Journal of Informatics and Mathematical Sciences
Vol. 10, Nos. 1 & 2, pp. 55–72, 2018
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications
http://www.rgnpublications.com
http://dx.doi.org/10.26713/jims.v10i1-2.674

Research Article

Influence of Poynting-Robertson Drag and Oblateness on Existence and Stability of Out of Plane Equilibrium Points in Spatial Elliptic Restricted Three Body Problem

A. Chakraborty* and A. Narayan

Department of Mathematics, Bhilai Institute of Technology, Durg, Chattish Garh 491001, India
Corresponding author: acbit10282@gmail.com

Abstract. This paper studies existence and stability of the out of plane equilibrium points $L_{6,7}$ analytically and numerically in the elliptical restricted three body problem, where both the primaries are radiating oblate spheroid, incorporating the effects of Poynting-Robertson (P-R) drag of the radiating primaries as well. It is observed that consideration of PR-drag forces results in the non-zero y-coordinate of the out of plane equilibrium points. The stability of the equilibrium points is studied in presence of PR-drag forces. Also the stability is further analyzed in the case when the PR-drag forces are neglected. We have explored the existence of out of plane equilibrium points and their stability around the binary systems: Luyten-726 and Sirius.

Keywords. P-R drag; Elliptic Restricted Three Body Problem; Out of plane equilibrium points; Binary systems

MSC. 70F07; 70E50

Received: August 16, 2017 **Accepted:** December 12, 2017

1. Introduction

The three body problem deals with three spherical masses, which are moving under the influence of forces due to mutual gravitational interactions as described by Newton's theory of gravity.

Here no restrictions are imposed on their initial positions and velocities. In restricted three body problem, the two primaries have dominant masses and move around their common center of mass, whereas, the third mass is very small and its gravitational influence on the primaries is negligible. The investigation of existence of five planar equilibrium points and their stability has been the area of research interest for many researchers. Radzievskii [11] formulated and discussed the problem for the three specific bodies: the Sun, a planet and a dust particle and was the first to find that an allowance for direct solar radiation pressure results in a change in the positions of the libration points and to the appearance of new libration points in comparison with the classical problem. Simmons *et al.* [18] extended the work in the circular restricted problem of three bodies taking both the gravitating bodies to be radiating. They obtained a complete solution of the problem of existence and linear stability of the equilibrium points for all values of radiation pressure of both luminous bodies and all values of mass ratios. It was found that there exists nine libration points, three collinear, two triangular and four out of the plane when the radiation pressure of the smaller mass is very high. They have shown that L_8 and L_9 are always linearly unstable, whereas L_6 and L_7 are stable for small range of radiation pressure provided that both large masses are luminous. Ragos and Zagouras [14] have studied the region of stability of these equilibrium points extensively by studying periodic solution of the system and presented the out of plane points as the saddle point of zero velocity surfaces [13]. Das *et al.* [3] presented a new approach by discussing the existence and stability of out of plane equilibrium points, taking the radiation parameter β_2 not as an independent parameter but as the parameter dependent upon β_1, mass ratio and luminosity of the binary components, while discussing the same problem, they also studied the effects of P-R drag on the system. Abouelmagd and Mostafa [1] found the location of out of plane equilibrium points and the forbidden movement region around these points in the spatial case of non-isotropic variation of mass in restricted three body problem. Singh and Vincent [24] studied the location and stability of out of plane equilibrium points for restricted four body problem. They [25] extended their work to encompass the effect of oblateness on the location and stability of out of plane equilibrium points for the same problem.

In the classical restricted three body problem the primaries were considered to be perfectly spherical in shape because of homogenous distribution of mass. But in reality, the rotation of stars and planets produces an equatorial bulge due to centrifugal force, as a result most of the planets (Earth, Jupiter and Saturn) and Stars (Archerner, Archid, Antares, Altair, Luyten etc) are sufficiently oblate. Hence, a generalization of the classical problem has been studied taking into account aspects such as shape of the bodies, eccentricity of the orbits and influence of perturbing forces other than gravitational forces. Many researchers have studied the generalized restricted three body problem by considering the shape of the bodies [4, 16, 17, 19].

The classical Restricted Three-Body Problem has also been generalized to include additional effects observed when the primaries follows not the circular but elliptical path. If the system performs motion about circular orbit, then it is circular restricted three body problem, where as the problem becomes elliptical restricted three body problem, when the eccentricity of the orbit is also considered. The elliptical restricted three body problem while generalizing the classical restricted three body problem still retain some exceptional and significant properties of circular model. The elliptical restricted three body problem has been described in considerable detail by various authors [2,8–10,15,20–23,26]and others. In these papers, authors have discussed the linear stability around the collinear and triangular equilibrium points. Gong and Li [6] gave certain analytical criteria of Hill stability based on the bifurcation of the extremum of Jacobi integral.

The present paper is a discussion of the location and stability of out of plane equilibrium points when both the primaries are oblate and luminous and effects of P-R drag are also taken into account. The problem is discussed analytically and numerically around the two binary systems: Luyten-726-8(AB) and Sirus B. The data relevant to the paper about the two binary systems [5,27] is shown in Table 1. The value of dimensionless velocity of light has been obtained using the following condition [12]:

$$c_d = \frac{c}{\sqrt{\gamma(m_1+m_2)/a}},$$

where

c = velocity of light in vacuum, a = semimajor axis, γ = gravitational constant,

m_1 = mass of bigger primary, m_2 = mass of smaller primary.

Table 1. Some relevant data of the binary systems.

Binary system	$M_1(M_\odot)$	$M_2(M_\odot)$	$a(AU)$	e	c_d
Luyten-726	0.101	0.99	1.95	0.62	13450.1
Sirius	2.15	1.05	7.5	0.59	15919.7

The paper is organized as follows: Section 1 gives the introduction. The equations of motion are described in Section 2. In Section 3 the out of plane equilibrium points are obtained. In Section 4 and its subsections, the analysis of stability is done. Numerical exploration of the problem is given in Section 5. Finally, conclusions are drawn in Section 6.

2. Equation of Motion

The differential equations of motion of the infinitesimal mass in the elliptical restricted three body problem under the effect of radiation pressure and oblateness of both the primaries with additional effects of P-R drag in a barycentric, pulsating, rotating and non dimensional coordinates system, Singh and Umar [20] are given in the following equations:

$$\begin{aligned} x'' - 2y' &= \frac{\partial\Omega}{\partial x}, \\ y'' + 2x' &= \frac{\partial\Omega}{\partial y}, \\ z'' &= \frac{\partial\Omega}{\partial z}, \end{aligned} \tag{2.1}$$

where $'$ denotes the differentiation with respect to eccentric anomaly E and

$$\Omega = \frac{1}{\sqrt{1-e^2}}\left[\frac{1}{2}(x^2+y^2)+\frac{1}{n^2}\Phi_1-\frac{\mathscr{W}_1}{n}\Phi_2-\frac{\mathscr{W}_2}{n}\Phi_3\right], \tag{2.2}$$

$$\Phi_1 = \frac{q_1(1-\mu)}{r_1}\left(1+\frac{A_1}{2r_1^2}-\frac{3A_1z^2}{2r_1^4}\right)+q_2\frac{\mu}{r_2}\left(1+\frac{A_2}{2r_2^2}-\frac{3A_2z^2}{2r_2^4}\right), \tag{2.3}$$

$$\Phi_2 = \frac{((x+\mu)x'+yy'+zz')}{r_1^2}-\arctan\left(\frac{y}{x+\mu}\right),$$

$$\Phi_3 = \frac{((x+\mu-1)x'+yy'+zz')}{r_2^2}-\arctan\left(\frac{y}{x+\mu-1}\right). \tag{2.4}$$

Here

$$\left.\begin{aligned} n^2 &= \tfrac{1}{a}\left(1+\frac{3e^2}{2}+\frac{3A_1}{2}+\frac{3A_2}{2}\right),\\ r_1 &= \sqrt{(x+\mu)^2+y^2+z^2}, \quad r_2=\sqrt{(x+\mu-1)^2+y^2+z^2},\\ \mathscr{W}_1 &= \frac{\beta_1(1-\mu)}{c_d} \end{aligned}\right\} \tag{2.5}$$

and

$$\mathscr{W}_2 = \frac{\beta_2\mu}{c_d}. \tag{2.6}$$

Also, $\mu = \frac{m_2}{m_1+m_2}$, where m_1 and m_2 are masses of bigger and smaller primaries respectively positioned at $(x_i,0,0)$, $i=1,2$. A_1 and A_2 denote the oblateness coefficients of the bigger and smaller primaries respectively such that $0<A_i\ll 1$, $(i=1,2)$. Furthermore, the radiation factor of the primaries are denoted by $q_i=(1-\beta_i)$ where $i=1,2$. We have considered the dimensionless velocity of light c_d, and $\mathscr{W}_1$ and $\mathscr{W}_2$ are the parameters representing the P-R drag due to the radiation of the first and second primary respectively, while e is the eccentricity of the orbit of either primary around the other. Figure 1 represents the non-planar depiction of the problem considered, with the position of all seven equilibrium points.

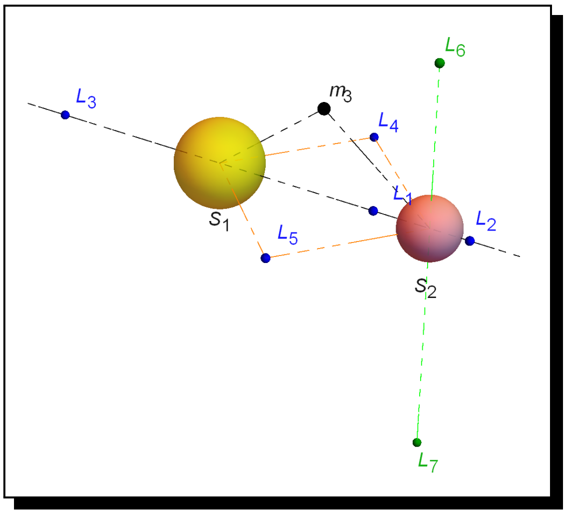


Figure 1. Position of the equilibrium points in non-planar ERTBP.

3. Out-of-Plane Equilibrium Points

The positions of the out-of-plane equilibrium points can be found from the equations of motion by putting all velocity and acceleration components equal to zero and solving the resulting system, i.e. $\Omega_x = 0$, $\Omega_y = 0$ and $\Omega_z = 0$, which are given as follows:

$$x - \frac{1}{n^2}\left[\frac{Q_1(x+\mu)}{r_1^3}\left(1+\frac{3A_1}{2r_1^2}-\frac{15A_1z^2}{2r_1^4}\right)+\frac{Q_2(x+\mu-1)}{r_2^3}\left(1+\frac{3A_2}{2r_2^2}-\frac{15A_2z^2}{2r_2^4}\right)\right] = -\frac{y}{n}\left(\frac{\mathscr{W}_1}{r_1^2}+\frac{\mathscr{W}_2}{r_2^2}\right), \quad (3.1)$$

$$y - \frac{1}{n^2}\left[\frac{Q_1 y}{r_1^3}\left(1+\frac{3A_1}{2r_1^2}-\frac{15A_1z^2}{2r_1^4}\right)+\frac{Q_2 y}{r_2^3}\left(1+\frac{3A_2}{2r_2^2}-\frac{15A_2z^2}{2r_2^4}\right)\right] = \frac{\mathscr{W}_1(x+\mu)}{nr_1^2}+\frac{\mathscr{W}_2(x+\mu-1)}{nr_2^2}, \quad (3.2)$$

$$-\frac{z}{n^2}\left[\frac{Q_1}{r_1^3}\left(1+\frac{9A_1}{2r_1^2}-\frac{15A_1z^2}{2r_1^4}\right)+\frac{Q_2}{r_2^3}\left(1+\frac{9A_2}{2r_2^2}-\frac{15A_2z^2}{2r_2^4}\right)\right] = 0, \quad (3.3)$$

where

$$Q_1 = q_1(1-\mu), \quad Q_2 = q_2\mu.$$

Now for $z \neq 0$, equation (3.3) yields

$$\frac{Q_1}{r_1^3}\left(1+\frac{3A_1}{2r_1^2}-\frac{15A_1z^2}{2r_1^4}\right)+\frac{Q_2}{r_2^3}\left(1+\frac{3A_2}{2r_2^2}-\frac{15A_2z^2}{2r_2^4}\right) = -\frac{3Q_1A_1}{r_1^5}-\frac{3Q_2A_2}{r_2^5}. \quad (3.4)$$

Consequently, equation (3.1) and (3.2) reduce to

$$x + \frac{1}{n^2}\left[(x+\mu)\left(\frac{3Q_1A_1}{r_1^5}+\frac{3Q_2A_2}{r_2^5}\right)+\frac{Q_2}{r_2^3}\left(1+\frac{3A_2}{2r_2^2}-\frac{15A_2z^2}{2r_2^4}\right)\right] = -\frac{y}{n}\left(\frac{\mathscr{W}_1}{r_1^2}+\frac{\mathscr{W}_2}{r_2^2}\right), \quad (3.5)$$

$$y\left[1+\frac{1}{n^2}\left(\frac{3Q_1A_1}{r_1^5}+\frac{3Q_2A_2}{r_2^5}\right)\right] = \frac{\mathscr{W}_1(x+\mu)}{nr_1^2}+\frac{\mathscr{W}_2(x+\mu-1)}{nr_2^2}. \quad (3.6)$$

Rearranging the terms of equation (3.4), we obtain the value of z as given below:

$$z^2 = \frac{2}{15}\cdot\frac{\frac{Q_1}{r_1^3}\left(1+\frac{9A_1}{2r_1^2}\right)+\frac{Q_2}{r_2^3}\left(1+\frac{9A_2}{2r_2^2}\right)}{\frac{Q_1A_1}{r_1^7}+\frac{Q_2A_2}{r_2^7}}. \quad (3.7)$$

Considering equation (3.5) and equation (3.6) as simultaneous equations, the values of x and y are expressed as functions of z, as given below:

$$x = \frac{A_{12}A_{23}-A_{13}A_{22}}{A_{12}A_{21}-A_{11}A_{22}}, \qquad y = \frac{A_{13}A_{21}-A_{11}A_{23}}{A_{12}A_{21}-A_{11}A_{22}}, \quad (3.8)$$

where

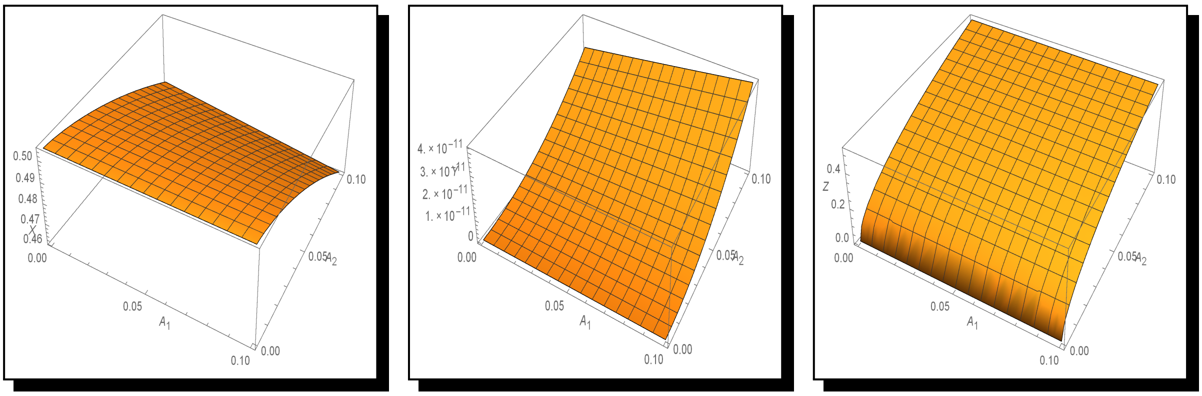
$$A_{11} = n^2+\frac{3A_1Q_1}{r_1^5}+\frac{3A_2Q_2}{r_2^5}, \quad A_{12} = n\left(\frac{\mathscr{W}_1}{r_1^2}+\frac{\mathscr{W}_2}{r_2^2}\right),$$

$$A_{13} = -\mu\left(\frac{3A_1Q_1}{r_1^5}+\frac{3A_2Q_2}{r_2^5}\right)-\frac{3A_2Q_2}{2r_2^5}+\frac{15A_2Q_2z^2}{2r_2^7}-\frac{Q_2}{r_2^3},$$

$$A_{21} = \frac{3A_1Q_1}{r_1^5}+\frac{3A_2Q_2}{r_2^5}+n^2, \quad A_{22} = -n\left(\frac{\mathscr{W}_1}{r_1^2}+\frac{\mathscr{W}_2}{r_2^2}\right), \quad A_{23} = n\mu\left(\frac{\mathscr{W}_1}{r_1^2}+\frac{\mathscr{W}_2}{r_2^2}\right)-\frac{n\mathscr{W}_2}{r_2^2}.$$

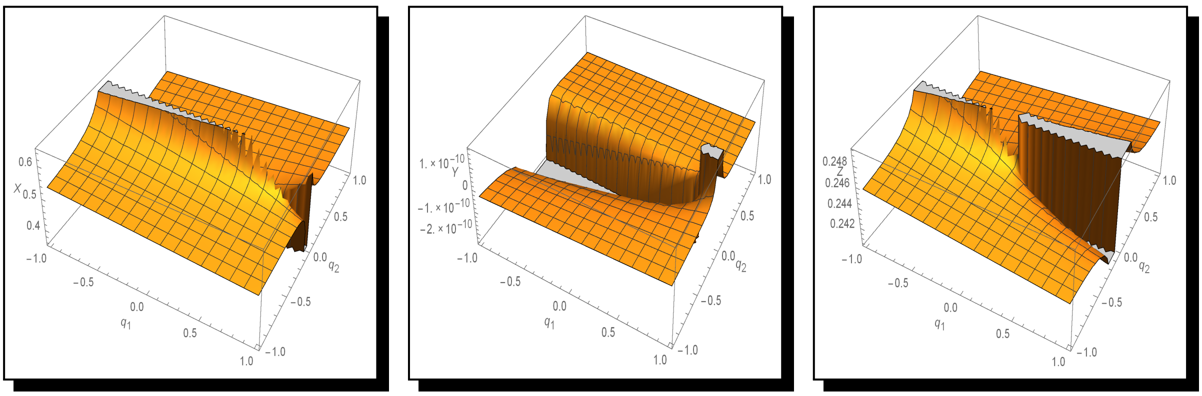
The out of plane equilibrium points are denoted as L_6 and L_7 and given as $(x^*, y^*, \pm z^*)$. To obtain the coordinates of L_6 and L_7, we take the initial approximation for these points as $(1-\mu, 0, \sqrt{3A_2})$ and use Software Mathematica. For both equations (3.7) and (3.8), r_1 and r_2 on the right hand side are replaced by the value of the distances according to the initial approximation that is $r_1 = r_{10} = \sqrt{1+3A_2}$ and $r_2 = r_{20} = \sqrt{3A_2}$, where $x_0 = 1-\mu$, $y_0 = 0$ and $z_0 = \sqrt{3A_2}$, however the values x^*, y^* and z^* are all assumed to be nonzero. Thus, the coordinates of the out of plane equilibrium points have been approximated in the form of power series as given below:

$$\begin{aligned}
x^* &= (1-\mu)\Bigg[1 + \frac{27\mathscr{W}_2^2(A_1+e^2)^2}{16aQ_2^2}A_2 + \frac{3\sqrt{3}(3A_1(aq_1-1)+2aq_1-3e^2-2)}{2aQ_2}A_2^{3/2} \\
&\quad - \frac{9\mathscr{W}_2(\mathscr{W}_1((24e^2-16)(\mu+1))+24A_1+6\mathscr{W}_2(A_1+e^2))}{16aQ_2^2}A_2^2\Bigg] + \mathscr{O}(A_2^{5/2}), \qquad (3.9)\\
y^* &= \frac{3\sqrt{3}\mathscr{W}_1(4+3A_1+3e^2)}{4\sqrt{a}Q_2}A_2^{3/2} \\
&\quad + \frac{9(1-\mu)\mathscr{W}_2(4+3A_1+3e^2)(3A_1(aq_1-1)+2aq_1-3e^2-2)}{8(a^{3/2}Q_2^2)}A_2^2 + \mathscr{O}(A_2^{5/2}),\\
z^* &= \sqrt{3A_2} + \frac{9Q_1}{10Q_2}(2+9A_1)A_2^2 + \mathscr{O}(A_2^{5/2}).
\end{aligned}$$



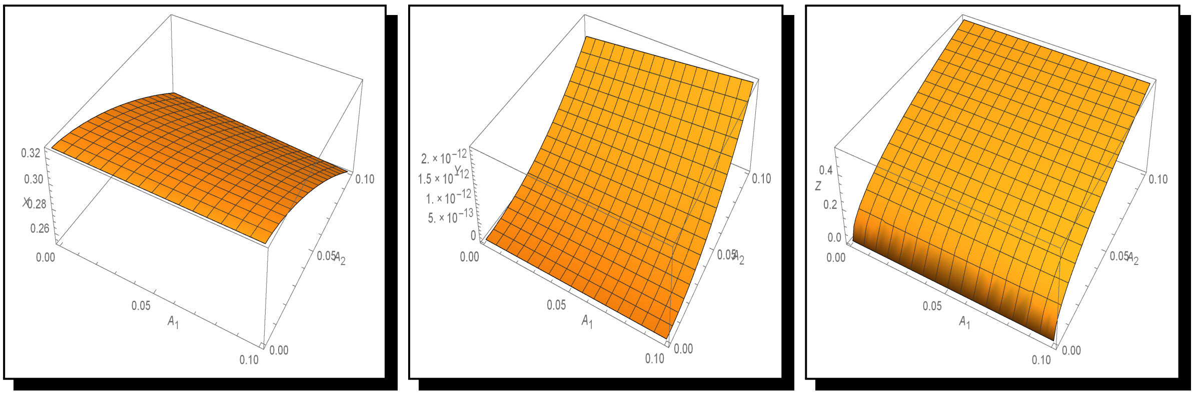
(a) x-coordinate (b) y-coordinate (c) z-coordinate

Figure 2. The variation in the coordinates of the out of plane equilibrium point L_6 with respect to oblateness for Luyten-726,taking $q_1 = 0.998$ and $q_2 = -0.998$



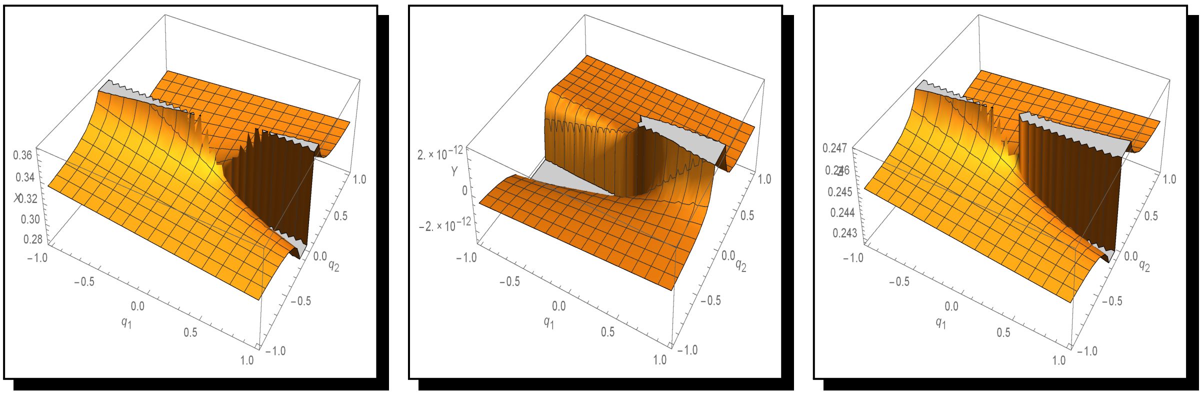
(a) x-coordinate (b) y-coordinate (c) z-coordinate

Figure 3. The variation in the coordinates of the out of plane equilibrium point L_6 with respect to radiation factor for Luyten-726, taking $A_1 = 0.01$ and $A_2 = 0.02$



(a) x-coordinate (b) y-coordinate (c) z-coordinate

Figure 4. The variation in the coordinates of the out of plane equilibrium point L_6 with respect to oblateness for Sirius,taking $q_1 = 0.998$ and $q_2 = -0.998$



(a) x-coordinate (b) y-coordinate (c) z-coordinate

Figure 5. The variation in the coordinates of the out of plane equilibrium point L_6 with respect to radiation factor for Sirius, taking $A_1 = 0.01$ and $A_2 = 0.02$

4. Stability of Out of Plane Equilibrium Points

4.1 Under the Effect of Drag Force

To study the stability of the equilibrium point denoted by (a_0, b_0, c_0) of an infinitesimal body, we displace it to the position (x, y, z) with a small displacement (u, v, w) from the point, such that $x = a_0 + u$, $y = b_0 + v$ and $z = c_0 + w$. Substituting these values in (2.1), we obtain the following linearized system of equations:

$$\begin{aligned}
u'' - 2v' &= (\Omega_{xx'})^0 u' + (\Omega_{xy'})^0 v' + (\Omega_{xz'})^0 w' + (\Omega_{xx})^0 u + (\Omega_{xy})^0 v + (\Omega_{xz})^0 w, \\
v'' + 2u' &= (\Omega_{yx'})^0 u' + (\Omega_{yy'})^0 v' + (\Omega_{yz'})^0 w' + (\Omega_{yx})^0 u + (\Omega^0_{yy})^0 v + (\Omega_{yz})^0 w, \\
w'' &= (\Omega_{zx'})^0 u' + (\Omega_{zy'})^0 v' + (\Omega_{zz'})^0 w' + (\Omega_{zx})^0 u + (\Omega^0_{zy})^0 v + (\Omega_{zz})^0 w.
\end{aligned} \tag{4.1}$$

here, the superscript 0 indicates that the derivatives are to be evaluated at the equilibrium points (a_0, b_0, c_0). Assuming the solution to the above system of equations to be $u = A\exp(\lambda f)$, $v = B\exp(\lambda f)$, $w = C\exp(\lambda f)$, system of equations (4.1) yields

$$A(\lambda^2 - \lambda(\Omega_{xx'})^0 - (\Omega_{xx})^0) + B(-2\lambda - \lambda(\Omega_{xy'})^0 - (\Omega_{xy})^0) + C(-\lambda(\Omega_{xz'})^0 - (\Omega_{xz})^0) = 0,$$

$$A(2\lambda-\lambda(\Omega_{yx'})^0-(\Omega_{yx})^0)+B(\lambda^2-\lambda(\Omega_{yy'})^0-(\Omega_{yy})^0)+C(-\lambda(\Omega_{yz'})^0-(\Omega_{yz})^0)=0,$$
$$A(-\lambda(\Omega_{zx'})^0-(\Omega_{zx})^0)+B(-\lambda(\Omega_{zy'})^0-(\Omega_{zy})^0)+C(\lambda^2-\lambda(\Omega_{zz'})^0-(\Omega_{zz})^0)=0.$$

For the above set of linear simultaneous equations, the characteristic equation is given as:

$$\lambda^6+a_0\lambda^5+a_1\lambda^4+a_2\lambda^3+a_3\lambda^2+a_4\lambda+a_5=0, \tag{4.2}$$

where

$$\begin{aligned}
a_0 &= -\Omega^0_{xx'}-\Omega^0_{yy'}-\Omega^0_{zz'},\\
a_1 &= \Omega^0_{xx'}\Omega^0_{yy'}+\Omega^0_{xx'}\Omega^0_{zz'}-\Omega^0_{xx}-\Omega^0_{xy'}\Omega^0_{yx'}+2\Omega^0_{xy'}-\Omega^0_{xy'}\Omega^0_{zx'}-2\Omega^0_{yx'}+\Omega^0_{yy'}\Omega^0_{zz'}-\Omega^0_{yy}\\
&\quad -\Omega^0_{yz'}\Omega^0_{zy'}-\Omega^0_{zz}+4,\\
a_2 &= -\Omega^0_{xx'}\Omega^0_{yy'}\Omega^0_{zz'}+\Omega^0_{yy}\Omega^0_{xx'}+\Omega^0_{xx'}\Omega^0_{yz'}\Omega^0_{zy'}+\Omega^0_{zz}\Omega^0_{xx'}+\Omega^0_{xx}\Omega^0_{yy'}+\Omega^0_{xx}\Omega^0_{zz'}\\
&\quad +\Omega^0_{xy'}\Omega^0_{yx'}\Omega^0_{zz'}-\Omega^0_{yx}\Omega^0_{xy'}-\Omega^0_{xy'}\Omega^0_{yz'}\Omega^0_{zx'}-2\Omega^0_{xy'}\Omega^0_{zz'}-\Omega^0_{xy}\Omega^0_{yx'}+2\Omega^0_{xy}\\
&\quad -\Omega^0_{xz'}\Omega^0_{yx'}\Omega^0_{zy'}+\Omega^0_{xz'}\Omega^0_{yy'}\Omega^0_{zx'}-\Omega^0_{zx}\Omega^0_{xz'}+2\Omega^0_{xz'}\Omega^0_{zy'}-\Omega^0_{xz}\Omega^0_{zx'}+2\Omega^0_{yx'}\Omega^0_{zz'}\\
&\quad -2\Omega^0_{yx}+\Omega^0_{zz}\Omega^0_{yy'}+\Omega^0_{yy}\Omega^0_{zz'}-2\Omega^0_{yz'}\Omega^0_{zx'}-\Omega^0_{zy}\Omega^0_{yz'}-\Omega^0_{yz}\Omega^0_{zy'}-4\Omega^0_{zz'},\\
a_3 &= -\Omega^0_{zz}\Omega^0_{xx'}\Omega^0_{yy'}-\Omega^0_{yy}\Omega^0_{xx'}\Omega^0_{zz'}+\Omega^0_{zy}\Omega^0_{xx'}\Omega^0_{yz'}+\Omega^0_{yz}\Omega^0_{xx'}\Omega^0_{zy'}-\Omega^0_{xx}\Omega^0_{yy'}\Omega^0_{zz'}\\
&\quad +\Omega^0_{xx}\Omega^0_{yy}+\Omega^0_{xx}\Omega^0_{yz'}\Omega^0_{zy'}+\Omega^0_{xx}\Omega^0_{zz}+\Omega^0_{zz}\Omega^0_{xy'}\Omega^0_{yx'}+\Omega^0_{yx}\Omega^0_{xy'}\Omega^0_{zz'}-\Omega^0_{zx}\Omega^0_{xy'}\Omega^0_{yz'}\\
&\quad -\Omega^0_{yz}\Omega^0_{xy'}\Omega^0_{zx'}-2\Omega^0_{zz}\Omega^0_{xy'}+\Omega^0_{xy}\Omega^0_{yx'}\Omega^0_{zz'}-\Omega^0_{xy}\Omega^0_{yx}-\Omega^0_{xy}\Omega^0_{yz'}\Omega^0_{zx'}-2\Omega^0_{xy}\Omega^0_{zz'}\\
&\quad -\Omega^0_{zy}\Omega^0_{xz'}\Omega^0_{yx'}-\Omega^0_{yx}\Omega^0_{xz'}\Omega^0_{zy'}+\Omega^0_{zx}\Omega^0_{xz'}\Omega^0_{yy'}+\Omega^0_{yy}\Omega^0_{xz'}\Omega^0_{zx'}+2\Omega^0_{zy}\Omega^0_{xz'}\\
&\quad -\Omega^0_{xz}\Omega^0_{yx'}\Omega^0_{zy'}+\Omega^0_{xz}\Omega^0_{yy'}\Omega^0_{zx'}-\Omega^0_{xz}\Omega^0_{zx}+2\Omega^0_{xz}\Omega^0_{zy'}+2\Omega^0_{zz}\Omega^0_{yx'}+2\Omega^0_{yx}\Omega^0_{zz'}\\
&\quad +\Omega^0_{yy}\Omega^0_{zz}-2\Omega^0_{zx}\Omega^0_{yz'}-2\Omega^0_{yz}\Omega^0_{zx'}-\Omega^0_{yz}\Omega^0_{zy}-4\Omega^0_{zz},\\
a_4 &= -\Omega^0_{yy}\Omega^0_{zz}\Omega^0_{xx'}+\Omega^0_{yz}\Omega^0_{zy}\Omega^0_{xx'}-\Omega^0_{xx}\Omega^0_{zz}\Omega^0_{yy'}-\Omega^0_{xx}\Omega^0_{yy}\Omega^0_{zz'}+\Omega^0_{xx}\Omega^0_{zy}\Omega^0_{yz'}\\
&\quad +\Omega^0_{xx}\Omega^0_{yz}\Omega^0_{zy'}+\Omega^0_{yx}\Omega^0_{zz}\Omega^0_{xy'}-\Omega^0_{yz}\Omega^0_{zx}\Omega^0_{xy'}+\Omega^0_{xy}\Omega^0_{zz}\Omega^0_{yx'}+\Omega^0_{xy}\Omega^0_{yx}\Omega^0_{zz'}\\
&\quad -\Omega^0_{xy}\Omega^0_{zx}\Omega^0_{yz'}-\Omega^0_{xy}\Omega^0_{yz}\Omega^0_{zx'}-2\Omega^0_{xy}\Omega^0_{zz}-\Omega^0_{yx}\Omega^0_{zy}\Omega^0_{xz'}+\Omega^0_{yy}\Omega^0_{zx}\Omega^0_{xz'}-\Omega^0_{xz}\Omega^0_{zy}\Omega^0_{yx'}\\
&\quad -\Omega^0_{xz}\Omega^0_{yx}\Omega^0_{zy'}+\Omega^0_{xz}\Omega^0_{zx}\Omega^0_{yy'}+\Omega^0_{xz}\Omega^0_{yy}\Omega^0_{zx'}+2\Omega^0_{xz}\Omega^0_{zy}+2\Omega^0_{yx}\Omega^0_{zz}-2\Omega^0_{yz}\Omega^0_{zx},\\
a_5 &= -\Omega^0_{xx}\Omega^0_{yy}\Omega^0_{zz}+\Omega^0_{xx}\Omega^0_{yz}\Omega^0_{zy}+\Omega^0_{xy}\Omega^0_{yx}\Omega^0_{zz}-\Omega^0_{xy}\Omega^0_{yz}\Omega^0_{zx}-\Omega^0_{xz}\Omega^0_{yx}\Omega^0_{zy}+\Omega^0_{xz}\Omega^0_{yy}\Omega^0_{zx}.
\end{aligned}$$

Here, the values of the second order partial derivatives of Ω^0_x, Ω^0_y and Ω^0_z at the equilibrium points are represented as follows,

$$\begin{aligned}
\Omega_{xx} &= \frac{1}{\sqrt{1-e^2}}\left[1-\frac{1}{n^2}\left\{Q_1(\mu+x)\left(-\frac{15A_1(\mu+x)}{2r_1^7}+\frac{105A_1z^2(\mu+x)}{2r_1^9}-\frac{3(\mu+x)}{r_1^5}\right)\right.\right.\\
&\quad +Q_2(\mu+x-1)\left(-\frac{15A_2(\mu+x-1)}{2r_2^7}+\frac{105A_2z^2(\mu+x-1)}{2r_2^9}-\frac{3(\mu+x-1)}{r_2^5}\right)\\
&\quad \left.+Q_1\left(-\frac{15A_1z^2}{2r_1^7}+\frac{3A_1}{2r_1^5}+\frac{1}{r_1^3}\right)+Q_2\left(-\frac{15A_2z^2}{2r_2^7}+\frac{3A_2}{2r_2^5}+\frac{1}{r_2^3}\right)\right\}\\
&\quad -\frac{\mathscr{W}_1}{n}\left(-\frac{4(\mu+x)^2((\mu+x)x'+yy'+zz')}{r_1^6}+\frac{2(\mu+x)y+yy'+zz'}{r_1^4}\right)
\end{aligned}$$

$$
-\frac{\mathscr{W}_2}{n}\left(-\frac{4(\mu+x-1)^2((\mu+x-1)x'+yy'+zz')}{r_2^6}+\frac{2(\mu+x-1)y+yy'+zz'}{r_2^4}\right)\Bigg],
$$

$$
\begin{aligned}
\Omega_{xy}=\frac{1}{\sqrt{1-e^2}}\Bigg[&-\frac{1}{n^2}\Bigg\{Q_1(\mu+x)\left(-\frac{15A_1y}{2r_1^7}+\frac{105A_1z^2y}{2r_1^9}-\frac{3y}{r_1^5}\right)\\
&+Q_2(\mu+x-1)\left(-\frac{15A_2y}{2r_2^7}+\frac{105A_2z^2y}{2r_2^9}-\frac{3y}{r_2^5}\right)\Bigg\}\\
&-\frac{\mathscr{W}_1}{n}\left(-\frac{4(\mu+x)y((\mu+x)x'+yy'+zz')}{r_1^6}+\frac{2(y-x')y+y'(x+\mu)}{r_1^4}-\frac{1}{r_1^2}\right)\\
&-\frac{\mathscr{W}_2}{n}\left(-\frac{4(\mu+x-1)y((\mu+x-1)x'+yy'+zz')}{r_2^6}+\frac{2(y-x')y+y'(x+\mu-1)}{r_2^4}-\frac{1}{r_2^2}\right)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{xz}=\frac{1}{\sqrt{1-e^2}}\Bigg[&-\frac{1}{n^2}\Bigg\{Q_1(\mu+x)\left(-\frac{15A_1z}{2r_1^7}+\frac{105A_1z^2z}{2r_1^9}-\frac{3z}{r_1^5}\right)\\
&+Q_2(\mu+x-1)\left(-\frac{15A_2z}{2r_2^7}+\frac{105A_2z^3}{2r_2^9}-\frac{3z}{r_2^5}\right)\Bigg\}\\
&-\frac{\mathscr{W}_1}{n}\left(-\frac{4(\mu+x)z((\mu+x)x'+yy'+zz')}{r_1^6}+\frac{2(y-x')z+z'(x+\mu)}{r_1^4}-\frac{1}{r_1^2}\right)\\
&-\frac{\mathscr{W}_2}{n}\left(-\frac{4(\mu+x-1)z((\mu+x-1)x'+yy'+zz')}{r_2^6}+\frac{2(y-x')z+z'(x+\mu-1)}{r_2^4}-\frac{1}{r_2^2}\right)\Bigg],
\end{aligned}
$$

$$
\Omega_{xx'}=\frac{-1}{n\sqrt{1-e^2}}\left[\mathscr{W}_1\left(\frac{(\mu+x)^2}{r_1^4}+\frac{1}{r_1^2}\right)+\mathscr{W}_2\left(\frac{(\mu+x-1)^2}{r_2^4}+\frac{1}{r_2^2}\right)\right],
$$

$$
\Omega_{xy'}=\frac{-1}{n\sqrt{1-e^2}}\left[\mathscr{W}_1\frac{(\mu+x)y}{r_1^4}+\mathscr{W}_2\frac{(\mu+x-1)y}{r_2^4}\right],
$$

$$
\Omega_{xz'}=\frac{-1}{n\sqrt{1-e^2}}\left[\mathscr{W}_1\frac{(\mu+x)z}{r_1^4}+\mathscr{W}_2\frac{(\mu+x-1)z}{r_2^4}\right],
$$

$$
\begin{aligned}
\Omega_{yx}=\frac{1}{\sqrt{1-e^2}}\Bigg[&-\frac{1}{n^2}\Bigg\{Q_1y(\mu+x)\left(-\frac{15A_1}{2r_1^7}+\frac{105A_1z^2}{2r_1^9}-\frac{3}{r_1^5}\right)\\
&+Q_2(\mu+x-1)y\left(-\frac{15A_2}{2r_2^7}+\frac{105A_2z^2}{2r_2^9}-\frac{3}{r_2^5}\right)\Bigg\}\\
&-\frac{\mathscr{W}_1}{n}\left(-\frac{4(\mu+x)y((\mu+x)x'+yy'+zz')}{r_1^6}+\frac{x'y-2(\mu+x)(\mu+x+y')}{r_1^4}\right)\\
&-\frac{\mathscr{W}_2}{n}\left(-\frac{4(\mu+x-1)y((\mu+x-1)x'+yy'+zz')}{r_2^6}+\frac{x'y-2(-1+\mu+x)(-1+\mu+x+y')}{r_2^4}\right)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{yy}=\frac{1}{\sqrt{1-e^2}}\Bigg[&1-\frac{1}{n^2}\Bigg\{Q_1y^2\left(-\frac{15A_1}{2r_1^7}+\frac{105A_1z^2}{2r_1^9}-\frac{3}{r_1^5}\right)+Q_2y^2\left(-\frac{15A_2}{2r_2^7}+\frac{105A_2z^2}{2r_2^9}-\frac{3}{r_2^5}\right)\\
&+Q_1\left(-\frac{15A_1z^2}{2r_1^7}+\frac{3A_1}{2r_1^5}+\frac{1}{r_1^3}\right)+Q_2\left(-\frac{15A_2z^2}{2r_2^7}+\frac{3A_2}{2r_2^5}+\frac{1}{r_2^3}\right)\Bigg\}
\end{aligned}
$$

$$-\frac{\mathscr{W}_1}{n}\left(-\frac{4y^2((\mu+x)x'+yy'+zz')}{r_1^6}+\frac{(\mu+x)(x'-2y)+zz'}{r_1^4}r_1^4\right)$$
$$\left.-\frac{\mathscr{W}_2}{n}\left(-\frac{4y^2((\mu+x-1)x'+yy'+zz')}{r_2^6}+\frac{(\mu+x-1)(x'-2y)+zz'}{r_2^4}\right)\right],$$

$$\Omega_{yz}=\frac{1}{\sqrt{1-e^2}}\left[-\frac{1}{n^2}\left\{Q_1z(\mu+x)\left(-\frac{15A_1}{2r_1^7}+\frac{105A_1z^2}{2r_1^9}-\frac{3}{r_1^5}\right)\right.\right.$$
$$\left.+Q_2(\mu+x-1)z\left(-\frac{15A_2}{2r_2^7}+\frac{105A_2z^2}{2r_2^9}-\frac{3}{r_2^5}\right)\right\}$$
$$-\frac{\mathscr{W}_1}{n}\left(-\frac{4zy((\mu+x)x'+yy'+zz')}{r_1^6}+\frac{z'y-2z(\mu+x+y')}{r_1^4}\right)$$
$$\left.-\frac{\mathscr{W}_2}{n}\left(-\frac{4zy((\mu+x-1)x'+yy'+zz')}{r_2^6}+\frac{z'y-2z(\mu+x+y'-1)}{r_2^4}\right)\right],$$

$$\Omega_{yx'}=\frac{-1}{n\sqrt{1-e^2}}\left[\mathscr{W}_1\frac{(\mu+x)y}{r_1^4}+\mathscr{W}_2\frac{(\mu+x-1)y}{r_2^4}\right],$$

$$\Omega_{yy'}=\frac{-1}{n\sqrt{1-e^2}}\left[\mathscr{W}_1\left(\frac{y^2}{r_1^4}+\frac{1}{r_1^2}\right)+\mathscr{W}_2\left(\frac{y^2}{r_2^4}+\frac{1}{r_2^2}\right)\right],$$

$$\Omega_{yz'}=\frac{-1}{n\sqrt{1-e^2}}\left[\mathscr{W}_1\frac{yz}{r_1^4}+\mathscr{W}_2\frac{yz}{r_2^4}\right],$$

$$\Omega_{zx}=\frac{1}{\sqrt{1-e^2}}\left[-\frac{1}{n^2}\left\{Q_1z(\mu+x)\left(-\frac{45A_1}{2r_1^7}+\frac{105A_1z^2}{2r_1^9}-\frac{3}{r_1^5}\right)\right.\right.$$
$$\left.+Q_2(\mu+x-1)z\left(-\frac{45A_2}{2r_2^7}+\frac{105A_2z^2}{2r_2^9}-\frac{3}{r_2^5}\right)\right\}$$
$$-\frac{\mathscr{W}_1}{n}\left(-\frac{4(\mu+x)z((\mu+x)x'+yy'+zz')}{r_1^6}+\frac{x'z-2(\mu+x)z'}{r_1^4}\right)$$
$$\left.-\frac{\mathscr{W}_2}{n}\left(-\frac{4(\mu+x-1)z((\mu+x-1)x'+yy'+zz')}{r_2^6}+\frac{x'z-2(-1+\mu+x)z'}{r_2^4}\right)\right],$$

$$\Omega_{zy}=\frac{1}{\sqrt{1-e^2}}\left[-\frac{1}{n^2}\left\{Q_1yz\left(-\frac{y5A_1}{2r_1^7}+\frac{105A_1z^2}{2r_1^9}-\frac{3}{r_1^5}\right)+Q_2yz\left(-\frac{45A_2}{2r_2^7}+\frac{105A_2z^2}{2r_2^9}-\frac{3}{r_2^5}\right)\right.\right.$$
$$\left.+Q_1\left(-\frac{15A_1z^2}{2r_1^7}+\frac{3A_1}{2r_1^5}+\frac{1}{r_1^3}\right)+Q_2\left(-\frac{15A_2z^2}{2r_2^7}+\frac{3A_2}{2r_2^5}+\frac{1}{r_2^3}\right)\right\}$$
$$-\frac{\mathscr{W}_1}{n}\left(-\frac{4yz((\mu+x)x'+yy'+zz')}{r_1^6}+\frac{zy'-2yz'}{r_1^4}r_1^4\right)$$
$$\left.-\frac{\mathscr{W}_2}{n}\left(-\frac{4yz((\mu+x-1)x'+yy'+zz')}{r_2^6}+\frac{zy'-2yz'}{r_2^4}\right)\right],$$

$$\Omega_{zz}=\frac{1}{\sqrt{1-e^2}}\left[1-\frac{1}{n^2}\left\{Q_1z^2\left(-\frac{75A_1}{2r_1^7}+\frac{105A_1z^2}{2r_1^9}-\frac{3}{r_1^5}\right)+Q_2z^2\left(-\frac{75A_2}{2r_2^7}+\frac{105A_2z^2}{2r_2^9}-\frac{3}{r_2^5}\right)\right.\right.$$

$$+Q_1\left(-\frac{15A_1z^2}{2r_1^7}+\frac{9A_1}{2r_1^5}+\frac{1}{r_1^3}\right)+Q_2\left(-\frac{15A_2z^2}{2r_2^7}+\frac{9A_2}{2r_2^5}+\frac{1}{r_2^3}\right)\Bigg\}$$
$$-\frac{\mathscr{W}_1}{n}\left(-\frac{4z^2((\mu+x)x'+yy'+zz')}{r_1^6}+\frac{(\mu+x)x+yy'}{r_1^4}\right)$$
$$-\frac{\mathscr{W}_2}{n}\left(-\frac{4z^2((\mu+x-1)x'+yy'+zz')}{r_2^6}+\frac{(\mu+x-1)x+yy'}{r_2^4}\right)\Bigg],$$
$$\Omega_{zx'}=\frac{-1}{n\sqrt{1-e^2}}\left[\mathscr{W}_1\frac{(\mu+x)z}{r_1^4}+\mathscr{W}_2\frac{(\mu+x-1)z}{r_2^4}\right],$$
$$\Omega_{zy'}=\frac{-1}{n\sqrt{1-e^2}}\left[\mathscr{W}_1\frac{yz}{r_1^4}+\mathscr{W}_2\frac{yz}{r_2^4}\right],$$
$$\Omega_{zz'}=\frac{-1}{n\sqrt{1-e^2}}\left[\mathscr{W}_1\left(\frac{z^2}{r_1^4}+\frac{1}{r_1^2}\right)+\mathscr{W}_2\left(\frac{z^2}{r_2^4}+\frac{1}{r_2^2}\right)\right].$$

Using these values, the coefficients of equation (4.2) are obtained.

4.2 When the Drag Forces are Neglected

When the drag forces are neglected, the coefficients of equation (4.2), when expanded in linear terms of A_1, A_2, q_1, q_2 and μ, reduces to the following values:

$$a_0=a_2=a_4=0,$$
$$a_1=4-\frac{2}{\sqrt{1-e^2}},$$
$$a_3=\frac{4}{n^2\sqrt{1-e^2}}(Q_1K_{31}+Q_2K_{32})+\frac{1}{1-e^2}\Bigg[1-\frac{1}{n^2}(Q_1K_{31}+Q_2K_{32})$$
$$+\frac{1}{n^4}\Bigg\{Q_1^2\left(\frac{54A_1z^2}{r_1^{10}}-\frac{18A_1}{r_1^8}-\frac{3}{r_1^6}\right)+Q_2^2\left(\frac{54A_2z^2}{r_2^{10}}-\frac{18A_2}{r_2^8}-\frac{3}{r_2^6}\right)$$
$$+Q_1Q_2\Bigg(-\frac{315A_1z^4}{2r_1^9r_2^5}-\frac{45}{2}\frac{(4(x+\mu)-5)A_1z^2}{r_1^7r_2^5}+\frac{45A_1z^2}{r_1^7r_2^3}+\frac{9A_1z^2}{r_1^5r_2^5}-\frac{18A_1}{r_1^5r_2^3}$$
$$-\frac{315A_2z^4}{2r_1^5r_2^9}+\frac{45}{2}\frac{A_2z^2(4(\mu+x)+1)}{r_1^5r_2^7}+\frac{9(1+A_2)z^2}{r_1^5r_2^5}+\frac{45A_2z^2}{r_1^3r_2^7}-\frac{18A_2}{r_1^3r_2^5}-\frac{6}{r_1^3r_2^3}\Bigg)\Bigg\}\Bigg],$$
$$a_5=\frac{1}{(1-e^2)^{3/2}}\Bigg[\frac{1}{n^2}(Q_1K_{31}+Q_2K_{32})+\frac{1}{n^4}\Bigg\{Q_1^2K_{41}+Q_2^2K_{42}$$
$$+Q_1Q_2\Bigg(-\frac{45A_1z_0^2(-4\mu-4x_0+z_0^2)}{2r_1^7r_2^5}-\frac{45A_2z_0^2(4\mu+4x_0+z_0^2)}{2r_1^5r_2^7}-\frac{105A_1z_0^4}{2r_1^9r_2^3}$$
$$+\frac{315A_1z_0^4}{2r_1^9r_2^5}-\frac{105A_2z_0^4}{2r_1^3r_2^9}+\frac{315A_2z_0^4}{2r_1^5r_2^9}+\frac{45A_1z_0^2}{2r_1^7r_2^3}-\frac{9(A_1+A_2+2)z_0^2}{2r_1^5r_2^5}-\frac{225A_1z_0^2}{2r_1^7r_2^5}$$
$$+\frac{45A_2z_0^2}{2r_1^3r_2^7}-\frac{45A_2z_0^2}{2r_1^5r_2^7}+\frac{9A_1}{r_1^5r_2^3}+\frac{9A_2}{r_1^3r_2^5}+\frac{3z_0^2}{r_1^5r_2^3}+\frac{3z_0^2}{r_1^3r_2^5}-\frac{1}{r_1^3r_2^3}+\frac{3}{r_1r_2^5}\Bigg)\Bigg\}$$
$$+\frac{1}{n^6}\Big\{Q_1^3K_{51}+Q_2^3K_{52}+Q_1Q_2^2K_{61}+Q_1^2Q_2K_{62}\Big\}\Bigg],$$

where

$$K_{31} = \frac{105A_1z^4}{2r_1^9} - \frac{45A_1z^2}{r_1^7} + \frac{3(3A_1-2z^2)}{2r_1^5} + \frac{1}{r_1^3},$$

$$K_{32} = \frac{105A_2z^4}{2r_2^9} - \frac{45A_2z^2}{r_1^7} + \frac{3(3A_2-2z^2)}{2r_2^5} + \frac{1}{r_2^3},$$

$$K_{41} = -\frac{75A_1z_0^4}{r_1^{12}} + \frac{18A_1z_0^2}{r_1^{10}} + \frac{3(z_0^2+3A_1)}{r_1^8} + \frac{1}{r_1^6},$$

$$K_{42} = -\frac{75A_2z_0^4}{r_2^{12}} + \frac{18A_2z_0^2}{r_2^{10}} + \frac{3(z_0^2+3A_2)}{r_2^8} + \frac{1}{r_2^6},$$

$$K_{51} = \frac{54A_1z_0^2}{r_1^{13}} - \frac{18A_1}{r_1^{11}} - \frac{2}{r_1^9}, \quad K_{52} = \frac{54A_1z_0^2}{r_2^{13}} - \frac{18A_2}{r_2^{11}} - \frac{2}{r_2^9},$$

$$K_{61} = -\frac{90A_1x_0z_0^2}{r_1^7r_2^8} - \frac{180A_2x_0z_0^2}{r_1^5r_2^{10}} - \frac{45A_1(4\mu-5)z_0^2}{2r_1^7r_2^8} - \frac{9A_2(20\mu+1)z_0^2}{r_1^5r_2^{10}} - \frac{315A_1z_0^4}{2r_1^9r_2^8}$$
$$- \frac{225A_2z_0^4}{r_1^5r_2^{12}} + \frac{45A_1z_0^2}{r_1^7r_2^6} + \frac{9(A_1+6A_2+1)z_0^2}{r_1^5r_2^8} + \frac{54A_2z_0^2}{r_1^3r_2^{10}} - \frac{18A_1}{r_1^5r_2^6} - \frac{36A_2}{r_1^3r_2^8} - \frac{6}{r_1^3r_2^6}$$

and

$$K_{62} = -\frac{90A_1x_0z_0^2}{r_1^{10}r_2^5} - \frac{180A_2x_0z_0^2}{r_1^8r_2^7} + \frac{18A_1(7-5\mu)z_0^2}{r_1^{10}r_2^5} - \frac{45A_2(8\mu+1)z_0^2}{2r_1^8r_2^7} - \frac{225A_1z_0^4}{r_1^{12}r_2^5}$$
$$- \frac{315A_2z_0^4}{2r_1^8r_2^9} + \frac{99A_1z_0^2}{r_1^{10}r_2^3} + \frac{9(A_1+6A_2+1)z_0^2}{r_1^8r_2^5} - \frac{36A_1}{r_1^8r_2^3} - \frac{18A_2}{r_1^6r_2^5} - \frac{6}{r_1^6r_2^3}.$$

Consequently, equation (4.2) reduces to

$$\lambda^6 + a_1\lambda^4 + a_3\lambda^2 + a_5 = 0. \tag{4.3}$$

Assuming $\lambda^2 = \rho$, the following cubic equation is obtained:

$$\rho^3 + a_1\rho^2 + a_3\rho + a_5 = 0. \tag{4.4}$$

The stability of the system will hold if the roots of equation (4.3) are purely imaginary, that is if $\rho_i < 0$, $i = 1,2,3$. This condition leads us to the following inequalities:

$$a_1 > 0,\ a_5 > 0 \quad \text{and} \quad \Delta < 0, \tag{4.5}$$

where

$$\Delta = \frac{(2a_1^3 - 9a_1a_3 + 27a_5)^2 + 4(3a_3 - a_1^2)^3}{27}. \tag{4.6}$$

Now, $a_1 > 0$, gives the condition $4 - \frac{2}{\sqrt{1-e^2}} > 0$, that is $e < 0.866$. Furthermore, expanding the terms of the above inequalities, we get the following conditions for the values of coefficient a_3 and a_5 as:

$$0 < a_3 \le \frac{1}{3}\left(4 - \frac{2}{\sqrt{1-e^2}}\right)^2 \le \frac{4}{3} \tag{4.7}$$

and

$$0 < a_5 < \frac{1}{27}(9a_1a_3 - 2a_1^3 - 2(a_1^2 - 3a_3)^{3/2}), \quad \text{for } a_3 \le 1,$$

$$\frac{1}{27}(9a_1a_3 - 2a_1^3 - 2(a_1^2 - 3a_3)^{3/2}) < a_5 < \frac{1}{27}(9a_1a_3 - 2a_1^3 + 2(a_1^2 - 3a_3)^{3/2}), \quad \text{for } a_3 > 1. \quad (4.8)$$

Thus the inequalities (4.7) and (4.8) are utilized to define the stability region for specific values of the various parameters such as oblateness, radiation pressure and so on. These conditions for stability are analogous to Ragos and Zagouras [14] when oblateness of the primaries are neglected.

5. Numerical Exploration

We numerically examined the analytical results obtained in the previous sections around two binary systems Luyten-726 and Sirius. Figures 2 and 3 are depicting the variation in the coordinates of the out of plane equilibrium point L_6 around Luyten-726, when the effects of oblateness and radiation factor respectively are taken into account. At the same time, Figures 4 and 5 are respectively representing the variation in the coordinates of the out of plane equilibrium point L_6 around Sirius when the above mentioned parameters are considered. On studying Figures 2 and 4, it was found that there is a smooth increase or decrease in each of the three coordinates with the variation in the value of oblateness parameter of second primary A_2 but overall effect of A_1 is negligible. Also, it was found, on studying the effect of radiation pressure separately as represented by Figures 3 and 5, a sharp and irregular variation in all the three coordinates are observed when the radiation pressure of the second primary q_2 is considered. However, there is no overall effect of q_1 on the system. Especially in the case of y-coordinates in the scale of minute variation (approx. in the order of 10^{-12}) stiff variations are observed.

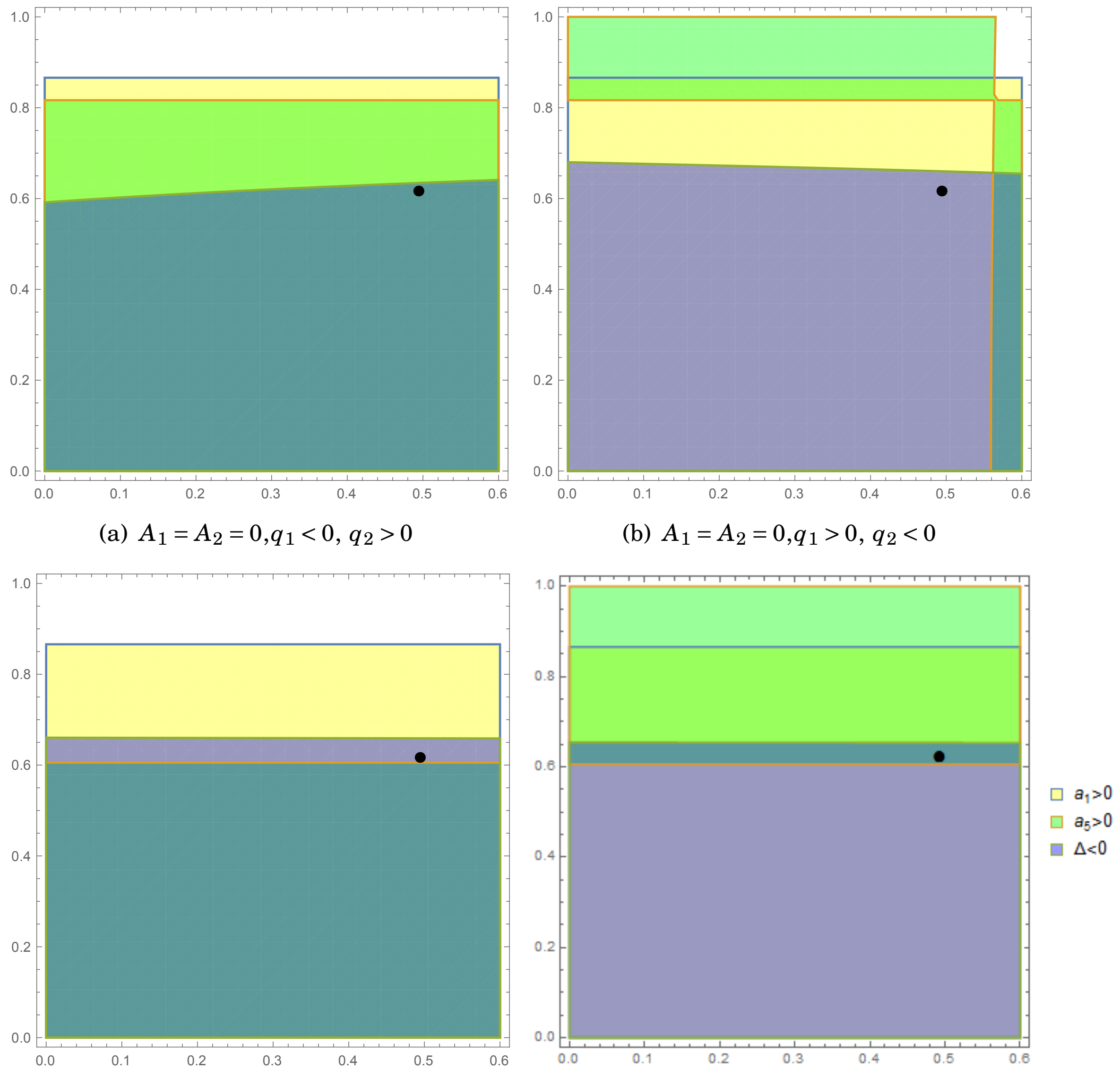
The effect of PR-drag on the stability of the out of plane equilibrium points has been investigated by finding the roots of the characteristic equation (4.2) for both the binary systems, varying the values of the radiation parameter q_1 and q_2, as shown in Table 2 and Table 3. It was observed that at least two of the six roots of the characteristic equation, have positive real part and hence the equilibrium points are unstable.

Table 2. Characteristic roots taking varying values of q_1 and q_2 for the out of plane equilibrium point L_6 for the binary system Luyten 726.

q_1	q_2	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
-0.013	0.3	$(30.75, \pm 1.00)$	$(-30.75, \pm 1.00)$	$(-1.28 \times 10^{-9}, \pm 43.45)$
	0.5	$(39.67, \pm 1.00)$	$(-39.67, \pm 1.00)$	$(-9.18 \times 10^{-10}, \pm 50.10)$
	0.9	$(53.22, \pm 1.00)$	$(-53.22, \pm 1.00)$	$(-7.53 \times 10^{-10}, \pm 75.27)$
0.008	-0.027	$(-9.43 \times 10^{-10}, \pm 7.91)$	$(-9.41 \times 10^{-10}, \pm 9.84)$	$(\pm 12.57, 0)$
	-0.125	$(-1.03 \times 10^{-9}, \pm 20.79)$	$(-1.03 \times 10^{-9}, \pm 18.80)$	$(\pm 28.01, 0)$
	-0.343	$(-1.23 \times 10^{-9}, \pm 31.84)$	$(-1.23 \times 10^{-9}, \pm 33.84)$	$(\pm 46.46, 0)$

Table 3. Characteristic roots taking varying values of q_1 and q_2 for the out of plane equilibrium point L_6 for the binary system Sirus.

q_1	q_2	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
	0.3	$(73.66, \pm 1.00)$	$(-73.66, \pm 1.00)$	$(-2.33 \times 10^{-10}, \pm 104.17)$
-0.013	0.5	$(95.10, \pm 1.00)$	$(-95.10, \pm 1.00)$	$(-1.67 \times 10^{-10}, \pm 134.49)$
	0.9	$(127.59, \pm 1.00)$	$(-127.59, \pm 1.00)$	$(-3.36 \times 10^{-11}, \pm 180.43)$
	-0.027	$(-1.71 \times 10^{-10}, \pm 23.08)$	$(1.71 \times 10^{-10}, \pm 21.08)$	$(\pm 31.24, 0)$
0.008	-0.125	$(-1.88 \times 10^{-10}, \pm 48.54)$	$(-1.87 \times 10^{-10}, \pm 46.54)$	$(\pm 67.24, 0)$
	-0.343	$(-2.24 \times 10^{-10}, \pm 77.76)$	$(-2.24 \times 10^{-10}, \pm 79.76)$	$(\pm 111.39, 0)$



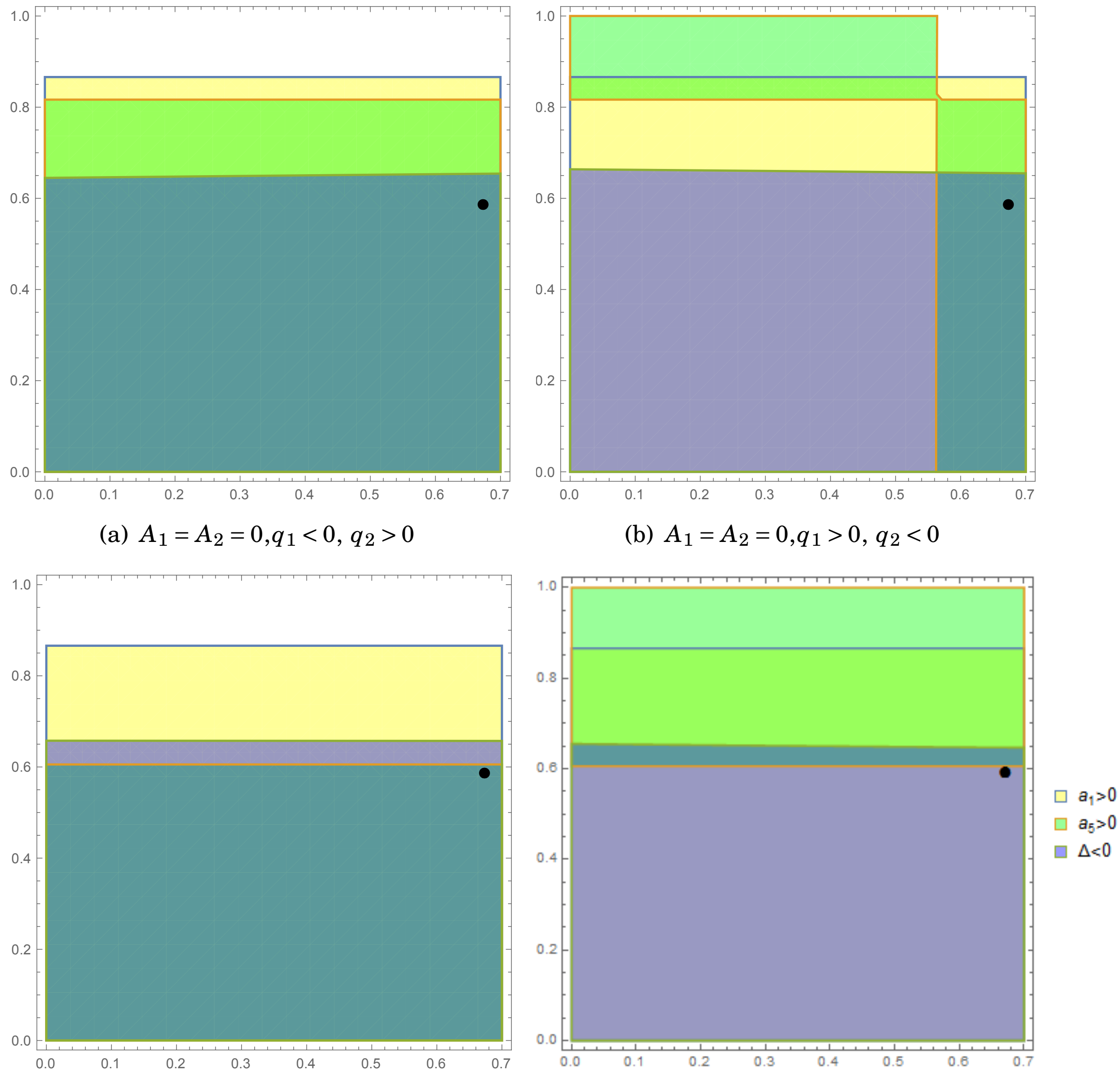
(a) $A_1 = A_2 = 0, q_1 < 0, q_2 > 0$

(b) $A_1 = A_2 = 0, q_1 > 0, q_2 < 0$

(c) $A_1 = 0.1, A_2 = 0.2, q_1 < 0, q_2 > 0$

(d) $A_1 = 0.1, A_2 = 0.2, q_1 > 0, q_2 < 0$

Figure 6. The region plot denoting the area satisfying the stability conditions as given in (4.5) in the $\mu - e$ plane for $a = 1.95$. The point denote the value corresponding to the binary system Luyten-726.



(a) $A_1 = A_2 = 0, q_1 < 0,\ q_2 > 0$

(b) $A_1 = A_2 = 0, q_1 > 0,\ q_2 < 0$

(c) $A_1 = 0.1, A_2 = 0.2, q_1 < 0,\ q_2 > 0$

(d) $A_1 = 0.1, A_2 = 0.2, q_1 > 0,\ q_2 < 0$

Figure 7. The region plot denoting the area satisfying the stability conditions as given in (4.5) in the $\mu - e$ plane for $a = 7.5$. The point denote the value corresponding to the binary system Sirius.

6. Conclusion and Discussion

The location and stability of out of plane equilibrium points in the photogravitational elliptical restricted three-body problem, where the primaries are taken to be oblate spheroid, has been investigated. The effect of PR-drag is incorporated in the analysis. The position of equilibrium points are analyzed in the form of series and found that inclusion of PR-drag results in a small but finite y-component for the out of plane equilibrium points. Also x-, y- and z- components are found to be affected by the eccentricity of the orbits, semi-major axis, radiation and oblateness parameters of both the primaries, while the x- and y- coordinates are also affected by the PR-drag forces. While studying the stability around the binary systems Luyten-726 and Sirius and found that at least two roots have positive real part so that the points are unstable when drag forces are considered.

The study of locations and stability of out of plane equilibrium points when both the primaries are radiating oblate spheroids in absence of PR-drag was undertaken by Singh and Umar [22]. However, the novelty of our paper lies in the different approach undertaken to study the stability. When the drag forces are neglected, the stability region in the $\mu - e$ plane has been plotted using the conditions given by the inequalities (4.5) and represented in Figures 5 and 6, taking $a = 1.95$ and $a = 7.5$, respectively. The area of intersection of all the three inequalities define the range of μ and e for which the equilibrium points are stable. Figures 6a and 6c depicts that for $q_1 < 0$ and $q_2 > 0$, the point denoting the value corresponding to Luyten-726 lies inside stable region when oblateness are negligible but the point lies outside the stability region for highly oblate primaries. On the contrary for $q_2 < 0$ and $q_1 > 0$ as shown in Figures 6b and 6d the point is stable when the primaries are assumed to be highly oblate but is unstable if the primaries are considered to be spherical. From Figures 7a and 7c, we observed that for $q_1 < 0$ and $q_2 > 0$, the point denoting the value corresponding to Sirius lies inside stable region whether the oblateness are taken to be negligible or high, where as in the case of $q_2 < 0$ and $q_1 > 0$ as shown in Figures 7b and 7d, the point denoting the value corresponding to Sirius lies inside stable region when oblateness are negligible but are outside the stability region for highly oblate primaries. We arrived at the conclusion that the oblateness factor reduced the stability region for all possible values of radiation parameter q_1 and q_2 for the out of plane equilibrium points around both the binary systems.

Taking $q_1 = 0$, $q_2 = q$, $A_1 = A$, $A_2 = 0$ and $e = 0$ our problem reduces to the problem discussed by Singh and Amuda [19]. However in this paper, we have discussed about the coordinates of the out-of-plane equilibrium as special cases for only two particular values of the y-coordinate $y^* = 0$ and $y^* = 0.1$. On neglecting the PR-drag forces, radiation pressure of the second primary and oblateness of the first primary, the stability condition are analogues to the conditions presented by Singh and Umar [20]. Taking $e = 0$ and $A_1 = 0$, neglecting the effect of the PR-drag forces, the results obtained for our problem corresponds to Shankaran *et al.* [16]. Taking $e = q_1 = q_2 = 0$, the problem corresponds to Douskos and Markellos [4]. In our future work, we plan to study the Hill's stability and Hamiltonian of the system.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] E.I. Abouelmagd and A. Mostafa, Out of plane equilibrium points locations and the forbidden movement regions in the restricted three-body problem with variable mass, *Astrophys Space Sci.* **357** (2015), 58.

[2] J.M.A. Danby, *Fundamentals of Celestial Mechanics*, 2nd edition, William-Bell Inc., Virginia (1988).

[3] M.K. Das, P. Narang, S. Mahajan and M. Yuasa, On out of plane equilibrium points in photo-gravitational restricted three-body problem, *J. Astrophys. Astr.* **30** (2009), 177 – 185.

[4] C.N. Douskos and V.V. Markellos, Out-of-plane equilibrium points in the restricted three-body problem with oblateness, *Astronomy and Astrophysics* **446** (2006), 357 – 360.

[5] D.W. Geyer, R.S. Harrington and C.E. Worley, Parallax, Orbit and Mass of the visual binary L726-8, *The Astronomical Journal* **95** (6) (1988), 1841 – 1842.

[6] S. Gong and J. Li, Analytical criteria of Hill stability in the elliptic restricted three body problem, *Astrophysics and Space Science* **358**, doi:10.1007/s10509-015-2463-y(2015).

[7] J. Liebert, P.A. Young, D. Arnett, J.B. Holberg and K.A. Williams, The age and progenitor mass of Sirius B, *The Astrophysical Journal* **630** (1) (2005), L69 – L72, arXiv:astro-ph/0507523.

[8] A. Narayan and C. Ramesh, Effects of photogravitation and oblateness on the triangular Lagrangian points in elliptic restricted three-body problem, *International Journal of Pure and Applied Mathematics* **68** (2011), 201 – 224.

[9] A. Narayan and C. Ramesh, Stability of triangular equilibrium points in elliptic restricted three-body problem under the effects of photogravitation and oblateness of the primaries, *International Journal of Pure and Applied Mathematics* **70** (2011), 735 – 754.

[10] A. Narayan, K.K. Pandey and S.K. Shrivastav, Effects of radiation and triaxiality on the triangular equilibrium points in elliptic restricted three-body problem, *International Journal of Advanced Astronomy* **3** (2) (2015), 97 – 106.

[11] V.V. Radzievskii, The restricted problem of three bodies taking account of light pressure, *Astron. Zh.* **27** (1950), 250 — 256.

[12] O. Ragos, E.A. Perdios, V.S. Kalantonis and M.N. Vrahatis, On the equilibrium points of the relativistic restricted three-body problem, *Nonlinear Analysis* **47** (2001), 3413 – 3418.

[13] O. Ragos and C. Zagouras, The zero velocity surfaces in the photogravitational restricted three body problem, *Earth, Moon and Planets* **41** (1988), 257 – 278.

[14] O. Ragos and C. Zagouras, Periodic solutions about the out of plane equilibrium points in the photogravitational restricted three body problem, *Celes. Mech.* **44** (1988), 135 – 154.

[15] S.K. Sahoo and B. Ishwar, Stability of collinear equilibrium points in the generalized photogravitational elliptic restricted three-body problem, *Bulletin of Astronomical Society of India* **28** (2000), 579 – 586.

[16] Shankaran, J.P. Sharma and B. Ishwar, Equilibrium points in the generalised photogravitational non-planar restricted three body problem, *International Journal of Engineering, Science and Technology* **3** (2) (2011), 63 – 67.

[17] R.K. Sharma and P.V. Subba Rao, Stationary solutions and their characteristic exponents in the restricted three body problem when the more massive body is an oblate spheroid, *Celestial Mechanics* **13** (1976), 137 – 149.

[18] J.F.L. Simmons, A.J.C. Mcdonald and J.C. Brown, The restricted 3-body problem with radiation pressure, *Celestial Mechanics* **35** (1985), 145 – 187.

[19] J. Singh and T.O. Amuda, Out-of-plane equilibrium points in the photogravitational CRTBP with oblateness and P-R drag, J. Astrophys. Astr. 36 (2015), 291 – 305.

[20] J. Singh and A. Umar, Motion in the photogravitational elliptic restricted three body problem under an oblate primary, *Astronomical Journal* **143** (2012), 109 – 130.

[21] J. Singh and A. Umar, On the stability of triangular equilibrium points in the elliptic R3BP under radiating and oblate primaries, *Astrophysics and Space Sciences* **341** (2012), 349 – 358.

[22] J. Singh and A. Umar, On 'out of plane' equilibrium points in the elliptic restricted three body problem with radiating and oblate primaries, *Astrophysics and Space Sciences* **344** (2013), 13 – 19.

[23] J. Singh and A. Umar, Collinear equilibrium points in the elliptic R3BP with oblateness and radiation, *Advances in Space Research* **52** (2013), 1489 – 1496.

[24] J. Singh and A.E. Vincent, Out-of-plane equilibrium points in the photogravitational restricted four-body problem, *Astrophys Space Sci.* **359** (2015), 38.

[25] J. Singh and A.E. Vincent, Out-of-plane equilibrium points in the photogravitational restricted four-body problem with oblateness, *British Journal of Mathematics & Computer Science* **19** (5) (2016), 1 – 15.

[26] T. Usha, A. Narayan and B. Ishwar, Effects of radiation and triaxiality of primaries on triangular equilibrium points in elliptic restricted three body problem, *Astrophysics and Space Sciences* **349** (2014), 151 – 164.

[27] W.H. Van Den Bos, The Orbit of Sirius, ADS 5423, *Journal des Observateurs* **43** (1960), 145 – 151.