy and Systems* **20** (1986), 87–96.

[3] K. Atanassov, New operations defined over the intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **61** (1994), 137–142.

[4] L.C. Barros and R. Marin, *Fuzzy Logic in Medicine*, Heidelbrg, Physica-Verlag, 2002.

[5] L.C. Barros, R.C. Bassanezi and P.A. Tonelli, Fuzzy modelling in population dynamics, *Ecol. Model.* **128** (2000), 27–33.

[6] D. Coker, An introduction to intuitionistic fuzzy topological spaces, *Fuzzy Sets and Systems* **88** (1997), 81–89.

[7] B.C. Dhage, Generalised metric spaces and mappings with fixed point, *Bull. Calcutta Math. Soc.* **84** (1992), 329–336.

[8] B.C. Dhage, A common fixed point principle in D-metric spaces, *Bull. Calcutta Math. Soc.* **91** (1999), 475–480.

[9] M.S. El Naschie, On uncertainty of Cantorian geometry and two-slit experiment, *Chaos Solitons and Fractals* **9** (1998), 517–529.

[10] M.S. El Naschie, On the unification of heterotic strings theory and $E^{(\infty)}$ theory, *Chaos Solitons and Fractals* **11**(14) (2000), 2397–2408.

[11] M.S. El Naschie, A review of E-infinity theory and the mass spectrum of high energy particle physics, *Chaos Solitons and Fractals* **19** (2004), 209–236.

[12] M.S. El Naschie, Quantum gravity, clifford algebras, fuzzy set theory and the fundamental constants of nature *Chaos Solitons and Fractals* **20** (2004), 437–450.

[13] M.S. El Naschie, On two new fuzzy K hler manifolds, Klein modular space and Hooft holographic principles, *Chaos Solitons and Fractals* **29** (2006), 876–881.

[14] M.S. El Naschie, Fuzzy dodecahedron topology and E-infinity space time as a model for quantum physics, *Chaos Solitons and Fractals* **30** (2006), 1025–1033.

[15] M.S. El Naschie, A review of applications and results of E-infinity theory, *Int. J. Nonlinear Sci. Numer. Simulat.* **8** (2007), 11–20.

[16] M.A. Erceg, Metric spaces in fezzy set theory, *J. Math. Anal. Appl.* **69** (1979), 205–230.

[17] J.X. Fang, A note on the completions of fuzzy metric spaces and fuzzy normed spaces, *Fuzzy Sets and Systems* **131** (2002), 399–407.

[18] C. Felbin, Fuzzy dimensional fuzzy normed linear space, *Fuzzy Sets and Systems* **48** (1992), 239–248.

[19] A.L. Fradkov and R.J. Evans, Control of chaos: methods and applications in engineering, *Chaos Solitons and Fractals* **29** (2005), 33–56.

[20] A. George and P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets and Systems* **64** (1994), 395–399.

[21] R. Giles, A computer program for fuzzy reasoning, *Fuzzy Sets and Systems* **4** (1980), 221–234.

[22] L. Hong and J.Q. Sun, Bifurcations of fuzzy nonlinear dynamical systems, *Commun. Nonlinear Sci. Number. Simul.* **1** (2006), 1–12.

[23] O. Kaleva and S. Seikkala, On fuzzy metric spaces, *Fuzzy Sets Syst* **12** (1984), 215–219.

[24] F.G. Lupia ez, Nets and filters in intuitionistic fuzzy topological spaces, *Information Sciences* **176** (2006), 2396–2404.

[25] J.H. Park, Intuitionistic fuzzy metric spaces, *Chaos Solitons and Fractals* **22** (2004), 1039–1046.

[26] B. Schweizer and A. Sklar, Statistical metric spaces, *Pacific J. Math.* **10** (1960), 313–334.

[27] S. Sedghi, N. Shobe and H. Zhou, A common fixed point theorem in $D^\star$-metric spaces, Hindawi Publishing Corporation, *Fixed Point Theory and Applications*, **Volume 2007**.

[28] J. Xiao and X. Zhu, On linearly topological structure and property of fuzzy normed linear space, *Fuzzy Sets and Systems* **125**(2002), 153–161.

[29] L.A. Zadeh, Fuzzy sets, *Inform. Control* **8** (1965), 338–353.

Masoumeh Vali, *Department of Mathematics, Islamic Azad University, Najafabad Branch, Najafabad, Isfahan, Iran*.
E-mail: vali.masoumeh@gmail.com