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Research Article

The Weighted Exponentiated Inverted Weibull Distribution: Properties and Application

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Abstract. The weighted distributions are widely utilized in numerous real life fields such as medicine, ecology, reliability, etc., for the assibilation of proper statistical model. This paper innovates and studies a new three parameter *Weighted Exponentiated Inverted Weibull Distribution* (WEIWD). The mathematical properties of thesuggested distribution including the cumulative distribution function, the moment generating function and the survival function are studied. The *maximum likelihood estimation* (MLE) method is implemented to estimate the proposed distribution parameters. Various candidate distributions are fitted on a data set of distance between crakes in a pipe. The result indicates that the WEIWD is the best fitted model for modeling of the real data set among the compared models.

Keywords. Weighted distribution; Exponentiated inverted Weibull distribution; Hazard function; Moment generating function

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1. Introduction

The theory of weighted distributions provides an integrative conceptualization for model stipulation and data representation problems. It also provides a unifying approach for correction of biases that exist in unequally weighted sample data. Also, the theory provides a means of fitting models to the unknown weighting function when samples can be taken both from the original distribution and the resulting 'biased' distribution. These problems exist in all disciplines of science and numerous ad hoc solutions have been developed. The concept of weighted distributions was given by Fisher [19] and Rao [40]. Rao [40] identified various situations that can be modeled by weighted distributions. These situations refer to instances where the recoded observations cannot be considered as a random sample from the original distributions. This may occur due to non observability of some events or damage caused to the original observation resulting in a reduced value, or adoption of a sampling procedure which gives unequal chances to the units in the original.

Weighted distributions are utilized to modulate the probabilities of the events as observed and transcribed. Patil and Rao [36] presented some useful concepts. The weighted version of the bivariate logarithmic series distribution was presented by Gupta and Tripathi [22] and a size biased sampling and related invariant weighted distributions are studied by Patil and Ord [37]. Castillo and Perez-Casany [14] introduced new exponential families that come from the concept of weighted distribution, that include and generalize the Poisson distribution.The applications of weighted distributions are also given by [17, 34, 35]. For more important results of weighted distribution you can see also [20, 25], who introduced the weighted inverse Weibull distribution and beta-inverse Weibull distribution.

For more results and applications of weighted distribution you can see also Al-Kadim Hussein [6–8, 15, 16, 18, 23, 30, 38, 42, 44], who proposed the length-biased Weighted Generalized Rayleigh distribution with its properties. Also they presented the length-biased form of the weighted Weibull distribution and its properties in detail (see Das and Roy, [15]).

Asgharzadeh [9] introduced the new weighted Lindley distribution with application. More work on weighted distributions and their applications in various fields includes [1–5, 10–13, 21, 24, 26–29, 31–33, 39, 41, 43, 45–47]. Weighted distributions have seen much use as a tool in the selection of appropriate models. Considering the importance of weighted distribution, we present a weighted Exponentiated Inverted Weibull distribution and the sub models which are the special cases of our proposed distribution. Various useful mathematical properties of the WEIWD are derived in the next sections. We also present the application of the proposed distribution on real life data set.

The rest of the article is organized as follows. In Section 2, we provide the definition and derivation of PDF and CDF of WEIWD. Some special sub-models are considered in Section 3 and different statistical properties of our proposed model are discussed in Section 4. Estimation of the unknown parameters is carried out in Section 5. The real data set has been analyzed in Section 6 and finally we conclude the article in Section 7.

2. Definition, Derivation of Weighted Exponentiated Inverted Weibull Distribution

In this section, first we provide the definition of the weighted distribution and then drive the weighted Exponentiated inverted Weibull distribution and also its cumulative distribution function. We also drive the shape of PDF and CDF of the WEIWD at various choices of parametric values.

Definition 1. Suppose X is a non negative random variable with its natural density function PDF $f(x)$. Let the weight function be $w(x)$ which is a non negative function then the random variable X having probability density function:

$$f_{w(x)} = \frac{w(x)f(x)}{E(w(x))}, \quad a < x < b \tag{2.1}$$

and

$$E(w(x)) = \int_0^\infty w(X)f(X)dX,$$

where $w(x)$ is a weight function and choice of this function determines the class of weighted distribution.

The probability density function and cumulative density function of the two parameters Exponentiated Inverted Weibull distribution are

$$f(\beta,\theta;x) = \theta\beta x^{-(\beta+1)}\{exp(-x^{-\beta})\}^{\theta}; \quad x > 0,\ \beta > 0,\ \theta > 0, \tag{2.2}$$

$$F(\beta,\theta;x) = \{exp(-x^{-\beta})\}^{\theta}. \tag{2.3}$$

In the past few years, several weighted statistical distributions have been proposed to model lifetime data. One of such distributions is the three-parameter *Weighted Exponentiated Inverted Weibull Distribution* (WEIWD) catheterized here. As many authors have utilized different weight functions e.g. $w(x) = x^c\beta$, $w(x) = x$, $w(x) = x^{2c-N}\exp\left[-x^2(c\sigma^{2-\frac{1}{2\sigma^2}})\right]$, $w(x) = e^n x$ etc for development of their weighted distributions but the question arises why they used these weight functions? If we select the cumulative distribution function of the distribution as a weight function then the value of our proposed distribution and the original distribution will remain almost same so we can employ $F(cx)$ as a weight function.

Remembering the PDF of the EIW distribution is

$$\{exp(-x^{-\beta})\}^{\theta}. \tag{2.4}$$

And the weight function used is as follows

$$w(x) = e^{-\theta(cx)^{-\beta}}. \tag{2.5}$$

Substitute (2.4) and (2.5) in (2.1), we get

$$g_w(x) = \frac{e^{-\theta(cx)^{-\beta}}\beta\theta x^{-(\beta+1)}\{exp(-x^{-\beta})\}^{\theta}}{\int_0^\infty e^{-\theta(cx)^{-\beta}}\beta\theta x^{-(\beta+1)}exp(-x^{-\beta})^{\theta}dx} = \frac{\beta\theta x^{-(\beta+1)}e^{-\theta x^{-\beta}(1+c^{-\beta})}}{(1+c^{-\beta})^{-1}},$$

$$g_w(x;\beta,\theta,c) = (1+c^{-\beta})\beta\theta x^{-(\beta+1)}e^{-\theta x^{-\beta}(1+c^{-\beta})} \tag{2.6}$$

and the corresponding *cumulative distribution function* (CDF) of the WEIW distribution denoted as $G_w(x;\beta,\theta,c)$.

Generally, the distribution function is expressed as:

$$G(x) = \int_0^x g(t)dt. \tag{2.7}$$

Substituting (2.6) into (2.7), we obtain

$$= \int_0^x \frac{\beta\theta}{(1+c^{-\beta})^{-1}} t^{-(\beta+1)} e^{-\theta t^{-\beta}(1+c^{-\beta})} dt.$$

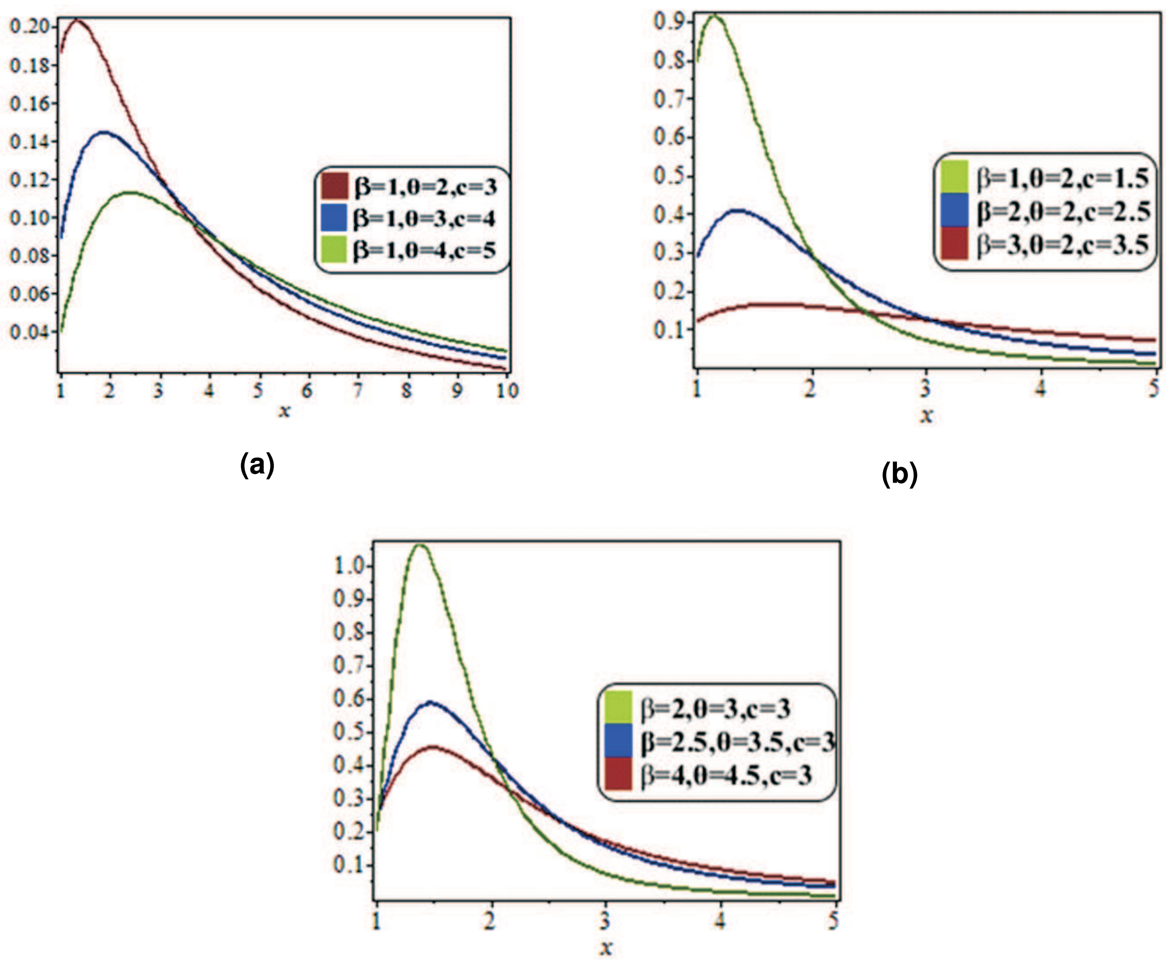
By setting $\theta t^{-\beta}(1+c^{-\beta}) = y$ and $\frac{\theta(1+c^{-\beta})}{x^\beta} < x < \infty$, we get finally

$$= \int_{\frac{\theta(1+c^{-\beta})}{x^\beta}}^{\infty} y^0 e^{-y} dy$$

$$= \int_{\frac{\theta(1+c^{-\beta})}{x^\beta}}^{\infty} y^{(1+0)-1} e^{-y} dy$$

$$G_w(x;\beta,\theta,c) = \Gamma\left(1, \frac{\theta(1+c^{-\beta})}{x^\beta}\right). \tag{2.8}$$

Figures 1 and 2 illustrate the PDF and CDF of the WEIW distribution respectively, for selected values of the parameters β, θ and c.



(c)

Figure 1. The probability density function of WEIW distribution at various parameters choices

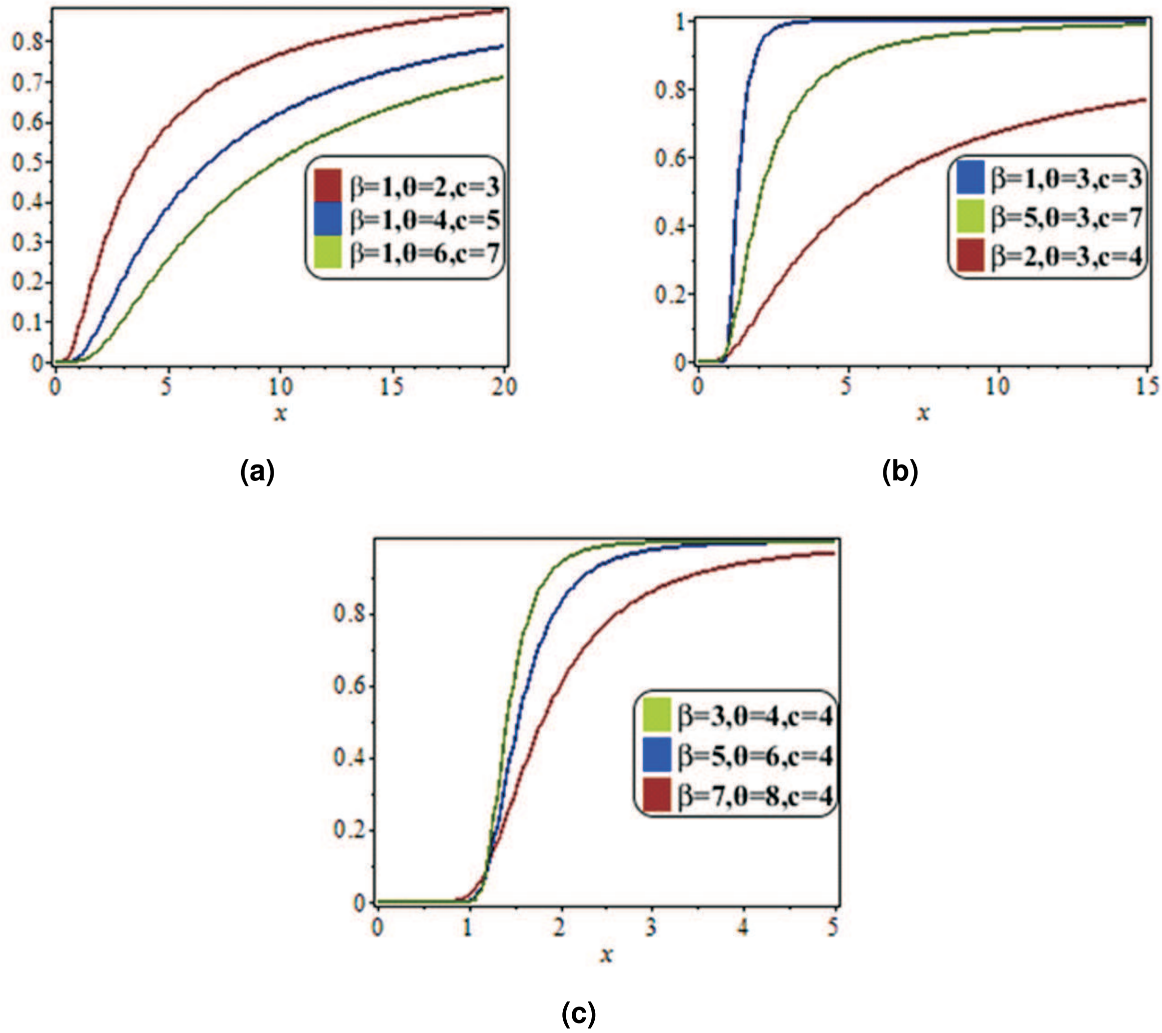


Figure 2. The distribution function of the WEIW distribution for selected values of β, θ and c.

3. Special Sub-Models

In this section, we meditate some special sub-models of the proposed WEIW distribution in the following five corollaries.

Corollary 1. *Putting* $\beta=\beta$, $\theta=-\theta$, $c^{-\beta}=-2$ *and multiplying a constant;* (-1)*; in* (2.6) *we get the Exponentiated Inverted Weibull distribution as*

$$g(\theta,\beta;x)=\theta\beta x^{-(\beta+1)}e^{-\theta x^{-\beta}}, \quad x>0,\ \theta>0,\ \beta>0.$$

Corollary 2. *Where* $\beta=-\beta$, $\theta=\theta$, $c=0$ *and multiplying a constant;* (-1)*; then WEIW distribution reduces to the Weibull density function as*

$$g(\theta,\beta;x)=\theta\beta x^{\beta-1}e^{-\theta x^{\beta}}, \quad x>0,\ \theta>0,\ \beta>0.$$

Corollary 3. *The case when* $\beta=-1$, $\theta=\theta$, $c=0$ *and multiplying a constant;* (-1)*; then* (2.6) *gives the Exponential density function of the form*

$$g(\theta;x)=\theta e^{-\theta x}, \quad x>0,\ \theta>0.$$

Corollary 4. *Putting* $\beta = 1$, $\theta = \frac{1}{\theta}$, $c = 0$ *in* (2.6) *then the resulted distribution is the Inverse Exponential distribution*

$$g(\theta; x) = \theta^{-1} x^{-2} e^{-(\theta x)^{-1}}, \quad x > 0,\ \theta > 0.$$

Corollary 5. *By substituting* $\beta = 2$, $\theta = \frac{1}{\theta^2}$, $c = 0$ *in* (2.6) *then we obtain the density function the Inverse Rayleigh distribution of the form*

$$g(\theta; x) = 2\theta^{-2} x^{-3} e^{-(\theta x)^{-2}}, \quad x > 0,\ \theta > 0.$$

4. Properties

This section derived the statistical properties of the WEIW distribution, specifically moments, mean variance, coefficient of variation, skewness, kurtosis and moment generating function, reliability function, hazard function and the reverse hazard function and mode of function is also discussed here as follow:

4.1 Moments

In this sub section we present the rth moment for the weighted Exponentiated Inverted Weibull distribution. If $X \sim$ WEIW distribution with parameters β, θ and c, then the rth moment of x, say μ'_r is given as

$$E_g(x^r) = \int_0^\infty x^r g(x) dx.$$

From the PDF of the WEIW distribution in (2.6) then $E_g(x^r)$ can b obtained as:

$$E_(g)(x^r) = \int_0^\infty x^r (1 + c^{-\beta}) \beta\theta x^{-(\beta+1)} e^{-\theta x^{-\beta}(1+c^{-\beta})}\, dx$$

$$= (1 + c^{-\beta}) \beta\theta \int_0^\infty x^{r-\beta-1} e^{-\theta x^{-\beta}(1+c^{-\beta})}\, dx.$$

By setting $\theta x^{-\beta}(1 + c^{-\beta}) = y$, after simplification we get

$$= \theta^{\frac{r}{\beta}} (1 + c^{-\beta})^{\frac{r}{\beta}} \Gamma\left(1 - \frac{r}{\beta}\right). \tag{4.1}$$

From the rth moment of the WEIW distribution, putting $r = 1$ in (4.1), and the Expected value of X is obtained as:

$$E_g(X) = \theta^1 \beta (1 + c^{-\beta})^1 \beta\Gamma\left(1 - \frac{1}{\beta}\right).$$

The first four central moments of WEIW distribution are given by

$$\mu_1 = 0,$$

$$\mu_2 = \mu'_2 - \mu'^2_1 = \theta^{\frac{2}{\beta}} (1 + c^{-\beta})^{\frac{2}{\beta}} \left[\Gamma\left(1 - \frac{2}{\beta}\right) - \Gamma^2\left(1 - \frac{1}{\beta}\right)\right],$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1 = \theta^{\frac{3}{\beta}} (1 + c^{-\beta})^{\frac{3}{\beta}} \left[\Gamma\left(1 - \frac{3}{\beta}\right) - 3\Gamma\left(1 - \frac{2}{\beta}\right)\Gamma\left(1 - \frac{1}{\beta}\right) + 2\Gamma^3\left(1 - \frac{1}{\beta}\right)\right],$$

$$\mu_4 = \mu'_4 - 4\mu'_1\mu'_3 + 6\mu'^2_1\mu'_2 - 3\mu'^4_1$$

$$= \theta^{\frac{4}{\beta}}(1+c^{-\beta})^{\frac{4}{\beta}}\left[\Gamma\left(1-\frac{4}{\beta}\right)-4\Gamma\left(1-\frac{1}{\beta}\right)\Gamma\left(1-\frac{3}{\beta}\right)+6\Gamma^2\left(1-\frac{1}{\beta}\right)\Gamma\left(1-\frac{2}{\beta}\right)-3\Gamma^4\left(1-\frac{1}{\beta}\right)\right].$$

Now we can also use (4.1) to obtain the mean, variance, coefficient of variation, coefficient of skewness and coefficient of kurtosis as follows:

$$\mu = E(x) = \theta^1\beta(1+c^{-\beta})^1\beta\Gamma\left(1-\frac{1}{\beta}\right), \tag{4.2}$$

$$\sigma^2 = E(x^2)-\mu^2 = \theta^{\frac{2}{\beta}}(1+c^{-\beta})^{\frac{2}{\beta}}\left[\Gamma\left(1-\frac{2}{\beta}\right)-\Gamma^2\left(1-\frac{1}{\beta}\right)\right], \tag{4.3}$$

$$\text{CV} = \frac{\sigma}{\mu} = \frac{\left[\Gamma\left(1-\frac{2}{\beta}\right)-\Gamma^2\left(1-\frac{1}{\beta}\right)\right]^{\frac{1}{2}}}{\Gamma\left(1-\frac{1}{\beta}\right)}, \tag{4.4}$$

$$\text{CS} = \frac{\mu^3}{\sigma^3} = \frac{\left[\Gamma\left(1-\frac{3}{\beta}\right)-3\Gamma\left(2-\frac{1}{\beta}\right)\Gamma\left(1-\frac{2}{\beta}\right)+2\Gamma^3\left(1-\frac{1}{\beta}\right)\right]}{\left[\Gamma\left(1-\frac{2}{\beta}\right)-\Gamma^2\left(1-\frac{1}{\beta}\right)\right]^{\frac{3}{2}}}, \tag{4.5}$$

$$\text{CK} = \frac{\mu^4}{\sigma^4} = \frac{\left[\Gamma\left(1-\frac{4}{\beta}\right)-4\Gamma\left(1-\frac{3}{\beta}\right)\Gamma\left(1-\frac{1}{\beta}\right)+6\Gamma\left(1-\frac{2}{\beta}\right)\Gamma^2\left(1-\frac{1}{\beta}\right)-3\Gamma^4\left(1-\frac{1}{\beta}\right)\right]}{\left[\Gamma\left(1-\frac{2}{\beta}\right)-\Gamma^2\left(1-\frac{1}{\beta}\right)\right]^2}. \tag{4.6}$$

4.2 Reliability Function

In this sub section, we present the Reliability function of WEIW distribution.

By definition, the survival function of the random variable X is given by:

$$S_w(x;\beta,\theta,c) = 1-G_w(x;\beta,\theta,c). \tag{4.7}$$

Using (2.8) into (4.7), the survival function of the WEIW distribution can be expressed by:

$$S_w(x;\beta,\theta,c) = 1-\Gamma\left(1,\frac{\theta(1+c^{-\beta})}{x^\beta}\right). \tag{4.8}$$

4.3 Hazard Function

This sub section presents the Hazard function of the WEIW distribution as

$$h_w(x;\beta,\theta,c) = \frac{g_w(x;\beta,\theta,c)}{S_w(x;\beta,\theta,c;)}. \tag{4.9}$$

Substituting (2.6) and (4.8) into (4.9), we obtain:

$$h_w(x;\beta,\theta,c) = \frac{(1+c^{-\beta})\beta\theta x^{-(\beta+1)}e^{-\theta x^{-\beta}(1+c^{-\beta})}}{1-\Gamma\left(1,\frac{\theta(1+c^{-\beta})}{x^\beta}\right)}. \tag{4.10}$$

4.4 Reversed Hazard Function

In this sub section, we derived the Reversed Hazard function of the WEIW distribution as:

$$\phi_w(x;\beta,\theta,c) = \frac{g_w(x;\beta,\theta,c)}{G_w(x;\beta,\theta,c)}. \tag{4.11}$$

Substituting (2.6) and (2.8) into (4.11)

$$\phi_w(x;\beta,\theta,c)=\frac{(1+c^{-\beta})\beta\theta x^{-(\beta+1)}e^{-\theta x^{-\beta}(1+c^{-\beta})}}{\Gamma\left(1,\frac{\theta(1+c^{-\beta})}{x^{\beta}}\right)}. \tag{4.12}$$

4.5 Moment Generating Function (m g f)

We start with the well-known definition of the moment generating function and we can provide m g f of WEIW distribution, is obtained by:

$$\begin{aligned}M_x(t)&=E(e^{tx})\\&=\int_0^{\infty}e^{tx}(1+c^{-\beta})\beta\theta x^{-(\beta+1)}e^{-\theta x^{-\beta}(1+c^{-\beta})}\,dx\\&=\beta\theta(1+c^{-\beta})\int_0^{\infty}e^{tx}x^{-(\beta+1)}e^{-\theta x^{-\beta}(1+c^{-\beta})}\,dx.\end{aligned}$$

By using Taylor's series expansion $e^{tx}=\sum\limits_{r=0}^{\infty}\frac{(tx)^r}{r!}$ and setting $\theta x^{-\beta}(1+c^{-\beta})=y$, we finally obtained

$$M_x=\sum_{r=0}^{\infty}\frac{t^r}{r!}\theta^{\frac{r}{\beta}}(1+c^{-\beta})^{\frac{r}{\beta}}\Gamma\left(1-\frac{r}{\beta}\right). \tag{4.13}$$

4.6 The Mode

We take density function of the weighted Exponentiated Inverted Weibull distribution is as follows:

$$g_w(x;\beta,\theta,c)=(1+c^{-\beta})\beta\theta x^{-(\beta+1)}e^{-\theta x^{-\beta}(1+c^{-\beta})}.$$

The limit of the density function is

(1) $\lim\limits_{x\to 0}g_w(x;\beta,\theta,c)=\lim\limits_{x\to 0}(1+c^{-\beta})\beta\theta x^{-(\beta+1)}e^{-\theta x^{-\beta}(1+c^{-\beta})}=0.$

(2) $\lim\limits_{x\to\infty}g_w(x;\beta,\theta,c)=\lim\limits_{x\to\infty}(1+c^{-\beta})\beta\theta x^{-(\beta+1)}e^{-\theta x^{-\beta}(1+c^{-\beta})}=0.$

Now the logarithm of the function $g_w(x;\beta,\theta,c)$ given by

$$\begin{aligned}\ln[g_w(x;\beta,\theta,c)]&=\ln\left[(1+c^{-\beta})\beta\theta x^{-(\beta+1)}e^{-\theta x^{-\beta}(1+c^{-\beta})}\right]\\&=\ln 1+\ln c^{-\beta}+\ln\beta+\ln\theta-\beta\ln x-\log x-\theta x^{-\beta}(1+c^{-\beta}).\end{aligned} \tag{4.14}$$

Differentiating (4.14) with respect to x, we obtain

$$\frac{\partial}{\partial x}[g_w(x;\beta,\theta,c)]=-\frac{\beta}{x}-\frac{1}{x}+\beta\theta x^{-\beta-1}(1+c^{-\beta}), \tag{4.15}$$

$$\frac{\partial^2}{\partial X^2}[g_w(x;\beta,\theta,c)]=-x^{-2}[-\beta-1+\theta\beta^2x^{-\beta}(1+c^{-\beta})+\theta\beta c^{-\beta}x^{-\beta}(1+c^{-\beta})]<0. \tag{4.16}$$

The mode of the WEIW distribution is obtained by solving the non linear equation (4.15) with respect to x, we get

$$x=\left[\frac{\theta\beta(1+c^{-\beta})}{\beta+1}\right]^{\frac{1}{\beta}}.$$

The mode of the proposed distribution is actually the maximum of the distribution function.

Since we have obtained the mode mean, variance, CV, CS and CK of WEIWD; we set the values of the parameters β, θ and c and compute the values of these quantities. Now for different values of parameters, we obtain Mode (x_0) of the WEIW distribution in Table 1 and Table 2 provided the values of the mean, variance, *standard deviation* (STD), *coefficient of variation* (CV), *coefficient of skewness* (CS), *coefficient of kurtosis* (CK) with some values of parameters β, θ and c of WEIW distribution.

Table 1. The Mode values of WEIWD at various parameters choices

β	θ	c	Mode (x_0)
2	1	1	1.1547
3	2	1	1.4422
4	3	1	1.4802
5	4	1	1.4614
6	5	1	1.4306
7	6	1	1.3417
8	7	1	1.3705

Table 2. The mean, variance, coefficient of variation, skewness, kurtosis of the WEIWD at various parameter choices.

β	θ	c	MEAN	VAR	STD	CV	CS	CK
5	2	3	1.3384	0.1768	0.4204	0.3141	3.5391	48.0909
6	2	4	1.2671	0.1006	0.3171	0.2721	2.8307	24.6947
7	2	4	1.2209	0.0647	0.2544	0.2084	2.4726	17.2330
8	2	5	1.1882	0.0454	0.2131	1.1793	2.1615	14.2197
9	2	6	1.1640	0.0334	0.1828	0.1571	1.9885	12.4474
10	2	7	1.1453	0.0089	0.0943	0.1397	1.8893	11.1114
11	2	8	1.1303	0.0065	0.0808	0.1262	1.7767	9.7362
12	2	9	1.1183	0.0049	0.0700	0.1149	1.6117	10.5206
13	2	10	1.1081	0.0137	0.1172	0.1057	1.5300	10.3725
14	2	11	1.0997	0.0115	0.1073	0.0976	1.3530	13.7084
15	2	12	1.0925	0.0098	0.0990	0.0907	1.4944	8.4913

From the above tables, we observe the following:

(1) when we fixed a parameter $c = 1$, the mode of WEIW distribution increases as the value of parameters β and θ increases and at a moment, the mode is becoming decreasing as we increase the value of parameters.

(2) When we fixed a parameter $\theta = 2$, mean, variance, standard deviation, coefficient of variation, skewness and kurtosis decreases as the value of parameters β and c increases.

Next, we have provided some plots of the Reliability function, Hazard function and Reverse Hazard function for selected values of parameters β, θ and c.

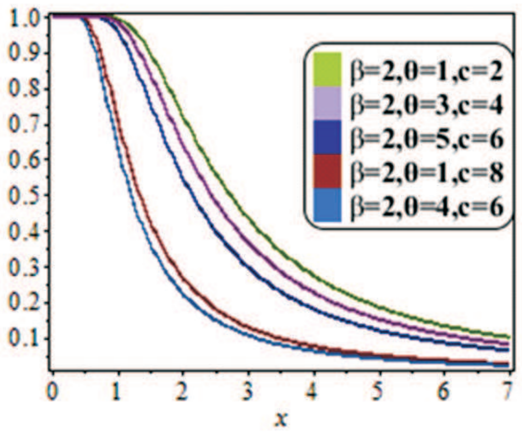


Figure 3. Reliability function of WEIWD

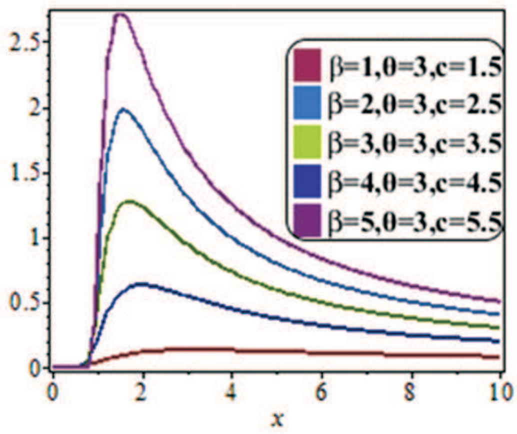


Figure 4. Hazard function of WEIWD

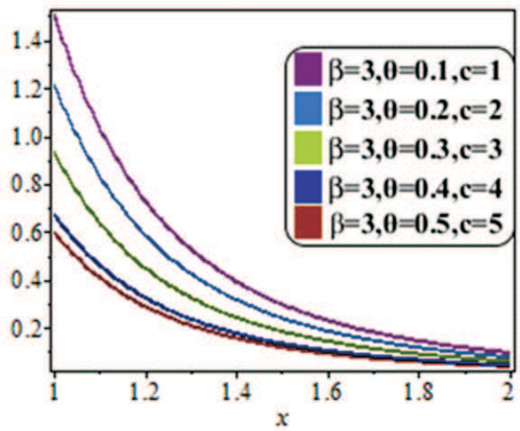


Figure 5. Reverse Hazard function of WEIWD

5. Parameter Estimation

Inthis section, we use maximum likelihood estimation (MLE) method to find the estimation of the unknown parameters of the WEIWD. This is the one of the most important methods for estimations of the parameters of a statistical model. Suppose that sample was drawn from (2.6) and we drive the non-linear equations for finding the maximum likelihood estimators of the parameters. The maximum likelihood estimates $\beta^\wedge$, $\theta^\wedge$ and $c^\wedge$ of β, θ and c are obtained by maximizing the log-likelihood

$$L(x;\beta,\theta,c)=(1+c^{-\beta})^n\theta^n\beta^n\prod_{i=1}^{n}x_i^{-(\beta+1)}e^{-\theta(1+c^{-\beta})\sum_{i=1}^{n}x_i^{-\beta}}. \tag{5.1}$$

The log-likelihood functions of n observations of x reduces to

$$\begin{aligned} l(x;\beta,\theta,c)&=\log L(x;\beta,\theta,c)\\ &=n\log\theta+n\log\beta-\beta\sum_{i=1}^{n}\log x_i-\sum_{i=1}^{n}\log x_i-\theta\sum_{i=1}^{n}x_i^{-\beta}-\theta c^{-\beta}\sum_{i=1}^{n}x_i^{-\beta}+n\log c^{-\beta}. \end{aligned} \tag{5.2}$$

The components corresponding to the model parameters are calculated by differentiating in (5.2) as

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - \left[\sum_{i=1}^{n} \ln(x_i)\right] - \theta\left[\sum_{i=1}^{n}(-x_i^{-\beta}\ln(x_i))\right]$$

$$+\theta c^{-\beta}\left[\sum_{i=1}^{n}(-x_i^{-\beta}\ln(x_i))\right] - \theta c^{-\beta}\left[\sum_{i=1}^{n}(-x_i^{-\beta}\ln(x_i))\right] - n\ln(c), \tag{5.3}$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \left[\sum_{i=1}^{n} x_i^{-\beta}\right] - c^{-\beta}\left[\sum_{i=1}^{n} x_i^{-\beta}\right], \tag{5.4}$$

$$\frac{\partial l}{\partial c} = \theta c^{-\beta-1}\beta\left[\sum_{i=1}^{n} x_i^{-\beta}\right] - \frac{n\beta}{c}. \tag{5.5}$$

The estimates can be obtained by setting the results equal to zero as;

$$\frac{n}{\beta} - \left[\sum_{i=1}^{n} \ln(x_i)\right] - \theta\left[\sum_{i=1}^{n}(-x_i^{-\beta}\ln(x_i))\right] + \theta c^{-\beta}\left[\sum_{i=1}^{n}(-x_i^{-\beta}\ln(x_i))\right]$$

$$-\theta c^{-\beta}\left[\sum_{i=1}^{n}(-x_i^{-\beta}\ln(x_i))\right] - n\ln(c) = 0, \tag{5.6}$$

$$\frac{n}{\theta} - \left[\sum_{i=1}^{n} x_i^{-\beta}\right] - c^{-\beta}\left[\sum_{i=1}^{n} x_i^{-\beta}\right] = 0, \tag{5.7}$$

$$\theta c^{-\beta-1}\beta\left[\sum_{i=1}^{n} x_i^{-\beta}\right] - \frac{n\beta}{c} = 0. \tag{5.8}$$

The non linear equations (5.6)-(5.8) does not seem to be have a closed form solution and must be solved iteratively to obtain the estimate of the parameters or the system of the three equations. We can use the Newton Raphson method for the above equations to obtain the $\beta^\wedge$, $\theta^\wedge$ and $c^\wedge$ the MLE of (β,θ,c), respectively.

6. Application to a real dataset

In this section, we determine the flexibility and potentiality of our proposed weighted Exponentiated Inverted Weibull distribution using the real data set. We provide an application of the WEIW distribution by considering the uncensored data on distance between cracks in a pipe dataset as follow:

30.94, 18.51, 16.92, 51.56, 22.85, 22.38, 19.08, 49.59, 17.12, 10.67, 25.43, 10.24, 27.47, 14.70, 14.10, 29.93, 27.98, 36.02, 19.40, 14.97, 22.57, 12.26, 18.14, 18.84.

The data can be modeled by Weighted Exponentiated Inverted Weibull Distribution and we estimate the unknown parameters β, θ and c by the maximum likelihood method. We apply the Newton Raphson procedure for simultaneously three equations, by taking the initial estimate $\beta^\wedge = 0.9$, $\theta^\wedge = 5$ and $c^\wedge = 0.01000$ then we obtained the estimate of parameters are: $\beta^\wedge = 2.8639$, $\theta^\wedge = 394.0386$ and $c^\wedge = 0.4815$.

This section also presents the comparison analysis of the WEIW, EIW, LBEIW, LBIW and Weibull distributions applying to the real datasets as given above. We apply the Kolmogorov-Smirnov test (KS test) for the goodness of fit purpose. Table 3 provides the MLE estimates of the parameters β, θ and c values of the test statistics which is KS test. The p-value of KS for the WEIW distribution is 0.9850.

The results in Table 3 shown that the WEIW distribution fits the data as well as EIW, LBEIW, LBIW and Weibull distribution. In this case the value of the K-S statistics is smaller for WEIW distribution as compared to those values of the other distributions. In fact, based on the value of the KS-statistic, we observe that the WEIW distribution provides the best fit for this data among all the models considered. So it is evident that the WEIW distribution is the best distribution and is a strong competitor to other distributions commonly used in literature for fitting lifetime data as given below.

Table 3. Goodness of fit summary of distance between cracks in a pipe data set.

Fitting models:	WEIW	EIW	LBEIW	LBIW	Weibull
Parameter Estimates:	$\beta^\wedge = 2.8639$ $\theta^\wedge = 394.0386$	$\beta^\wedge = 2.7347$ $\theta^\wedge = 2384.5601$	$\beta^\wedge = 3.3891$ $\theta^\wedge = 9508.9505$	$\beta^\wedge = 1.3484$ $\theta^\wedge = 26.0230$	$\beta^\wedge = 2.3089$ $c^\wedge = 0.4815$
K.S statistic P-value	0.0865 0.9850	0.0891 0.9822	0.1031 0.9376	0.5129 0.0000	0.1436 0.6532

7. Conclusions

In this paper, we have introduced a new three parameter *weighted Exponentiated Inverted Weibull distribution* (WEIWD). We use cumulative distribution function as a weight function in our proposed distribution and we derive various mathematical properties of the WEIWD. It is a weighted distribution and in fact contains a fairly large class of distributions with potential applications to a wide area of probability and statistics and it has also some special sub-models. We also analyzed the behavior of the PDF, CDF, survival function, hazard function and reversed hazard function by plotting the functions for different values of parameters.

We examine the maximum likelihood estimation of the models parameters. An application of this new distribution to real data set on distance between crakes in a pipe is given to demonstrate that it can be used quite effectively to provide better fits than other available models that might be of use for practitioners in the applied sciences and this show that the fit of the WEIWD is best fit to the data with highest p value. We hope that the WEIWD provides a rather general and flexible framework for statistical analysis and may attract the extensive application in life time data analysis and other fields.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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