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Fekete-Szeg Problem for α-Quasi Convex Functions of Order β

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Abstract. For $f \in Q^c_{\alpha,\beta}$, sharp bounds are obtained for the Fekete-Szeg functional $|a_3 - \mu a_2^2|$, where μ is real.

1. Introduction

Let S denote the class of normalized analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

where $z \in D = \{z : |z| < 1\}$. We denote by C the subclass of S consisting of functions which are convex in D. A classical result of Fekete and Szeg [4] determines the maximum value of $|a_3 - \mu a_2^2|$ as a function of the real parameter μ, for functions belonging to S.

Also every quasi-convex function is close-to-convex and hence univalent in D. Many authors [1, 2, 3, 5, 7] have got the estimate for the functional $|a_3 - \mu a_2^2|$ for different classes. In this paper, we give an estimate for the functional $|a_3 - \mu a_2^2|$ for the class $Q^c_{\alpha,\beta}$.

Special Cases

(1) When $\beta = 0$ we get the results of [1].
(2) When $\alpha = 0$, $\beta = 0$, then $f \in K$, an close-to-convex functions and we have a result given in [6].
(3) When $\alpha = 1$, $\beta = 0$, then $f \in Q$, the class of quasi-convex functions introduced by Noor [9].

Definition 1.1. Let f be given by (1.1) and $0 \le \alpha < 1$, $0 \le \beta < 1$. Then $f \in Q^c_{\alpha,\beta}$ if and only if there exists $g \in C$ such that for $z \in D$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ satisfying

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the condition.

$$Re\left\{(1-\alpha)\frac{f'(z)}{g'(z)}+\frac{\alpha(zf'(z))'}{g'(z)}\right\}>\beta\,. \tag{1.2}$$

Here C denotes the class of convex functions that is $g \in C$ if and only if g is analytic in D and

$$Re\left\{1+\frac{zg''(z)}{g'(z)}\right\}>0, \quad z\in D\,. \tag{1.3}$$

We note that by using a lemma due to Miller and Mocanu [8] it can easily be shown that $Q^c_{\alpha,\beta} \subset Q^c$, the class of quasi-convex functions for $0<\alpha<1$ and hence $f \in Q^c_{\alpha,\beta}$ means f is univalent.

We now state some preliminary lemmas that are required for proving our results.

2. **Preliminary Results**

Lemma 2.1 ([10]). *Let h be analytic in D with $Re\ h(z) > 0$ and be given by $h(z) = 1+c_1z+c_2z^2+\cdots$ for $z\in D$. Then*

$$\left|c_2-\frac{1}{2}c_1^2\right|\le 2-\frac{|c_1|^2}{2}.$$

Lemma 2.2 ([6]). *Let $g\in C$ with $g(z)=z+b_2z^2+b_3z^3+\cdots$. Then for μ real* $|b_3-\mu b_2^2|\le \max\left\{\frac{1}{3},|\mu-1|\right\}$.

Lemma 2.3. *Let $f\in Q^c_{\alpha,\beta}$ and be given by* (1.1)*. Then*

(i) $(\alpha+1)|a_2|-\beta\le 2$,

(ii) $(2\alpha+1)|a_3|-\beta\le 3$.

Proof. Since $g\in C$, it follows from (1.3)

$$g'(z)+zg''(z)=g'(z)p(z) \tag{2.1}$$

for $z\in D$ with $Re\ p(z)>0$ given by $p(z)=1+p_1z+p_2z^2+\ldots$.
Equating the coefficients we get

$$2b_2=p_1\,, \tag{2.2}$$

$$6b_3=p_2+2b_2p_1\,. \tag{2.3}$$

It follows from (1.2) that

$$(1-\alpha)f'(z)+\alpha(zf'(z))'-\beta g'(z)=g'(z)h(z) \tag{2.4}$$

where $h(z)=1+c_1z+c_2z^2+\cdots$ with $Re\ h(z)>0$.
Equating coefficients we get

$$2(\alpha+1)a_2-2\beta b_2=2b_2+c_1 \tag{2.5}$$

and

$$3(2\alpha+1)a_3-3b_3\beta=c_2+2b_2c_1+3b_3\,. \tag{2.6}$$

On using classical inequalities $|p_1| \le 2$, $|p_2| \le 2$, $|c_1| \le 2$, $|c_2| \le 2$, $|b_1| \le 1$, $|b_2| \le 1$, the required result follows from (2.5) and (2.6). □

3. Main Results

Theorem 3.1. *Let f be given by* (1.1) *and belong to the class $Q^c_{\alpha,\beta}$. Then for $0 \le \alpha < 1$, $0 \le \beta < 1$,*

$$3(2\alpha+1)(\alpha+1)^2(1+\beta)|a_3-\mu a_2^2|$$

$$\le \begin{cases} (\alpha+1)^2(3+2\beta)^2 - 3(2\alpha+1)(1+\beta)(2+\beta)^2\mu, & \mu \le \dfrac{(\alpha+1)^2}{3(2\alpha+1)(1+\beta)} \\ (5+3\beta)(1+\beta)(1+\alpha)^2 - 3(2\alpha+1)(1+\beta)^3\mu + \dfrac{(1+\beta)[2(1+\alpha)^2-3(2\alpha+1)\mu(1+\beta)]^2}{3(2\alpha+1)\mu}, & \dfrac{(\alpha+1)^2}{3(2\alpha+1)(1+\beta)} \le \mu \le \dfrac{2(\alpha+1)^2}{3(2\alpha+1)(1+\beta)} \\ (1+\beta)(1+\alpha)^2(3+\beta), & \dfrac{2(\alpha+1)^2}{3(2\alpha+1)(1+\beta)} \le \mu \le \dfrac{(\alpha+1)^2}{(2\alpha+1)(1+\beta)} \\ 3(2\alpha+1)(1+\beta)(2+\beta)^2\mu - (\alpha+1)^2(2\beta^2+8\beta+9), & \mu \ge \dfrac{(\alpha+1)^2}{(2\alpha+1)(1+\beta)} \end{cases}$$

All the inequalities are sharp.

Proof. From (2.2), (2.3), (2.5) and (2.6) it is easily established that

$$\begin{aligned} &3(2\alpha+1)(1+\beta)(a_3-\mu a_2^2) \\ &\le (1+\beta)\left\{3\left[b_3(1+\beta) - \frac{(2\alpha+1)}{(\alpha+1)^2}(1+\beta)^2\mu b_2^2\right]\right. \\ &\quad + \left[c_2 + \left(\frac{2(\alpha+1)^2-3(2\alpha+1)\mu}{4(\alpha+1)^2} - \frac{1}{2}\right)c_1^2\right] \\ &\quad \left. + \left[1 - \frac{3(2\alpha+1)\mu(1+\beta)}{2(\alpha+1)^2}\right]p_1c_1\right\}. \end{aligned} \tag{3.1}$$

First consider

$$\frac{(\alpha+1)^2}{3(2\alpha+1)(1+\beta)} \le \mu \le \frac{2(\alpha+1)^2}{3(2\alpha+1)(1+\beta)}.$$

Equation (3.1) gives

$$\begin{aligned} &3(2\alpha+1)(1+\beta)|a_3-\mu a_2^2| \\ &\le (1+\beta)\left\{3\left|b_3(1+\beta) - \frac{(2\alpha+1)}{(\alpha+1)^2}(1+\beta)^2\mu b_2^2\right|\right. \\ &\quad + \left|c_2 - \frac{c_1^2}{2}\right| + \left[\frac{2(\alpha+1)^2-3(2\alpha+1)\mu}{4(\alpha+1)^2}\right]|c_1|^2 \\ &\quad \left. + \left[\frac{2(\alpha+1)^2-3(2\alpha+1)\mu(1+\beta)}{2(\alpha+1)^2}\right]|c_1|\right\} \\ &= \phi(x), \quad \text{with } x = |c_1| \end{aligned}$$

where we have used Lemma 2.1 and the inequality $|p_1| \leq 2$. Elementary calculation shows that the function ϕ attains its maximum value at

$$x_0 = 2\left[\frac{2(\alpha+1)^2 - 3(2\alpha+1)\mu(1+\beta)}{3(2\alpha+1)\mu}\right]$$

and hence

$$3(2\alpha+1)(\alpha+1)^2(1+\beta)|a_3 - \mu a_2^2| \leq \phi(x_0),$$

i.e.,

$$\begin{aligned}
&3(2\alpha+1)(\alpha+1)^2(1+\beta)|a_3 - \mu a_2^2| \\
&\quad \leq (5+3\beta)(1+\beta)(\alpha+1)^2 - 3(2\alpha+1)(1+\beta)^3\mu \\
&\qquad + \frac{(1+\beta)}{3(2\alpha+1)\mu}[2(\alpha+1)^2 - 3(2\alpha+1)\mu(1+\beta)]^2 \quad \text{(using Lemma 2.2).}
\end{aligned}$$

Next, since $|x_0| \leq 2(1+\beta)$ we have $\mu \geq \dfrac{(\alpha+1)^2}{3(2\alpha+1)(1+\beta)}$ and hence completing the proof for the case

$$\frac{(\alpha+1)^2}{3(2\alpha+1)(1+\beta)} \leq \mu \leq \frac{2(\alpha+1)^2}{3(2\alpha+1)(1+\beta)}.$$

Letting $c_1 = 2\left[\dfrac{2(\alpha+1)^2 - 3(2\alpha+1)\mu(1+\beta)}{3(2\alpha+1)\mu}\right]$, $c_2 = 2$, $p_1 = 2$, $p_2 = 2$, $b_2 = 1$ and $b_3 = 1$ in (3.1) shows that the result is sharp.

Secondly, consider the case $\mu \leq \dfrac{(\alpha+1)^2}{3(2\alpha+1)(1+\beta)}$. Write

$$a_3 - \mu a_2^2 = a_3 - \frac{(\alpha+1)^2 a_2^2}{3(2\alpha+1)(1+\beta)} + \left[\frac{(\alpha+1)^2}{3(2\alpha+1)(1+\beta)} - \mu\right] a_2^2 .$$

Since $|a_2| \leq \dfrac{2+\beta}{\alpha+1}$ it follows that

$$\begin{aligned}
&3(2\alpha+1)(\alpha+1)^2(1+\beta)|a_3 - \mu a_2^2| \\
&\quad \leq 3(2\alpha+1)(\alpha+1)^2(1+\beta)\left|a_3 - \frac{(\alpha+1)^2 a_2^2}{3(2\alpha+1)(1+\beta)}\right| \\
&\qquad + \left[\frac{(\alpha+1)^2}{3(2\alpha+1)(1+\beta)} - \mu\right]|a_2|^2 3(2\alpha+1)(\alpha+1)^2(1+\beta) \\
&\quad \leq (1+\alpha)^2(3+2\beta)^2 - 3(2\alpha+1)(1+\beta)(2+\beta)^2\mu .
\end{aligned}$$

Here we have used the result that is proved for $\mu = \dfrac{(\alpha+1)^2}{3(2\alpha+1)(1+\beta)}$. Equality is attained by choosing $c_1 = c_2 = p_1 = p_2 = 2$, $b_2 = b_3 = 1$ in (3.1).

Next assume that

$$\frac{2(\alpha+1)^2}{3(2\alpha+1)(1+\beta)} \leq \mu \leq \frac{(\alpha+1)^2}{(2\alpha+1)(1+\beta)}.$$

First we consider the case $\mu = \dfrac{(\alpha+1)^2}{(2\alpha+1)(1+\beta)}$. It follows from (2.1), (2.2), (2.3) and (3.1) that

$$\begin{aligned} &3(2\alpha+1)(1+\beta)(1+\alpha)^2|a_3-\mu a_2^2| \\ &\quad \le (1+\beta)(1+\alpha)^2(3+\beta) - \frac{(\alpha+1)^2}{4}[|c_1|-(1+\beta)|p_1|]^2 \\ &\quad = \psi(|c_1|,|p_1|), \;\; \text{say} \end{aligned}$$

we can show that ψ attains maximum value when $|c_1| = (1+\beta)|p_1|$ and so

$$3(2\alpha+1)(1+\beta)(1+\alpha)^2|a_3-\mu a_2^2| \le (1+\beta)(1+\alpha)^2(3+\beta)\,.$$

Next write

$$\begin{aligned} a_3-\mu a_2^2 = {} & \frac{3(2\alpha+1)(1+\beta)\mu-2(\alpha+1)^2}{(\alpha+1)^2}\left[a_3-\frac{(\alpha+1)^2a_2^2}{(2\alpha+1)(1+\beta)}\right] \\ & +3\left[\frac{(\alpha+1)^2-(2\alpha+1)(1+\beta)\mu}{(\alpha+1)^2}\right]\left[a_3-\frac{2(\alpha+1)^2a_2^2}{3(2\alpha+1)(1+\beta)}\right] \end{aligned}$$

and the result follows at once by using results already established for $\mu = \dfrac{2}{3}\dfrac{(\alpha+1)^2}{(2\alpha+1)(1+\beta)}$ and $\mu = \dfrac{(\alpha+1)^2}{(2\alpha+1)(1+\beta)}$. The result is sharp for $p_1 = c_1 = 0$, $p_2 = c_2 = 2$, $b_2 = 0$ and $b_3 = \dfrac{1}{3}$ in (3.1). Finally consider $\mu \ge \dfrac{(\alpha+1)^2}{(2\alpha+1)(1+\beta)}$. Write

$$a_3-\mu a_2^2 = a_3 - \frac{(\alpha+1)^2}{(2\alpha+1)(1+\beta)}a_2^2 + \left(\frac{(\alpha+1)^2}{(2\alpha+1)(1+\beta)}-\mu\right)a_2^2$$

and thus

$$\begin{aligned} &3(2\alpha+1)(\alpha+1)^2(1+\beta)|a_3-\mu a_2^2| \\ &\quad \le 3(2\alpha+1)(1+\alpha)^2(1+\beta)\left|a_3-\frac{(\alpha+1)^2}{(2\alpha+1)(1+\beta)}a_2^2\right| \\ &\qquad + 3(2\alpha+1)(1+\alpha)^2(1+\beta)\left[\mu-\frac{(\alpha+1)^2}{(2\alpha+1)(1+\beta)}\right]|a_2|^2 \\ &\quad \le 3(2\alpha+1)(1+\beta)(2+\beta)^2\mu-(\alpha+1)^2(2\beta^2+8\beta+9), \end{aligned}$$

where results for $\mu = \dfrac{(\alpha+1)^2}{(2\alpha+1)(1+\beta)}$ and the inequality $|a_2| \le \dfrac{2+\beta}{\alpha+1}$ has been used. By choosing $c_1 = p_1 = 2i$, $c_2 = p_2 = -2$, $b_2 = i$, $b_3 = -1$ in (3.1) equality is obtained. □

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