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Complementary Eccentric Uniform Labeling Graphs

Research Article

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Abstract. Given a graph $G=(V,E)$, a set $M\subset V$ is called Complementary Eccentric Uniform (CEU), if the M-eccentricity labeling $e_M(u)=\max\{d(u,v): v\in M\}$ is identical for all $u\in V-M$. The least cardinality of a CEU set is called the CEU number of the graph G. In this paper we initiate a study on CEU labelled graphs and obtain bounds for certain graphs.

Keywords. CEU set; CEU number

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1. Introduction

For all terminology and notation in graph theory, we refer the reader to F. Harary [5]. Unless mentioned otherwise, all graphs considered in this paper are finite, simple and connected.

The eccentricity distribution over all nodes in a graph is an important property which has been studied in [8]. In [7], Linda Lesniak studied various properties of eccentricity sequences. The distance labeling of graphs has been widely studied in [4]. In network analysis there are situations in which a set of nodes are in equal distance from some other nodes or we want to keep some nodes at a particular distance from a set of nodes. Motivated from this we initiate a study on uniform eccentricity labeling in graphs.

2. Definitions and Results

Given a graph $G=(V,E)$, the distance between two vertices $d(u,v)$ is the length of the shortest $u-v$ path in G. The eccentricity $e(v)$ of a vertex v is $\max\limits_{u\in V} d(u,v)$. The radius $\operatorname{rad} G$ is $\min\limits_{v\in V} e(v)$

and the diameter diam$G$ is $\max_{v\in V} e(v)$. A vertex $v \in V(G)$ is called an eccentric point of the vertex $u \in V(G)$ if $d(u,v) = e(u)$. If rad$G$ = diam$G$, then the graph G is called self-centered. If $e(u) = \text{diam}G$, then u is called a peripheral vertex of G.

Definition 2.1. Let $G = (V,E)$ be a (p,q) graph and M be any nonempty proper subset of $V(G)$. Then, the M-eccentricity of u is the number $e_M(u) = \max\{d(u,v) : v \in M\}$. If $e_M(u)$ is identical for all $u \in V - M$, then we say M is a Complementary Eccentric Uniform(CEU) set and G is called Complementary Eccentric Uniform Labeled graph. If the common value $e_M(u) = k$, then we say G is Complementary Eccentric k-uniform graph or k-CEU graph .

The following is an immediate observation.

Observation 2.2. For any connected graph $G = (V,E)$ and $M \subset V$, $1 \le e_M(u) \le \text{diam}(G)$, for every $u \in V - M$.

Problem 1. For a graph $G = (V,E)$, find or characterize $M \subset V$, such that $e_M(v) = e(v)$ for every $v \in V - M$?

Let $G = (V,E)$ be a connected graph with at least 2 vertices. Then for any $v \in V$, the set $M = V - \{v\}$ is a CEU set. Hence every connected graph has a CEU set. In a (p,q) graph G, a CEU set with cardinality $p-1$ is called trivial CEU set. Hence we are interested in finding the non-trivial CEU sets.

Definition 2.3. The least cardinality of the CEU set in G is called the CEU number of G and is denoted by $\eta(G)$.

Example 2.4. The following is an example of a graph with CEU labeling. Here $M = \{u,v\}$ is the minimum CEU set.

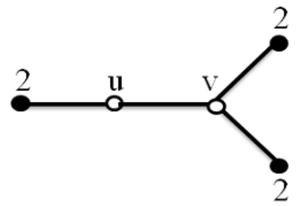

Figure 1. CEU labelled graph

Theorem 2.5. *For a (p,q) graph $G = (V,E)$, $\eta(G) = 1$ if and only if there exists $v \in V$ such that $d_G(v) = p-1$.*

Proof. Suppose there exists a vertex $v \in V$ such that $d_G(v) = p-1$. Take $M = \{v\}$. Then for all $u \ne v$, $e_M(u) = d(u,v) = 1$, so that M is a CEU set. Hence $\eta(G) = 1$.

Conversely suppose that $\eta(G) = 1$. Then there exists $v \in V$ such that $M = \{v\}$ is a CEU set. Since $\eta(G) = 1$, $e_M(u) = 1$ for all $u \ne v$. i.e, $d(u,v) = 1$, for all $u \ne v$. Hence $d_G(v) = p-1$. □

Remark 2.6. In a graph $G = (V,E)$ of order n, a vertex $v \in V$ with $\deg(v) = n-1$ is called a full degree vertex. Hence the above theorem can be restated as follows.

Theorem 2.7. *For any graph G, $\eta(G) = 1$ if and only if G has atleast one full degree vertex.*

Corollary 2.8. *Complete graph K_n is 1-CEU.*

Corollary 2.9. *For $m,n \geq 2$, $\eta(K_{m,n}) = 2$.*

Proof. Let $V(K_{m,n}) = X \cup Y$. Take $M = \{x, y\}$ where $x \in X$ and $y \in Y$. Then for every $u \in V - M$, $e_M(u) = d(u,y) = 2$, if $u \in Y$ and $e_M(u) = d(u,x) = 2$, if $u \in X$. Hence M is a CEU set. Since there are no full degree vertex in $K_{m,n}$, by Theorem 2.7, $\eta(K_{m,n}) = 2$. □

Definition 2.10. [3] A tree containing exactly two vertices that are not end vertices is called a bistar.

Corollary 2.11. *For a bistar $B_{m,n}, m,n \geq 1$, $\eta(B_{m,n}) = 2$.*

Proof. Let M be the central vertices in $B_{m,n}$. Then for all $u \in V(B_{m,n}) - M$, $e_M(u) = 2$. Since $B_{m,n}$ has no full degree vertex, by Theorem 2.7, $\eta(B_{m,n}) = 2$. □

Remark 2.12. In a non-selfcentered graph $G = (V,E)$, the relation $\sim$ on V given by $u \sim v$ if and only if $e(u) = e(v)$ is an equivalence relation. Let the equivalence classes corresponding to the eccentricities $e_1, e_2, \dots, e_k$ be denoted by $[e_1], [e_2], \dots, [e_k]$.

Proposition 1. In a non-selfcentered graph $G = (V,E)$ with eccentricities $e_1, e_2, \dots, e_k$, the sets $M_i = V - [e_i]$ for $i = 1,2,\dots,k$ are CEU sets in G.

Proof. Assume that $e_1 < e_2 < \cdots < e_k$. We consider two cases.
Case 1. When $i = k$.
Since e_k is the diameter of G, $[e_k]$ is the peripheral vertices of G. Therefore for all $u \in V - M_k$, $e_{M_k}(u) = \max\{d(u,v) : v \in M_k\} = e_k - 1$. Hence M_k is a CEU set in G.
Case 2. When $i \neq k$.
In this case, for all $u \in V - Mi$, $e_{M_i}(u) = e(u) = e_i$. Hence in either case M_i is a CEU set. □

Remark 2.13. The converse of the above proposition need not be true. That is, if $M \subset V$ is a CEU set in G, then its complement $V - M$ need not be an equivalence class in G.

For example consider the graph shown in figure 2. Here $M = \{v_4, v_5\}$ is a CEU set in G, but its complement $\{v_1, v_2, v_3, v_6\}$ is not an equivalence class since $e(v_1) = 2$ and $e(v_3) = 3$.

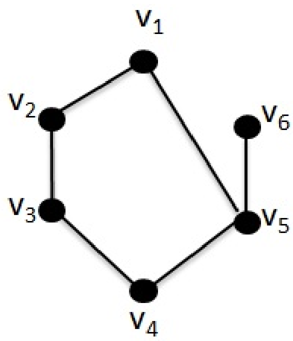

Figure 2

Problem 2. Characterize graphs whose CEU sets are precisely the complement of equivalence classes under ∼?

Proposition 2. For path P_{2n}, the nontrivial CEU sets are precisely the complement of the equivalence classes under ∼.

Proof. Each equivalence class in P_{2n} has exactly two elements and their complement is clearly a CEU set. Therefore if $E \subset V$ is an equivalence class in P_{2n}, then $M = V - E$ is a CEU set whose cardinality is $n-2$. To prove that they are the only CEU sets in P_{2n}, we consider two cases. Let $M \subset V$.

Case 1. Let the cardinality of M be less than $n-2$.

Then $V - M$ has atleast three vertices. Let $v_1, v_2, v_3 \in V - M$. Since there is a unique path between any two vertices in P_{2n}, atleast one of the $e_M(v_i)$, $i = 1,2,3$ is different from the others. Hence M is not a CEU set.

Case 2. Let the cardinality of M be $n-2$ and $M \neq V - E$ for any equivalence class E.

Then in $V - M$ there are exactly two vertices u and v, they are either adjacent or non-adjacent. If they are adjacent then they cannot be the central vertices since the central vertices form an equivalence class. Also $e_M(u) = e(u)$ and $e_M(v) = e(v)$ and either $e_M(u) = e_M(v) - 1$ or $e_M(v) = e_M(u) - 1$. Hence M is not a CEU set. If they are nonadjacent then both cannot be the peripheral vertices. Let u be a peripheral vertex. Then $e_M(u) = \text{diam}(P_{2n})$ and $e_M(v) < \text{diam}(P_{2n})$ so that M is not a CEU set. If $u, v \in V - M$ are non-adjacent non-peripheral vertices then $e_M(u) = e(u) \neq e(v) = e_M(v)$ so that M is not a CEU set. Hence the result. □

Remark 2.14. For the path P_{2n+1}, there are n equivalence classes of which one is a singleton set consisting of the central vertex and all other classes contains exactly two vertices. Hence the nontrivial CEU sets are precisely the complement of the equivalence classes under ∼ of cardinality 2.

Hence we have the following result.

Corollary 2.15. *For path P_n, $n > 2$, $\eta(P_n) = n-2$.*

Theorem 2.16. *For cycle C_n,* $\eta(C_n) = \begin{cases} \frac{n}{3}, & \text{if } n \equiv 0 \pmod 3 \\ \lfloor \frac{n}{2} \rfloor, & \text{if } n \not\equiv 0 \pmod 3 \end{cases}$

Proof. **Case 1.** If $n \equiv 0 \pmod 3$

Let $n = 3k$, for some $k \in \mathbb{N}$. Let $C_n = v_1, v_2, \ldots, v_{3k}, v_{3k+1} = v_1$. Consider $M = \{v_i, v_{i+3}, \ldots, v_{i+3(k-1)}\}$, for any $i = 1,2,\ldots,3k$. Then for any $v_j \in V(C_n) - M$, either $v_{j-1} \in M$ or $v_{j+1} \in M$. Hence $e_M(v_j) = \max\{1,2,4,5,\ldots,\frac{n-4}{2},\frac{n-2}{2}\} = \frac{n-2}{2}$, if n is even and $e_M(v_j) = \max\{1,2,4,5,\ldots,\frac{n-1}{2}\} = \frac{n-1}{2}$, if n is odd. Hence M is a CEU set and $\eta(C_n) \leq \frac{n}{3}$.

To prove the equality, first assume n is odd. Let $M_1 = \{u_1, u_2, \ldots, u_j\} \subset V(C_n)$ such that $j < k$ where each u_i is some v_t, for $t = 1,2,\ldots,3k$. Since n is odd, to each vertex in C_n there are exactly two eccentric vertices. Let $M_{1e} = \{u_{11}, u_{12}, u_{21}, u_{22} \ldots, u_{j1}, u_{j2}\}$ be the set of eccentric points of vertices in M_1. Note that they may not be distinct and some of them may be vertices in M_1. But cardinality of M_{1e} is atmost $2j$. Since $j < k$, there are vertices which does not belong to $M_1 \cup M_{1e}$. Choose such a vertex u which is not in $M_1 \cup M_{1e}$ and which is adjacent to a vertex $v \in M_{1e}$. Then clearly $e_{M_1}(v) = \text{diam}\, C_n$ and $e_{M_1}(u) = e_{M_1}(v) - 1$. Since $u, v \in V(C_n) - M_1$, it follows that M_1 is not a CEU set. Hence in this case $\eta(C_n) = \frac{n}{3}$.

Now assume n is even. Then to each vertex in C_n there is a unique eccentric point. Let $M_{1e} = \{u_{11}, u_{21}, \dots, u_{j1}\}$ be the set of eccentric points of vertices in M_1. Note that they must be distinct but some of them may elements in M_1. Now cardinality of M_{1e} is atmost j so that there are vertices which does not belong to $M_1 \cup M_{1e}$. Let u be a vertex which is not in $M_1 \cup M_{1e}$ and which is adjacent to a vertex $v \in M_{1e}$. But then $e_{M_1}(v) = \operatorname{diam} C_n$ and $e_{M_1}(u) = e_{M_1}(v) - 1$ and hence M_1 is not a CEU set. Hence, in this case also $\eta(C_n) = \frac{n}{3}$.

Case 2. If $n \not\equiv 0 \pmod 3$

Subcase 1. If n is even.

Let $n = 2k$, for some $k \in \mathbb{N}$ and let $C_n = v_1, v_2, \dots, v_{2k}, v_{2k+1} = v_1$. Let $M = \{v_i, v_{i+1}, \dots, v_{i+k-1}\}$, for any $i = 1, 2, \dots, 2k$. Then for any $v \in V(C_n) - M$, $e_M(v) = \operatorname{diam}(C_n)$ so that M is a CEU set. Hence $\eta(C_n) \le k = \frac{n}{2}$.

To prove the equality, let $M_1 = \{u_1, u_2, \dots, u_j\} \subset V(C_n)$ be such that $j < \frac{n}{2}$. Let $M_{1e} = \{u_{11}, u_{21}, \dots, u_{j1}\}$ be the set of eccentric points of vertices in M_1. Now M_{1e} has at most j elements. Since $j < \frac{n}{2}$, there are vertices in C_n which does not belong to $M_1 \cup M_{1e}$. Let u be a vertex which is not in $M_1 \cup M_{1e}$ and which is adjacent to a vertex $v \in M_{1e}$. Then clearly $e_{M_1}(u) = \operatorname{diam} C_n$ and $e_{M_1}(v) = e_{M_1}(u) - 1$. Hence M_1 is not a CEU set and $\eta(C_n) = \frac{n}{2}$.

Subcase 2. If n is odd.

Let $n = 2k+1$, for some $k \in \mathbb{N}$ and let $C_n = v_1, v_2, \dots, v_{2k+1}, v_{2k+2} = v_1$. Let $M = \{v_i, v_{i+1}, \dots, v_{i+k-1}\}$, for any $i = 1, 2, \dots, 2k+1$. Since n is odd each vertex in C_n has precisely two eccentric points. Since the vertices in M are k adjacent vertices in C_n their eccentric points are the remaining $k+1$ vertices in C_n. Hence for all $v \in V(C_n) - M$, $e_M(v) = \operatorname{diam} C_n$. Hence M is a CEU set and $\eta(C_n) \le k = \frac{n-1}{2}$.

As in the above case we can show that any subset of $V(C_n)$ with less than k elements is not a CEU set. Hence $\eta(C_n) = \frac{n-1}{2}$. Thus combining these two cases we get $\eta(C_n) = \lfloor \frac{n}{2} \rfloor$. □

Theorem 2.17. *In a tree T with a full degree vertex v, the CEU sets are precisely subsets of $V(T)$ which contains v.*

Proof. Tree with a full degree vertex is isomorphic to $K_{1,n}$. Let v be the central vertex and $v_1, v_2, \dots, v_n$ be the leaves of $K_{1,n}$. Then for any $M \subset V(T)$ which contains v, $e_M(v_i) = \max\{d(v, v_i), d(v_j, v_i) : v_j \in M\} = 2$ for any $v_i \in V - M$. Hence every subset of V which contains v is a CEU set. Now if $M \subset V$ does not contain v, then $e_M(v) = 1 \neq 2 = e_M(v_i)$ for any $v_i \in V - M$. Hence the result. □

Lemma 2.18 ([7]). *Let T be a tree and $P : u_0, u_1, \dots, u_l$ a longest path in T. If $u \in V(T)$, then $e(u) = \max\{d(u, u_0), d(u, u_l)\}$.*

Corollary 2.19. *Let T be a tree and $M \subset V(T)$. Then for any $u \in V(T) - M$, $e_M(u) = \max\{d(u, u_0), d(u, u_l)\}$ where $P_M = u_0, u_1, \dots, u_l$ be the longest path in T which starts and ends in M.*

Proof. Proof follows from the proof of Lemma 2.18. □

Corollary 2.20. *If M is the set of all peripheral vertices of T, then $e_M(u) = e(u)$ for every $u \in V - M$.*

Proof. Since M is the set of all peripheral vertices, each element of M is the end points of some longest path in T. Hence the result follows from Lemma 2.18. □

Theorem 2.21. *For a tree T on n vertices with eccentricities $e_1, e_2, \dots, e_k$, $\eta(T) = n - \|[e_i]\|$, where $\|[e_i]\| \geq \|[e_j]\|$ for every $j \neq i$.*

Proof. For a tree T on n vertices with eccentricities $e_1, e_2, \dots, e_k$, by Proposition 1, $V(T) - [e_i]$ is a CEU set for $i = 1, 2, \dots, k$. If $\|[e_i]\| \geq \|[e_j]\|$, then $n - \|[e_i]\| \leq n - \|[e_j]\|$ so that $\eta(T) \leq n - \|[e_i]\|$. Let $\|[e_i]\| = m_i$, for $i = 1, 2, \dots, k$ so that by assumption $m_i \geq m_j$, for every $j \neq i$. Without loss of generality assume that $e_1 < e_2 < \cdots < e_k$. Let $M = \{u_1, u_2, \dots, u_l\} \subset V(T)$ be such that $l < n - m_i$. Then in $V - M$ there are $n - l$ vertices. Since $l < n - m_i$, $n - l > m_i$. Thus $V - M$ has at least $m_i + 1$ elements. Now the centre of T is either a vertex v or K_2 which corresponds to the class $[e_1]$. Since $n - l > m_i$, there exist at least two vertices u_s and u_t for $l + 1 \leq s, t \leq n$ such that $u_s \in [e_p]$ and $u_t \in [e_q]$ for some $1 \leq p \neq q \leq k$. But then $e_M(u_s) = \max\{d(u_s, v_0), d(u_s, v_h)\}$ where $v_0, v_1, \dots, v_h$ is the longest path in T which starts and ends in M. Then either $e_M(u_t) < e_M(u_s)$ or $e_M(u_t) > e_M(u_s)$. Thus M is not a CEU set and hence $\eta(T) = n - m_i$. □

3. Conclusion

As pointed out already, the concept under study has important applications in the field of network analysis. In a network there are situations to keep a set of nodes at a particular distance from another set of nodes. So Complementary eccentric uniform sets allow set of points in a graph to be in a particular distance from another set of points. In this paper we have identified the CEU sets in many graphs and CEU number of certain well known graphs.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam (1973).

[2] F. Buckley and F. Harary, *Distance in Graphs*, Addison Wesley Publishing Company (1990).

[3] G. Chartrand and P. Zhang, *Introduction to Graph Theory*, McGraw-Hill International Edition, Singapore (2005).

[4] C. Gavoille, D. Peleg, S. Perennes and R. Raz, Distance labeling in graphs, *Journal of Algorithms* **53** (2004), 85–112.

[5] F. Harary, *Graph Theory*, Addison-Wesley Publishing Company, Massachusetts (1969).

[6] K.R. Parthasarathy and R. Nandakumar, Unique eccentric point graphs, *Discrete Mathematics* **46** (1983), 69–74.

[7] L. Lesniak, Eccentric sequences in graphs, *Period. Math. Hung.* **6** (1975), 287–293.

[8] F.W. Takes and W.A. Kosters, Computing the eccentricity distribution of large graphs, *Algorithms* **6** (2013), 100–118.