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# Spectral Analysis and Moment Functions

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**Abstract.** We show that on commutative hypergroups the existence of nonzero moment functions in a variety is closely related to spectral analysis for the variety.

## 1. Introduction

Spectral analysis is related to the description of translation invariant function spaces over topological groups. Having translation operators similar investigations are meaningful on hypergroups. For the definition of hypergroup the reader should refer to [1]. A nonzero, closed, tranlation invariant linear subspace of the space in $\mathscr{C}(K)$ is called a *variety*. A complex valued continuous function on a hypergroup $K$ is an *exponential function* if $m(x*y)=m(x)m(y)$ holds for each $x,y$ in $K$. *Spectral analysis* for a variety means that the variety contains an exponential function. We say that spectral analysis holds for the hypergroup, if it holds for each nonzero variety. This is the case on some types of hypergroups, but here we present another approach to the spectral analysis problem for varieties. Our idea is based on the concept of moment functions. For any nonnegative integer $n$ the complex valued continuous function $\varphi$ on $K$ is called a *generalized moment function of order $n$*, if there are complex valued continuous functions $\varphi_k : K \to \mathbb{C}$ for $k=0,1,\dots,n$ such that $\varphi_0 \neq 0$, $\varphi_n = \varphi$ and

$$\varphi_k(x*y)=\sum_{i=0}^{k}\binom{k}{i}\varphi_i(x)\varphi_{k-i}(y) \tag{1}$$

holds for $k=0,1,\dots,n$ and for all $x,y$ in $K$. In this case we say that the functions $\varphi_k$ $(k=0,1,\dots,n)$ form a *generalized moment function sequence of order $n$*. For a detailed study of generalized moment functions and moment function sequences on polynomial and Sturm–Liouville hypergroups the reader should refer to [3], [5] and [4].

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In [6] linear independence of exponential monomials on commutative hypergroups has been studied. In the following section we show that nonzero generalized moment functions are linearly independent. We emphasize that we always suppose that in any generalized moment function sequence $\varphi_0$ is not identically zero.

## 2. **Linear independence of moment functions**

**Theorem 1.** *Let $K$ be a commutative hypergroup, $n \geq 1$ an integer and $(\varphi_k)_{k=0}^n$ a sequence of generalized moment functions with $\varphi_1 \neq 0$. Then the generalized moment function $\varphi_n$ is not the linear combination of the generalized moment functions $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$.*

***Proof.*** We prove our statement by induction on $n$. Let first $n = 1$ and suppose that $\varphi_1 = \lambda\,\varphi_0$ with some nonzero complex $\lambda$. Then, by (1), it follows

$$\begin{aligned}\varphi_0(x)\varphi_0(y) &= \varphi_0(x * y)\\ &= \frac{1}{\lambda}\,\varphi_1(x * y)\\ &= \frac{1}{\lambda}\,\varphi_1(x)\varphi_0(y) + \frac{1}{\lambda}\,\varphi_0(x)\varphi_1(y)\\ &= 2\,\varphi_0(x)\varphi_0(y),\end{aligned}$$

a contradiction.

Now let $n \geq 2$ be any integer and suppose that we have proved our statement for all integers not greater than $n$. For $n+1$ we suppose the contrary, that is, that there are complex numbers $c_i$ $(i = 0, 1, \ldots, n)$ such that

$$\varphi_{n+1}(x * y) = \sum_{i=0}^{n} c_i \varphi_i(x * y) \tag{2}$$

holds for each $x, y$ in $K$. By (1) we have

$$\sum_{j=0}^{n+1} \binom{n+1}{j} \varphi_j(x)\varphi_{n+1-j}(y) = \sum_{i=0}^{n} \sum_{j=0}^{i} c_i \binom{i}{j} \varphi_j(x)\varphi_{i-j}(y) \tag{3}$$

for each $x, y$ from $K$. Using (1), (2) and reordering the sum on the right hand side after simplification we get

$$\sum_{j=0}^{n} \left[ \binom{n+1}{j} \varphi_{n+1-j}(y) + c_j \varphi_0(y) - \sum_{i=j}^{n} c_i \binom{i}{j} \varphi_{i-j}(y) \right] \varphi_j(x) = 0\,.$$

By our assumption the coefficient of $\varphi_n$ must be zero for each $y$ in $K$, that is

$$(n+1)\varphi_1(y) = 0\,,$$

which is impossible. The theorem is proved. □

This result has the following consequence.

**Theorem 2.** *Let $K$ be a commutative hypergroup, $n \geq 1$ be an integer and $(\varphi_k)_{k=0}^n$ be a sequence of generalized moment functions with $\varphi_1 \neq 0$. Then the functions $\varphi_0, \varphi_1, \dots, \varphi_n$ are linearly independent. In particular, none of them is identically zero.*

## 3. Spectral analysis using moment functions

As the main result of this paper we show that if a variety $V$ in $\mathscr{C}(K)$ contains certain nonzero generalized moment functions, then spectral analysis holds for this variety.

**Theorem 3.** *Let $K$ be a commutative hypergroup, $n$ a nonnegative integer and $\varphi_0, \varphi_1, \dots, \varphi_n$ a generalized moment function sequence with $\varphi_1 \neq 0$. If $V$ is a variety in $\mathscr{C}(K)$ and $\varphi_n$ belongs to $V$, then $\varphi_k$ belongs to $V$ for $k = 0, 1, \dots, n$. In particular, spectral analysis holds for $V$.*

***Proof.*** By the previous theorem the functions $\varphi_0, \varphi_1, \dots, \varphi_n$ are linearly independent. Hence there are elements $y_0, \dots, y_n$ in $K$ for which the matrix $A = \big(\varphi_{n-i}(y_j)\big)_{i,j=0}^n$ is regular. Now we fix the element $x$ from $K$ to get the following inhomogeneous system of linear equations for unknowns $\varphi_i(x)$ $(i = 0, 1, \dots, n)$:

$$\varphi_n(x * y_j) = \sum_{i=0}^n \binom{n}{i} \varphi_i(x) \varphi_{n-i}(y_j) \qquad (j = 0, 1, \dots, n).$$

As the fundamental matrix of the previous system is the matrix $A$, the system has a unique solution. By Cramer's rule, the function $\binom{n}{i}\varphi_i(x)$, hence also the function $\varphi_i(x)$ (($i = 0, 1, \dots, n$) is a linear combination of translates of $\varphi_n(x)$. As $V$ is translation invariant, hence the translates of $\varphi_k$ belong to $V$, which implies that $\varphi_0, \varphi_1, \dots, \varphi_n$ are also in $V$.

Since $\varphi_0(x)$ is an exponential, our theorem is proved. □

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