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On Integer Additive Set-Valuations of Finite Jaco Graphs

Research Article

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Abstract. Let X denote a set of non-negative integers and $\mathcal{P}(X)$ be its power set. An integer additive set-labeling (IASL) of a graph G is an injective set-valued function $f : V(G) \to \mathcal{P}(X) - \{\emptyset\}$ where induced function $f^+ : E(G) \to \mathcal{P}(X) - \{\emptyset\}$ is defined by $f^+(uv) = f(u) + f(v)$, where $f(u) + f(v)$ is the sumset of $f(u)$ and $f(v)$. Let $f(x) = mx + c$; $m \in \mathbb{N}$, $c \in N_0$. A finite linear Jaco graph, denoted by $J_n(f(x))$, is a directed graph with vertex set $\{v_i : i \in \mathbb{N}\}$ such that (v_i, v_j) is an arc of $J_n(f(x))$ if and only if $f(i) + i - d^-(v_j) \geq j$. In this paper, we discuss the admissibility of different types of integer additive set-labeling by finite linear Jaco graphs.

Keywords. Integer additive set-labeled graphs; Weak integer additive set-labeled graphs; Arithmetic integer additive set-labeled graphs; Dispensing number of a graph; Finite linear Jaco graph

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1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [2, 3, 5, 25]. Unless mentioned otherwise, all graphs considered here are simple, finite, non-trivial and connected.

Motivated by several problems in social interactions, the notion of the *set-valuation* or *set-labeling* of graphs was introduced in [1] as an injective set assignment of G in which the vertices of G are labeled by the subsets of a ground set X according certain rules. A graph with a set-valuation is called a *set-valued graph* or a *set-labeled graph*. Several studies on different types of set-valued graphs have been taken place since then.

1.1 Basics of Integer Additive Set-Labeled Graphs

The *sumset* of two sets A and B of integers, denoted by $A+B$, is defined as $A+B=\{a+b : a\in A, b\in B\}$. If A or B is countably infinite, then their sumset $A+B$ will also be countably infinite. Hence, all sets we consider here are finite sets of non-negative integers.

Definition 1.1 ([4, 10])**.** Let X be a non-empty finite set of non-negative integers and let $\mathcal{P}(X)$ be its power set. An *integer additive set-valuation* or an *integer additive set-labeling* (IASL) of a graph G is an injective function $f : V(G) \to \mathcal{P}(X)-\{\emptyset\}$ such that the induced function $f^+ : E(G) \to \mathcal{P}(X)-\{\emptyset\}$ is defined by $f^+uv = f(u)+f(v)$ $\forall$ $uv\in E(G)$. A graph G which admits an IASL is called an *integer additive set-labeled graph* (IASL-graph).

Definition 1.2 ([4, 10])**.** An *integer additive set-labeling* of a graph G is said to be an *integer additive set-indexer* (IASI) if the induced function f^+ is also a injective. A graph G which admits an IASI is called an *integer additive set-indexed graph* (IASI-graph).

The cardinality of the set-label of an element (vertex or edge) of a graph G is called the *set-labeling number* of that element. An element of a given graph G is said to be a *mono-indexed element* of G if its set-labelling number is 1.

1.2 Jaco Graphs – A Revisit

Definition 1.3 ([6])**.** An infinite linear Jaco graph, denoted by $J_\infty(f(x))$, is a directed graph with vertex set $\{v_i : i\in\mathbb{N}\}$ such that (v_i, v_j) is an arc of $J_\infty(f(x))$ if and only if $f(i)+i-d^-(v_j)\geq j$.

A Jaco graph $J_n(f(x))$ has the following four fundamental properties (see [6]).

(i) $V(J_\infty(f(x)))=\{v_i : i\in\mathbb{N}\}$,

(ii) if v_j is the head of an edge (arc), then the tail is always a vertex v_i, $i<j$,

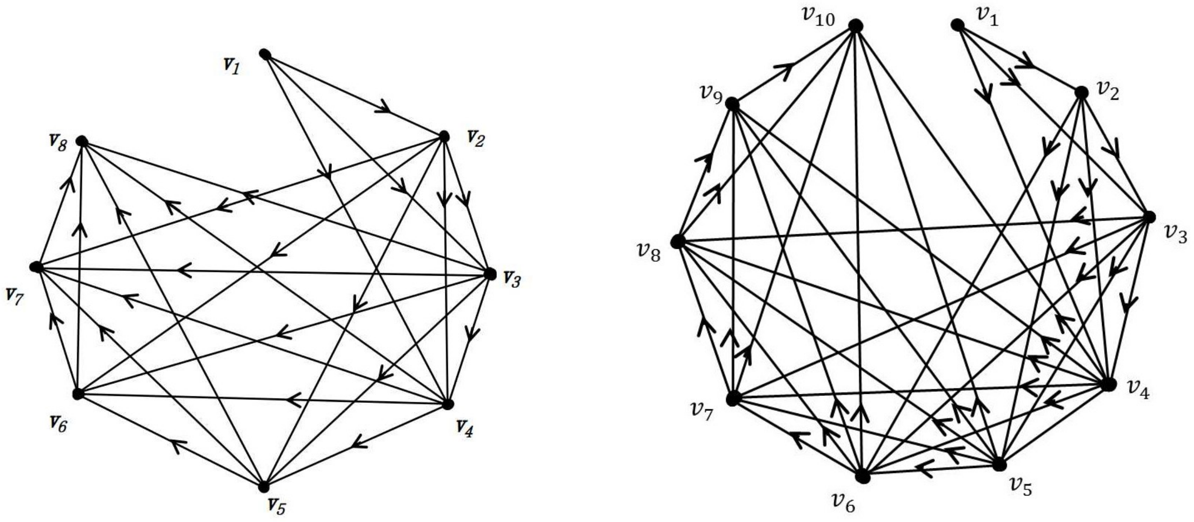
(iii) if v_k, for smallest $k\in\mathbb{N}$ is a tail vertex then, all vertices, v_ℓ, $k<\ell<j$ are tails of arcs to v_j,

(iv) the degree of the k-th vertex v_k is $mk+c$ for $1\leq k\leq n$.

Hence, trivially we have $d(v_i)\leq i$ for $i\in\mathbb{N}$.

Definition 1.4 ([6])**.** A *finite linear Jaco graph*, denoted by $J_n(mx+c) : x, m\in\mathbb{N}$, $c\in\mathbb{N}_0$, is a directed graph whose vertex set is $V=V(J_n(mx+c))=\{v_i : i\in\mathbb{N}, i\leq n\}$ and the edge set is $A(J_n(mx+c))\subseteq V\times V$ such that $(v_i, v_j)\in A(J_n(mx+c))$ if and only if $(f(i)+i)-d^-(v_i)\geq j$.

Some illustrations to certain finite linear Jaco graphs are given in Figure 1(a,b).



(a) A linear Jaco graph $J_8(3x)$ **(b)** A linear Jaco graph $J_{10}(2x+1)$

Figure 1

If the context is clear, the underlying graph of $J_n(mx+c)$ may also be referred to as a finite Jaco graph of order n with the same notation.

Different types of integer additive set-labelings of graphs have been introduced and studied the properties and structural characteristics of various classes which admit these integer additive set-labelings of graphs in [4, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. Motivated by these studies, in this paper, we discuss the admissibility of these types IASLs by the family of finite Jaco graphs.

2. IASLs of Jaco Graphs

Integer additive set-labelings of graphs are classified in to certain types based on the cardinality or the properties of the set-labels of the set-labels of elements of the graphs under consideration. In the first section, we consider the case when the collection of set-labels have some structural properties.

Definition 2.1 ([17])**.** Let X be a non-empty set of non-negative integers. An IASI $f : V(G) \to \mathcal{P}(X) - \{\emptyset\}$ of a graph G is said to be a *topological integer additive set-labeling* of G if the collection of all vertex set-labels of G together with the null set is a topology on the ground set X; that is, if $f(V(G)) \cup \{\emptyset\}$ is a topology on X.

Theorem 2.1 ([17])**.** *A necessary and sufficient condition for a graph G to the admit a topological IASL is that G has at least one pendant vertex.*

Invoking the above theorem, a necessary and sufficient condition for the admissibility of topological IASL by finite Jaco graphs can be established as follows.

Theorem 2.2. *A finite Jaco graph $J_n(mx+c)$ admits a topological IASL if and only if $m=1$ and $c=0$.*

Proof. By Theorem 2.1, a graph G admits a topological IASL if and only if it has at least one end vertex. For a finite Jaco graph $J_n(mx+c)$, we note that if a pendant vertex exists, that vertex is the very first vertex v_1 of $J_n(mx+c)$. Moreover, we can also note that the first few initial vertices, say v_k, have degree $mk+c$ (see the fourth fundamental property of Jaco graphs). Therefore, v_1 is a pendant vertex if and only if $m=1$ and $c=0$. Hence, $J_n(mx+c)$ admits a topological IASL if and only if $m=1$ and $c=0$. □

Another type of IASL of a graph G is the graceful integer additive set-labeling, which is defined as follows.

Definition 2.2 ([22])**.** An integer additive set-indexer $f: V(G) \to \mathcal{P}(X)-\{\emptyset\}$ is said to be an *integer additive set-graceful labeling* (IASGL) or a *graceful integer additive set-labeling* of G if $f^+(E(G)) = \mathcal{P}(X)-\{\emptyset,\{0\}\}$. A graph G which admits an integer additive set-graceful labeling is called an *integer additive set-graceful graph* (in short, IASG-graph).

If $f: V(G) \to \mathcal{P}(X)-\{\emptyset\}$ is an integer additive set-graceful labeling on a given graph G, then $\{0\}$ must be a set-label of one vertex of G. That is, G has even number of edges. A star graph $K_{1,m}$ admits an integer additive set-graceful labeling if and only if $m = 2^n-2$ for any integer $n>1$. For a positive integer $m>3$, the path P_m does not admit an integer additive set-graceful labeling. Let G be an IASG-graph which admits an IASGL f with respect to a finite non-empty set X. Then, G must have at least $|X|-1$ pendant vertices.

The following theorem discusses a necessary and sufficient condition for a Jaco graph to admit a graceful IASL.

Theorem 2.3. *A finite Jaco graph $J_n(mx+c)$ admits a graceful IASL if and only if $J_n(mx+c)$ is isomorphic to a path of order 3.*

Proof. Assume that a finite Jaco graph $J_n(mx+c)$ admits a graceful IASL f. It is proved in [22] that a graceful IASL-graph G must have at least $|X|-1$ pendant vertices, where X is the ground set for labeling. Then, as stated in the previous theorem, if the Jaco graph $J_n(mx+c)$ has a pendant vertex, then $m=1$ and $c=0$. By the definition of a finite Jaco graphs, we note that the Jaco graph $J_n(mx+c)$ (in this case, $J_n(x)$) can have at most 3 vertices. It is also proved in [22] that the number of edges in any graceful IASL-graph is $\mathcal{P}(X)|-2 = 2^{|X|}-2 = 2(2^{|X|-1}-1)$. That is, $J_n(x)$ must have even number of edges. It is possible only when $n=3$. But, if $n=3$, $m=1$ and $c=0$, then it is clear that $J_n(mx+c)$ is isomorphic to P_3.

Conversely, assume that $J_n(mx+c) \cong P_3 = v_1v_2v_3$, where $d(v_1)=d(v_3)=1$ and $d(v_2)=2$. Now, choose the set $X=\{0,1\}$. Label v_2 by the set $\{0\}$, v_1 by $\{1\}$ and v_3 by X. Therefore, $f^+(v_1v_2)=\{1\}$ and $f^+(v_2v_3)=X$. Clearly, $f^+(E)=\mathcal{P}(X)-\{\emptyset\}$ and hence this labeling is a graceful IASL for $J_n(mx+c)$. □

Another type of integer additive set-labeling of graphs that attracted much interest is *arithmetic integer additive set-labeling* (AIASL) with respect to which the set-labels of all its elements are arithmetic progressions (see [21]).

If the context is clear, the common difference of the set-label of an element of an AIASL-graph may be called the common difference of that element. The *deterministic ratio* of an edge of an AIASL-graph is the ratio $k > 1$ of the common differences of its end vertices.

The following theorem described a necessary and sufficient condition for a graph to admit an AIASL.

Theorem 2.4. [21] *A graph G admits an arithmetic IASI f if and only if the deterministic ratio of every edge of G is a positive integer, which is less than or equal to the set-labeling number of its end vertex having smaller common difference.*

If the deterministic ratio of every edge of an AIASL-graph G is 1, then the common difference of all elements of G will be the same and such an AIASL is called *isoarithmetic IASL* (IIASL) of G (see [16]). Hence, we have the following straight forward proposition.

Proposition 2.5. *Every finite Jaco graph $J_n(mx+c)$ admits an isoarithmetic integer additive set-labeling.*

The proof of the above result does depend only on the choice of the ground set X, which enables us to find n distinct subsets of X, which are arithmetic progressions with the same common difference, say d.

If the deterministic ratio of every edge of an AIASL-graph G is greater than 1, then such an arithmetic IASL is called a *biarithmetic IASL* (BISAL) of G (see [16]). As stated earlier, we have the following result also.

Proposition 2.6. *Every finite Jaco graph $J_n(mx+c)$ admits a biarithmetic integer additive set-labeling.*

In view of the above two propositions, we can also establish the following theorem on arithmetic IASL-graphs.

Theorem 2.7. *Every finite Jaco graph $J_n(mx+c)$ admits an arithmetic integer additive set-labeling.*

If the deterministic ratio of all edges of an AIASL-graphs G is a constant $k > 1$, then such an AIASL is called an *identical arithmetic IASL* (IAIASL) of G (see [16]). The following theorem described a necessary and sufficient condition for a graph to an IAIASL.

Theorem 2.8. *A graph G admits an IAIASL if and only if it is bipartite.*

Invoking this theorem, we propose the following theorem as a necessary and sufficient condition for the admissibility of an IAIASL by a Jacograph.

Theorem 2.9. *A finite Jaco gtaph $J_n(mx+c)$ admits an IAIASL if and only if $J_n(mx+c) \cong P_n$; $n \le 4$.*

Proof. If $J_n(mx+c) \cong P_n$; $n \le 4$, then it is a bipartite graph and hence by Theorem 2.8, $J_n(mx+c)$ admits an IAIASL.

If possible, assume that $J_n(mx+c)$ is not isomorphic to P_n, for any $n \le 4$. Then, we have $n \ge 4$ and hence any Jaco graph $J_n(mx+c)$ contains at least one triangle in $J_n(mx+c)$ irrespective of the values of m and c. Hence, by Theorem 2.8, $J_n(mx+c)$ does not admit an IAIASL. □

An important parameter defined on AIASL-graph is their *dispensing number*, which was defined in [24] as the minimum number of edges to be removed for the given graph G so that it admits an IAIASL. The dispensing number of a graph is denoted by $\vartheta(G)$. The following theorem determines the dispensing number of finite linear Jaco graphs.

Theorem 2.10. *The dispensing number of a linear Jaco graph $J_n(mx+c)$ is given by*

$$\vartheta(J_n(mx+c)) = \begin{cases} n-3, & \text{if } m=1 \text{ and } c=0, \\ n-1 & \text{otherwise.} \end{cases}$$

Proof. Note that the dispensing number of a graph is the minimum number of edges in G, the removal of which makes the graph bipartite. Let $V = \{v_i : i = 1,2,3,\ldots,v_n\}$ be the vertex set of a finite linear Jaco graph $J_n(mx+c)$ and let P denotes the maximal path $v_1v_2v_3\ldots v_n$ in $J_n(mx+c)$, which has the length $n-1$. Note that P is also a spanning tree of $J_n(mx+c)$.

If $m = 1$ and $c = 0$, every edge of the path $P' = P - \{v_1v_2, v_2v_3\}$ are contained in some triangles in $J_n(x)$. Also, it can be noted that all every odd cycle in G contains some of the edges in the path P'. Hence, $J_n(x) - P'$ is free from odd and hence is bipartite. Therefore, by Theorem 2.8, $J_n(x) - P'$ admits an IAIASL and then the dispensing number of $J_n(x)$ is $\vartheta(J_n(x)) = |E(P')| = n-3$.

For all other positive integral values of m and c, we can see that every edge of the path P is contained in some triangles in $J_n(mx+c)$. Also, note that every odd cycle of $J_n(mx+c)$ contains at least one edge of P and hence the subgraph $J_n(mx+c) - P$ contains no odd cycles. Therefore, $J_n(mx+c) - P$ is a bipartite graph. Then, by Theorem 2.8, $J_n(mx+c) - P$ admits an IAIASL. Hence, $\vartheta(J_n(x)) = |E(P)| = n-1$. □

The two cases mentioned in the above proof can be verified from the graphs provided in Figure 1.

A *weak integer additive set-labeling* (WIASL)of a graph G is an IASL $f : V(G) \to \mathcal{P}(X) - \{\emptyset\}$, where induced function $f^+ : E(G) \to \mathcal{P}(X) - \{\emptyset\}$ is defined by $f^+(uv) = f(u) + f(v)$ such that either $|f^+(uv)| = |f(u)|$ or $|f^+(uv)| = |f(v)|$, where $f(u) + f(v)$ is the sumset of $f(u)$ and $f(v)$.

A necessary and sufficient condition for a graph to admit a WIASL was proved in [12] as follows.

Theorem 2.11 ([12]). *An IASL f of a graph G is a weak IASL of G if and only if at least one end vertex of every edge of G has a singleton set-label.*

In view of Theorem 2.11, we have the following result.

Theorem 2.12. *Every finite Jaco graph admits a weak integer additive set-labeling.*

Proof. Let I be a maximal independence set in $J_n(mx+c)$ such that maximum number of edges in $J_n(mx+c)$ have one end vertex in I. Choose a ground set X with sufficient cardinality so that we can label the vertices in I by distinct non-singleton subsets of X in injective manner and all other vertices in $J_n(mx+c)$ can be labeled by distinct singleton sets in an injective manner. Clearly, this labeling will be a weak IASL of $J_n(mx+c)$. □

3. Scope for Further Studies

In this paper, we discussed the admissibility of various types of integer additive set-labelings by a particular type of graph classes called Jaco graphs. There are more open problems in this area which seem to be promising and interesting for further investigation. We mention some of these open problems, we have identified during our present study.

Problem 3.1. An IASI $f : V(G) \to \mathcal{P}(X) - \{\emptyset\}$ of a graph G is said to be a *topogenic integer additive set-labeling* of G if the collection of the set-labels of all elements of G together with the null set is a topology on the ground set X (see [20]). Determining suitable topogenic IASLs for different Jaco graphs is a challenging problem for further studies.

Problem 3.2. An IASI $f : V(G) \to \mathcal{P}(X) - \{\emptyset\}$ of a graph G is said to be a *sequential integer additive set-labeling* of G if $f(V) \cup f^+(E) \cup \{\emptyset\} = X$, the ground set (see [18]). Determining a suitable sequential IASLs for different Jaco graphs is a challenging problem for further studies.

Problem 3.3. An integer additive set-labeling $f : V(G) \to \mathscr{P}(X)$ is said to be an *integer additive set-filter labeling* (IASFL, in short) of G if $\mathscr{F} = f(V)$ is a proper filter on X (see [23]). A graph G which admits an IASFL is called an *integer additive set-filter graph* (IASF-graph). Determining a suitable integer additive set-filter labelings for different Jaco graphs is a promising problem for further studies.

Problem 3.4. The number of edges in a WIASL- graph, whose both end vertices have singleton set-labels is called the *sparing number* of that graph G. Since all finite Jaco graphs admit a weak integer additive set-labeling and almost all Jaco graphs are not bipartite, some edges will have their both end vertices with singleton set-labels. Determining the sparing number of various Jaco graphs is a worthy problem for further discussions.

Problem 3.5. The *set-indexing number* of a graph G is the minimum cardinality required for a ground set X with respect to which G admit an integer additive set-labeling. Determining the set-indexing numbers of Jaco graphs with respect to different IASls are promising problems for future investigations.

Problem 3.6. An integer additive set-labeling of a graph G is said to be a *uniform IASL* if the cardinality of the set-labels of all edges of G is the same. Investigating the conditions required for various graph classes to admit different types of uniform IASLs offers a strong platform for further research.

Further studies on many other characteristics of different IASL-graphs are also interesting and challenging. All these facts highlight the scope for further studies in this area.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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