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Financial and Economic Aspects of St. Petersburg Paradox

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Abstract. This paper reviews some aspects of the history and the economic application of the St. Petersburg paradox. In the early 1700s Nicolaus and Daniel Bernoulli formulated a problem in the theory of games of chance, now known as the St. Petersburg paradox. The mathematically correct infinity entrance fee for the game is not acceptable for a real person. Examining the paradox we can see, that there are many-many interesting, economic aspects of it. Some catastrophic financial situations could have been avoided using the analysis and application of the St. Petersburg paradox. In this work I show some generalizations and applications of the paradox mainly in the field of finance and economics. The studying of the paradox and the applications of it is very useful in the teaching of probability and statistics in any college of economics. If we see the scientific aspect, we can say that there are many possibilities, unresolved issues for the financial researchers, mathematicians in the field of St. Petersburg game.

1. The St. Petersburg Paradox

First let us see the interpretation of this paradox. Peter and Paul play a game with a regular coin. (The probability of head-tosses and the probability of tail-tosses are the same: 50-50%.) Paul tosses the coin and Peter pays 2 ducats (or dollars, or euros, or what you like) if it shows heads on the first toss, 4 ducats if the first head appears on the second toss, 8 ducats if the first head appears on the third toss, and so on. So Peter pays 2^k ducats if the first head appears on the kth toss. How much should Peter charge Paul as an entrance fee to this game so that the game will be fair? (Fair game means that neither of gamblers wins or loses any money on average.) Surprisingly, the game cannot be made fair, no matter how large the entrance fee is. Paul is always in a winning position, but as Bernoulli wrote: “there ought not be a sane man who would not happily sell his chance for forty ducats”. How can we resolve this conflict? Let us see our data in Table 1, where x_i means the amount of ducats if the first head appears on the ith toss and p_i means the probability of this event.

Key words and phrases. St. Petersburg paradox; Utility; Stock exchange; portfolio.

Table 1. Payoff values and probabilities

x_i	2	4	8	16	32	...	2^k	...
p_i	1/2	$(1/2)^2$	$(1/2)^3$	$(1/2)^4$	$(1/2)^5$	...	$(1/2)^k$	...

For example, let us see the first column. Paul gets 2 ducats if the first tossing is head. The probability of this equals 0.5. On the second column we can see that Paul gets 4 ducats if the first toss is tail and only the second one is head. The probability of this equals 0.5 times 0.5 (because the tosses are independent). Following this, Paul gets 2^k ducats if the first $k-1$ tosses are tails but the kth toss is head. The probability of this equals $(1/2)^{k-1} \cdot (1/2) = (1/2)^k$. And we can continue it to infinite. This is right, because the sum of the probabilities equals 1: $\sum_{k=1}^{\infty} (1/2)^k = 1$. Now we can calculate the expected value of the payoffs, which will be infinite.

$$\begin{aligned} E(x) &= \sum_{i=1}^{\infty} x_i p_i \\ &= 2 \cdot 1/2 + 4 \cdot (1/2)^2 + 8 \cdot (1/2)^3 + \ldots + 2^k \cdot (1/2)^k + \ldots \\ &= \sum_{i=1}^{\infty} 1 = \infty. \end{aligned} \tag{1}$$

So this means that Paul needs to pay an infinite value to Peter as an entrance fee. However, this is a requirement to which almost no rational person would agree to or be able to satisfy. We can see this, because if $x \geq 2$ then the probability of winning at least x value equals the following: $P(X \leq x) = \sum_{k:2^k \leq x} (1/2)^k = \sum_{k=1}^{[\log_2 x]} (1/2)^k = 1 - (1/2)^{[\log_2 x]}$ where $[a]$ means the integer part of a: $[a] = \max\{b \epsilon Z, b \leq a\}$. Using this, we can give the distribution function:

$$F(x) = P(X \leq x) \begin{cases} 0, & \text{if } x < 2, \\ 1 - 2^{-[\log_2 x]}, & \text{if } x \geq 2. \end{cases} \tag{2}$$

So, $P(X > x) = 2^{-[\log_2 x]}$ as for example the probability of winning greater than 40 ducats equals $P(X > 40) = 1/32 \approx 0.03125$, or the probability of winning a "much bigger" value, for example the probability of winning a value which is greater than $32,000$ ducats equals about 0.00006. So Paul does not want to risk a big value (not even 40 ducats!) to enter the game. Although the calculation of Paul's expectation is mathematically correct, the paradoxical conclusion was regarded by many early researchers of probability as unacceptable. It is worth to mention Keynes's words [13]: "We are unwilling to be Paul, partly because we do not believe Peter will pay us if we have good fortune in the tossing, partly because we do not know what we should do with so much money... if we won it, partly because we do not believe we should ever win it, and partly because we do not think would be a rational act to risk an infinite sum or even a very large sum for an infinitely larger one, whose attainment is infinitely unlikely".

2. **The history of the paradox and some solutions of it**

In the early 1700s Nicolaus Bernoulli formulated the above-mentioned problem, now known as the St. Petersburg paradox. In this early period, the theory of games of chance concentrated heavily on notions of fairness, equity, and examples of fair games, and so Bernoulli was surprised and concerned by his example of a game in which the price of entry appeared to be grossly unfair. On September 9, 1713, Bernoulli communicated his problem by letter to Pierre Reymond de Montmort, and this was followed by a series of letters between them. Montmort could not solve (perhaps did not understand either) the problem, but later published his correspondence with Bernoulli in the second edition of his book on games of chance [15]. There, the world could read about Bernoulli's paradox for the first time, and since then the problem has fascinated some of the world's greatest intellects, for it was clear that standard mathematical approaches to this problem were not in harmony with common sense reasoning. Gabriel Cramer (the originator of Cramer's rule for solving systems of linear equations) in a letter to Bernoulli written on May 21, 1728, rephrased the problem from one involving dice to coins. (So this composition of the paradox belongs to his name.) His main idea was that the feeling of satisfaction does not increase linear-proportionately with the increase of the amount of money. (Or: the larger the size of a person's fortune, the smaller the "moral value" of a given increment in that fortune.) Cramer proposed alternatives under which the value of a sum of money is measured through various utility functions, such as the square root, or the inverse of the amount, or by placing a limit on payout, all of these devices being chosen so as to lead to a finite expectation. So he estimated the entrance fee between 6 and 25 ducats. Nicolaus was not satisfied with this result so he communicated the problem to his cousin Daniel.

In 1738, Daniel Bernoulli presented to the Imperial Academy of Sciences in St. Petersburg an article that announced the paradox to the world. (Some researchers mistakenly believe that the name of the paradox dates back to this.) In this historically memorable article Daniel also proposed a solution to the paradox and expanding Cramer's thoughts put up the terminology of utility. Economics and Psychology use and develop this notion permanently nowadays, too. In that paper, Daniel Bernoulli concluded that the natural choice of utility function should be the logarithm function [1]. In his opinion if x ducat increases with dx, then the increase of the utility (feeling of satisfaction) equals only: $du = b \cdot dx/x$ where $b > 0$ constant. (If you have got more and more money, a little increase of it will cause less and less feeling of satisfaction.) Thus, if the gambler has got α starting ducat, than the moral benefit of winning x amount equals: $u(x) = b \cdot \ln([\alpha + x]/\alpha)$.

So the expected value is not infinite, but $E(u(X)) = b \cdot E(\ln([\alpha + x]/\alpha))$. So for example if we have got 0 starting fund ($\alpha = 0$), than the entrance fee of the game equals 4 ducats, but if the starting fund is $1,000$ ducats than the fee of the game equals 11 ducats only.

Euler gave a similar solution, but he did not want to interfere with the war of the Bernoulli family, so he did not publish it. (Daniel's father was his teacher and Daniel got his workplace in St. Petersburg.) In 1754, d'Alembert started to deal with this problem and published on it many times. He was not on friendly terms with Daniel Bernoulli, so he did not mention any Bernoulli names in his works. When he wrote about the paradox, first he named it at full length as "probleme propose dans le Tome V des Memoires de l'academie de Petersbourg" and later he omitted one word after the other, so finally the "probleme de Petersbourg" expression had formulated. So it's very likely that the name of the paradox dates back to this. In a letter to Lagrange he admitted his failure in solving the paradox. There were some other mathematicians who tried to find a good solution, but the main problem was the following. They tried to use the mathematical laws of probability to the study of individual events. They tried to guess what the result of the next, concrete trial was. The only researcher who approached the problem in the correct way was Condorcet. In his opinion, Paul has to play the game infinite times to get infinite wins. The problem is not apprehensive, but it is the limes of similar, apprehensive problems. If we play the game n times, than the amount of wins depends on n. Buffon took the next step, he had a child play the St. Petersburg game 2048 times. The sum of Paul's winning was $20,104$ ducats, so $20104/2048 = 9.81$ so about 10 ducats per game. The results of this simulation are in the Table 2.

Table 2. Buffon's results

The first head occurs on the kth tossing (k)	1	2	3	4	5	6	7	8	9
Frequency	1061	494	232	137	56	29	25	8	6
Payoff (2^k)	2	4	8	16	32	64	128	256	512

So, based on these empirical frequencies, Buffon concluded that despite the theoretically expected infinite expectation the St. Petersburg game in practice becomes fair with an entrance fee of approximately 10 ducats. He thought and calculated that Paul has to pay $n \log_2 n$ ducats for n games, so $\log_2 n$ ducats are spent per game. Seeing the simulation, Paul has to pay $\log_2 2048 = 11$ ducats per game, that is almost equal to 10, the error is only $\frac{1}{11}$. But if Paul would like to play the game 1 048 575 times, he would have to pay 20 ducats per game. As these games would take almost 30 years, we do not need to study this situation. Therefore, about 10 ducats per game is an acceptable amount.

Whitworth [21] assumed that prudent gamblers would place at risk a fixed percentage, rather than a fixed amount, of their funds, and he developed a procedure for analyzing ventures that involve risk of ruin. The Bernoulli-type investigations continue in two ways. In the first one Fechner (the founder of the experimental psychology) set up the Weber-Fechner empirical law. The other way gave new results in the field of economics, Menger's study led Neumann to the formation of the axiom-system of utility. The next mathematical result belongs to Feller, who proposed a different method to determine entrance fees which would make the St. Petersburg game fair. Suppose Paul chooses to play the game repeatedly. After n games have been played, let R_n denote the total entrance fees and let S_n denote Paul's accumulated receipts. Let us call the game asymptotically fair if the ratio $\frac{S_n}{R_n}$ converges to 1 in probability as n tends to infinity. Feller proved that the St. Petersburg game becomes asymptotically fair if $R_n = n \cdot \log_2 n$ [10] and [11], so if one only allows a finite number n of trials, then for any fixed $\varepsilon > 0$, $P\{|\frac{S_n}{n \log_2 n} - 1| > \varepsilon\} \to 0$ as $n \to \infty$, where S_n denotes the accumulated winnings in n trials of the St. Petersburg game. Thereby suggesting that the fair price for n games is $n \log_2 n$ for large n. Following on the idea of varying entrance fees as initiated by Feller, a deterministic sequence of entrance fees for the St. Petersburg game was given by Steinhaus [17]. To construct this sequence, place twos in alternating empty places, 2_2_2_2_2_2_2_2_2_2_2_2_2_2_2_... then fill every second empty place by a four, 242_242_242_242_242_242_242_24... next fill every second remaining empty space by an eight, and so on 242824216242824232242824216242824.... Denote by $a_1, a_2, \ldots$ the members of this sequence, and let a_n be the entrance fee at the nth repetition of a St. Petersburg game. Steinhaus proved, that the sample distribution function of $a_1, a_2, \ldots$ converges to the distribution function of our variable $F(x)$. More recently, Cs rg and Simons provided an extensive discussion of Steinhaus's sequence of entrance fees, and there now exists an extensive literature on asymptotic theory for St. Petersburg games [6]. It is explained in [4] why $n \log_2 n$ will not satisfy the banker, and generally no satisfactory solution can be based on laws of large numbers. S_n has no asymptotic distribution for any centered and normalized sequences as $n \to \infty$ over the entire sequence N of natural numbers. We can remark that in n games $\log_2 n$ fee per game is too little, if Paul passes up his biggest win, but it is too much if he passes up the biggest two wins. The remarkable article by them reveals new paradoxes within the structure of the classical St. Petersburg paradox. Their results show that (paradoxically) there may be a very different outcome if n distinct Pauls play one St. Petersburg game each than if one Paul plays n games [5]. Many points were raised by researchers on the problem that "the St. Petersburg paradox enjoys an honored corner in the memory bank of the cultured analytic mind". [16] See more historical notes for example in [8], notes about the limiting distribution are for example in [19] and [14]. Nowadays there are some simulating results either, for example in [20].

3. **Some economic and financial aspects of the paradox**

Many examples of St. Petersburg games and their generalizations are studied in the statistics, economics, and mathematics literature and in many fields of life. As I have mentioned, Daniel Bernoulli's work in 1738 was the basis of the formation of the modern utility concept that the economists and psychologists and other researchers use even today [2]. But for example an interesting application of a modified St. Petersburg game is a popular television game show, Who Wants to be a Millionaire? I think that everybody knows it. Moreover, we can mention the so-called "matringale strategy", that many gamblers like to use. In this game we play against the bank with a 50% chance to win. If we lose in the first game, we double the bet. If we lose in the second game again, we also double the previous bet, and so on, until we win. If our first bet equals A dollar and we lose in the first $n-1$ game but we win in the nth game, then our winning amount equals $A\cdot 2^n$, but we had to pay out $A(1+2+4+...+2^{n-1}) = A(2^n-1)$ dollars. So our benefit is A dollar. It seems to be a good strategy, but there are some problems. Usually we have not got almost infinite money (perhaps we win in "infinite"), and if we have a lot of money then to win A dollars, does not matter. Last but not least, of course the casinos know about this strategy so they limit the amount of bets.

Let us see some special St. Petersburg games. If there are two players ("two Pauls") we can see, that the averaging strategy is better for both Pauls than the individualistic strategy. It means that $\frac{1}{2}X_1 + \frac{1}{2}X_2$ are stochastically larger than the individual winnings X_1 and X_2. How much better is the averaging strategy? The averaging strategy provides an extra ducat of added value for each of both Pauls in comparison with their individualistic strategy. Is this (averaging) strategy good for three players either? Surprisingly the answer is not, or not in this way. Seeing the previous averaging strategy we can say that the results are incomparable. But we (or the three Pauls) can use other pooling strategies. In the first one each Paul gives all of his winnings to the other two Pauls, half to each. Under this strategy $Paul_1$ ends up with $\frac{1}{2}X_2 + \frac{1}{2}X_3$, $Paul_2$ with $\frac{1}{2}X_1 + \frac{1}{2}X_3$ and $Paul_3$ with $\frac{1}{2}X_1 + \frac{1}{2}X_2$. This strategy provides one ducat of added value for each of the three Pauls. Or, there is another strategy. In this each Paul shares one-half of his winnings evenly with the other two Pauls. Under this strategy, $Paul_1$ ends up with $\frac{1}{2}X_1 + \frac{1}{4}X_2 + \frac{1}{4}X_3$, $Paul_2$ with $\frac{1}{2}X_2 + \frac{1}{4}X_1 + \frac{1}{4}X_3$ and $Paul_3$ with $\frac{1}{2}X_3 + \frac{1}{4}X_1 + \frac{1}{4}X_2$. This strategy provides 1.5 ducats of added value for each of the three Pauls. What about if we have got more than 3 players? Are there any good strategies for each Pauls? Let n players (Pauls) be, and their individual winnings are $X_1, X_2, \ldots, X_n$. The focus of attention here is on a pooling strategy $\mathbf{p}_n = (p_{1,n}, \ldots, p_{n,n})$ consisting of non-negative components that sum to unity, to which all players agree before any of them plays. A strategy

$\mathbf{p}_n$ is called admissible if each of its components is either zero or an integer power of 2. Cs rg and Simons proved in [7] that if $\mathbf{p}_n$ is admissible, then the added value equals the entropy of $\mathbf{p}_n$: $H(\mathbf{p}_n) = -\{p_{1,n}\log_2 p_{1,n} + \ldots + p_{n,n}\log_2 p_{n,n}\}$. Moreover, the entropy is bounded above by $H_n = [\log_2 n] + 2^{\langle \log_2 n\rangle} - 1$. (Here [.] means the integer part and $\langle . \rangle$ means the fractional part of an expression.) The first ten values of H_n, $n \geq 2$, with their maximizing strategies are in Table 3. The evolving pattern as n grows is that every time n increases by one, a single component $\frac{1}{2^j}$ with smaller exponent j is replaced by two components, each equals to $\frac{1}{2^{j+1}}$. Significantly, the added value represents, simultaneously for all Pauls, a genuine anticipated benefit, arising solely from their agreement to use pooling strategy $\mathbf{p}_n$, in no way is this "magic". Paradoxically, all Pauls and Peter know that Pauls' total winnings are $S_n = X_1 + X_2 + \ldots + X_n$, the same amount with or without the pooling strategy. So we can say in economic terms: through cooperation, the microeconomic perspective is sweetened for all Pauls while the macroeconomic perspective is unaltered.

Table 3. Some maximizing strategies

$H_2 = 1$	$p_2^* = (\frac{1}{2}, \frac{1}{2})$
$H_3 = 1\frac{1}{2}$	$p_3^* = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$
$H_4 = 2$	$p_4^* = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
$H_5 = 2\frac{1}{4}$	$p_5^* = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})$
$H_6 = 2\frac{2}{4}$	$p_6^* = (\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$
$H_7 = 2\frac{3}{4}$	$p_7^* = (\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$
$H_8 = 3$	$p_8^* = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$
$H_9 = 3\frac{1}{8}$	$p_9^* = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16})$
$H_{10} = 3\frac{2}{8}$	$p_{10}^* = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16})$
$H_{11} = 3\frac{3}{8}$	$p_{11}^* = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16})$

The study of another version of the paradox leads us the so called constantly rebalanced portfolio. Of course the best strategy depends on the optimality criteria. Breiman [3] introduced the so called *log-optimal portfolio* as a good optimality criterion. In each round we maximize the expectation $E\ln\langle \mathbf{b}, \mathbf{X}_n\rangle$. Where $\mathbf{b}$ is the portfolio vector, the jth component of it denotes the proportion of the investor's capital invested in financial instrument j. The market vector in the nth round is $\mathbf{X}_n$, and $\langle . \rangle$ means the inner product. Gy rfi and Kevei examined the optimality in [12]. Now let us suppose that a player starts with initial capital $S_0 = 1$ and there is a sequence of simple St. Petersburg games, where for each simple game the player reinvest his capital. The name of this problem is sequential St. Petersburg Game. If $S_{n-1}^{(c)}$ is the capital after the $(n-1)$-th simple game then the invested capital is $S_{n-1}^{(c)}(1-c)$, while $S_{n-1}^{(c)}c$ is the proportional cost of the simple game with commission factor $0 < c < 1$. So the capital after the nth round equals $S_n^{(c)} = S_{n-1}^{(c)}(1-c)X_n = S_0(1-c)^n \prod_{i=1}^{n} X_i = (1-c)^n \prod_{i=1}^{n} X_i$. Because of this (multiplicative definition) $S_n^{(c)}$ has exponential trend $S_n^c = 2^{nW_n^{(c)}} \approx 2^{nW^{(c)}}$, with average growth rate $W_n^{(c)} := \frac{1}{n}\log_2 S_n^{(c)}$ and with asymptotic average growth rate $W^{(c)} := \lim_{n\to\infty} \frac{1}{n}\log_2 S_n^{(c)}$. Using the strong law of large numbers $W^{(c)} = \log_2(1-c) + E\{\log_2 X_1\}$. The commission factor c is called fair if $W^{(c)} = 0$ (the growth rate of the sequential game equals 0), where from we can calculate the value of c. $\log_2(1-c) = -E\{\log_2 X_1\} = -\sum_{k=1}^{\infty} k \cdot 2^{-k} = -2$, so $c = \frac{3}{4}$. They studied the portfolio game with two, three and more St. Petersburg components either [12].

If instead of tossing coins, Paul organizes a corporation in a growth industry and offers Peter stock, the latter might be deterred from paying the full discounted value by any of the considerations that would deter him from paying the full mathematical expectation to enter the Petersburg game. After Sz kely and Richards [18] if we consider the financial aspect of the paradox, our conclusion can be that the run-up in stock prices and the subsequent declines in 2000 could have been avoided by an analysis and application of the paradox. Let us consider a modified St Petersburg game in which Peter is a growth company and Paul is a prospective purchaser of Peter's stock. We assume that the probability of tossing head is $\frac{1}{(1+i)}$, $i > 0$, and the probability of tossing tail is $\frac{i}{(1+i)}$. Next suppose that the payoffs are a series of increasing payments in which Peter pays Paul D ducats if the first toss is head, $D(1+g)$ ducats if the second toss is head, $D(1+g)^2$ if the third toss is head, and so on, and this continues until the toss result is head, at which point the game ends. If k tosses are needed for the game to end then the total payment to Paul is $\sum_{j=0}^{k-2} D(1+g)^j = \frac{D[(1+g)^{k-1}-1]}{g}$. This payment occurs with probability $\frac{i}{(1+i)^k}$. As Durand [9] observed, Paul's expected payoff is given by the following double summation $\sum_{k=1}^{\infty} \frac{i}{(i+1)^k} \sum_{j=0}^{k-2} D(1+g)^j$. This is evaluated by substituting for the inner sum the closed form expression, so we find that Paul's expected payoff is the following

$$\sum_{k=1}^{\infty} \frac{D(1+g)^{k-1}}{(1+i)^k} = \begin{cases} \frac{D}{i-g}, & \text{if } g < i, \\ \infty, & \text{if } g \geq i. \end{cases} \tag{3}$$

In the context of appraising the values of financial securities, the parameter i represents a compound interest rate, equivalently, the present value of a loan of one dollar to be repaid one year in the future is $\frac{1}{(1+i)}$. In the appraisal of fair value for a company's shares, g represents the growth rate of the company as measured by the compound increase in revenue per share. Denote by E_n Peter's earnings (or profits) per share in year n, and let B_n denote Peter's book value (or net asset value) per share in the same year. Let D_n be the total of Peter's paid-out dividends per share in year n. It is clear that changes in book value from year-to-year are equal to the difference between earnings and dividends paid. So, for all $n \geq 1$, $B_{n+1} - B_n = E_n - D_n$. In estimating fair value for Peter's stock, it is common practice for Paul to assume that the ratios $r = \frac{E_n}{B_n}$ and $p = \frac{D_n}{E_n}$ are independent of n. This assumption implies that the change in book value from year n to year $n+1$ is a constant multiple of E_n. So $B_{n+1} - B_n = E_n - D_n = (1-p)E_n = (1-p)rB_n$. Therefore, Peter's dividends, book value, and earnings all are growing at a constant rate, $g = (1-p)r$. In this context the sum represents a perpetual series of dividend payments, starting at D ducats, growing at a constant rate g, and discounted at rate i in perpetuity. So, if $i > g$, then the sum converges to $\frac{D_1}{(i-g)} = \frac{pE_1}{(i-g)}$, which represents an estimate of fair value for one share of Peter's stock. If $i \leq g$, then the formula diverges, and we now have a form of the St. Petersburg paradox in which the practice of discounting future dividends at a uniform rate in perpetuity leads to a paradoxical result. The St. Petersburg paradox explains some of the unprecedented increases in the prices of high-tech growth stocks in the late 1990s. During that period the Federal Reserve System's discount rate was near a historical low, so i was very small. Moreover, purchasers of growth stocks assumed that g, the growth rate of a typical high-tech company, would remain high in perpetuity. The outcome was that $i \ll g$, indeed, even more extreme was that for many high-tech companies, $\frac{i}{g} \approx 0$. Having applied our formula to obtain exorbitant estimated valuations for many high-tech growth stocks, stock purchasers bought avidly, thereby forcing prices to extreme levels. By late 2000, stock prices underwent the "prolonged contractions" with subsequent unprecedented losses to corporate and individual stock buyers. Three years later, many formerly avidly sought-after high-tech companies and mutual funds were defunct. See more applications in insurance for example in [9].

I close this paper with a comment from Mark Twain. He wrote in Pudd'nhead Wilson's Calendar: "October. This is one of the peculiarly dangerous months to speculate in stocks in. The others are July, January, September, April, November, May, March, June, December, August and February".

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