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Research Article

Viscosity Approximation Method for Split Common Null Point Problems between Banach Spaces and Hilbert Spaces

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Abstract. We study an iterative scheme to approximation the split common null point problems for set-valued maximal monotone operators which combine viscosity method and some fixed point technically proving method between Banach spaces and Hilbert spaces, without using the metric projection. We prove that strong convergence theorem. Also, we show that our result can be solves the split minimization problems.

Keywords. Iterative method; Viscosity approximation method; Fixed point problems; Split common null point problems; A zero point; Nonexpansive operator; (metric) resolvent operator

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1. Introduction

Many important problems in mathematics, sciences, engineering and other can be reformulation which require finding zero points or null point of nonlinear operators i.e. equation of the form $0 \in Ax$, where $x \in X$ such that X be a vector space, and A is a maximal monotone operator.

The inverse problem can be split into two inverse problems, which is called *Split Inverse Problem*(*SIP*). Let IP_1, IP_2 be two inverse problems. First one is formulated in the space X and the second one is formulated in space Y. This problem concern in a model which there are given two vector spaces X, Y, and a linear operator $T : X \to Y$. The *Split Inverse Problem* is formulated as follows.

$$\begin{aligned} &\text{Find a point } x^* \in X \text{ that solves } IP_1\\ &\quad \text{and } y^* = Tx^* \in Y \text{ that solves } IP_2. \end{aligned} \tag{1.1}$$

The *Split Convex Feasibility Problem* is the first case of the *SIP*, which is introduced by Censor and Elfving [1]. The two inverse problems IP_1 and IP_2 there are of the Convex Feasibility Problem (CFP) type. However, someone called the *Split Convex Feasibility Problem* is the *Split Feasibility Problem* (SFP). The *Split (Convex) Feasibility Problem* is formulated as follows.

$$\text{Find a point } x^* \in C \text{ such that } Tx^* \in Q, \tag{1.2}$$

where C, Q are nonempty closed convex subspace of Hilbert space H_1, H_2, respectively. The set solution of Split Feasibility Problem is denoted by $\Gamma := \{x^* \in C \text{ such that } Tx^* \in Q\} = C \cap T^{-1}(0)$.

Assume that the *SFP* has a solution, then $x \in C$ solves (1.1) if and only if it solves the following *fixed point equation*:

$$x = P_C(I - \gamma T^*(I - P_Q)T)x, \quad x \in C, \tag{1.3}$$

where γ is any positive constant, P_C is metric projections of H_1 onto C, and P_Q is metric projections of H_2 onto Q, and T^* is the adjoint of T.

Later, many researcher have studies *SIP* in Hilbert spaces, for instance, [2, 3, 4] Takahash have studies *SIP* in Banach spaces [5].

Byrne el al. [4] introduced the *Split Common Null Point Problem* (*SCNPP*) for set-valued maximal monotone operators in Hilbert spaces. They given two set-valued operators $A : H_1 \to 2^{H_1}$, $B : H_2 \to 2^{H_2}$ and let $T : H_1 \to H_2$ be a bounded linear operator. They consider *SCNPP* as the following.

$$\begin{aligned} &\text{Find a point } x^* \in H_1 \text{ that solves } 0 \in Ax^*\\ &\quad \text{and } y^* = Tx^* \in H_2 \text{ that solves } 0 \in By^* \end{aligned} \tag{1.4}$$

where $A^{-1}(0)$ and $B^{-1}(0)$ are null point set of A and B, respectively. They given $x_0 \in H_1$, define iterative scheme by the following:

$$x_{n+1} = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \tag{1.5}$$

where A^* is the adjoint of $A, L = \|A^*A\|$ and $\gamma \in (0, \frac{2}{L})$, and $\lambda > 0$.

In 2015, Takahashi [6] have studied *SCNPP* and he is the first that extended *SCNPP* form Hilbert space to Banach space by using metric resolvent and metric projections with applies the hybrid method. Recently, Takahashi and Yao [7] considered *SCNPP* in Hilbert spaces and Banach spaces by using the hybrid method as the following theorem:

Theorem 1.1 ([7]). *Let H be a Hilbert space and let E be a uniformly convex and uniformly smooth of Banach spaces. Let J_E be the duality mapping on E. Let $A : H \to 2^H$ and $B : E \to 2^{E^*}$ be maximal monotone operator such that $A^{-1}(0) \neq \emptyset$ and $B^{-1}(0) \neq \emptyset$. Let J_λ^A be the resolvent of A for $\lambda > 0$ and let Q_μ^B be the metric resolvent of B for $\mu > 0$. Let $T : H \to E$ be a bounded linear operator* *such that $T \neq \emptyset$ and T^* be an adjoint operator of T. Suppose that $A^{-1}(0) \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in H$, and let $\{x_n\}$ be a sequence generated by the following.*

$$\begin{cases} z_n = J^A_{\lambda_n}(x_n - \lambda_n T^* J_E(Tx_n - Q^B_{\mu_n} Tx_n)), \\ y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x_n 1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1; \quad \forall\, n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\alpha_n \subset [0,1]$, and $\lambda_n, \mu_n \subset (0,\infty)$ satisfies that $0 \leq \alpha_n \leq a < 1$, $0 < b \leq \mu_n$, and $0 < c \leq \lambda_n \|T\|^2 \leq d < 2$ for some $a,b,c,d \in \mathbb{R}$. Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in A^{-1}(0) \cap T^{-1}(B^{-1}0)$, where $z_0 = P_{A^{-1}(0)\cap T^{-1}(B^{-1}0)} x_1$.

A viscosity approximation method is a well-known iterative method for solving a fixed point of nonexpansive mappings. Moudafi [8] is first person that proposed viscosity approximation method by combing the nonexpansive mapping and a contraction mapping. He proposed the following iterative scheme in Hilbert spaces:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S x_n, \tag{1.7}$$

where f is a contraction mapping and $\{\alpha_n\} \subseteq (0,1)$ satisfies some conditions, he proved that $\{x_n\}$ converges strongly to a fixed point of S.

Further, Xu [9] developed the viscosity approximation method for solves the zero points of monotone operators in a Banach space. He proved that strong convergent theorem.

Motivated by the problems of split common null point, and result of Takahashi [7], these is interesting that the result of [7] is formulated in two different spaces. Moreover, a well-known viscosity approximation method of Moudafi [8] and Xu [9] are still effective and interesting for solving a fixed point problem. Then we consider *SCNPP* related between a Hilbert space and a Banach space by using the viscosity approximation method. We consider *SCNPP* as the following.

$$\begin{aligned} &\text{Find a point } x^* \in H \text{ that solves } 0 \in Ax^* \\ &\quad \text{and } y^* = Tx^* \in E \text{ that solves } 0 \in By^*, \end{aligned} \tag{1.8}$$

where $A^{-1}(0)$ and $B^{-1}(0)$ are null point set of A and B, respectively. While, we let H, E are Hilbert spaces and Banach spaces, respectively.

2. Preliminaries

Let E be a real Banach space with the norm $\|\cdot\|_E$ and the dual space E^* of E is the space of continuous linear functional on E .

The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x\| = \|x^*\|\}, \quad \forall\, x \in E.$$

In the case if E is Hilbert space, then the normalized duality mapping is linear and it is just the identity mapping, i.e. $J = I$.

A Banach space E is said to satisfy *Opial's condition* if for each sequence $\{x_n\}_{n=0}^{\infty}$ in E such that $\{x_n\}$ converges weakly to some x in E, the inequality

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|$$

holds for all $y \in E$ with $y \neq x$. In fact, a Banach space with a weakly sequentially continuous duality mapping has the Opial's condition; see [10]. It known that every Hilbert satisfies the Opial's condition.

A Banach space E is called a *strictly convex* if it satisfies the following condition

$$\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\| \quad \forall\, x, y \in E \text{ and } 0 < \lambda < 1 \implies x = y.$$

Let $S(E) = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. E is said to be *uniformly convex* if for each $\epsilon \in (0,2]$ there exists a constant $\delta = \delta(\epsilon) > 0$ such that for all $x, y \in S(E)$, if $\|x - y\| \geq \epsilon$ then $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$. It is well known that uniformly convex is strictly convex.

A Banach space E is said to be *smooth* if the limit

$$\lim_{t\to 0}\frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S(E)$.

The norm of E is said to be *uniformly Gâteaux differentiable norm* if for each $y \in S(E)$, the limit (2.1) is attained uniformly for all $x \in S(E)$ and it is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit (2.1) is attained uniformly for all $y \in S(E)$. Moreover, it is said to be *uniformly smooth* if the limit (2.1) is attained uniformly for all $(x, y) \in S(E) \times S(E)$.

The *modulus of smoothness* of E is the function $\rho : [0,\infty) \to [0,\infty)$ defined by $\rho(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = t\right\}$. A Banach space E is an *uniformly smooth* if and only if $\lim_{t\to 0}\frac{\rho(t)}{t} = 0$.

A Banach space E is said to be q-*uniformly smooth* if for $1 < q \leq 2$ be a fixed real number. There exists a constant $c > 0$ such that $\rho(t) \leq ct^q$ for all $t > 0$. In the case $q = 2$, E is said to be 2-*uniformly smooth* if there exists a constant $c > 0$ such that $\rho(t) \leq ct^2$ for all $t > 0$.

The examples of 2-uniformly smooth and uniformly convex Banach space that

$$L_p,\ l_p, \text{ or the sobolev space } W_m^p \text{ is } \begin{cases} \text{2-uniformly smooth,} & \text{if } p \geq 2; \\ q\text{-uniformly smooth,} & \text{if } 1 < p \leq 2. \end{cases}$$

Note that no a Banach space is q-uniformly smooth for $q > 2$. It is known that a Hilbert space is 2-uniformly convex and 2-uniformly smooth. By [11, 12] we know that if E be a q-uniformly smooth Banach space, then for all $x,\ y \in E$ there exists a constant $c_1 > 0$ such that $\|J(x) - J(y)\| \leq c_1\|x - y\|^{q-1}$. Hence if E be a 2-uniformly smooth Banach space, then there exists a constant $c_1 > 0$ such that $\|J(x) - J(y)\| \leq c_1\|x - y\|$ for all $x, y \in E$. If E be a q-uniformly smooth Banach space for $1 < q < 2$, then there exists the constant $c_1 > 0$ such that $\|J(x) - J(y)\| \leq c_1\|x - y\|^{q-1}$. Then we can see that $\|J(x) - J(y)\| \leq c_1\|x - y\|$ too. For instance in L_p is 2-uniformly smooth Banach space for $2 \leq p < \infty$ and we know that $\|J(x) - J(y)\| \leq (p-1)\|x - y\|$. For $1 \leq q \leq 2$, then L_q is q-uniformly smooth Banach space. So that $\|J(x) - J(y)\| \leq 2q^{-1}K_q\|x - y\|^{q-1}$ and also we have $\|J(x) - J(y)\| \leq 2q^{-1}K_q\|x - y\|$, where K_q is q-uniformly smooth constant.

Remark 2.1 (See [12]). (1) If E is a *uniformly smooth*, then E is smooth and reflexive.

(2) If E is a *uniformly convex*, then E is strictly convex.

(3) If E is a smooth, reflexive and strictly convex then the normalized duality mapping J is single-valued, one-to-one and onto. Then $J^{-1}: E^* \to E$ is single-valued, bijective, that is the inverse mapping $J^{-1}: E^* \to (E)$ and also $JJ^{-1} = I_{E^*}$, $J^{-1}J = I_E$.

(4) A normed space E is reflexive, if and only if E is bounded sequence has a weakly convergent subsequence.

Next, we recall some definitions. Let C be a nonempty close convex subset of E. Let $S: C \to C$ be an operator. If there exists a coefficient $k \in (0,1)$ such that $\|Sx - Sy\| \le k\|x - y\|$ for all $x, y \in C$, then S is called k-contraction. If $k = 1$ that is $\|Sx - Sy\| \le \|x - y\|$ for all $x, y \in C$, then we called *nonexpansive*. In this paper, fixed point of an operator S is denoted by $Fix(S)$, i.e, $Fix(S) := \{x \in C : Sx = x\}$.

Let $B: E \to 2^{E^*}$ be a set-value operator, B is said to be *monotone* if $\langle x - y, u^* - v^* \rangle \ge 0$, for all $x, y \in D(B)$, $u^* \in Bx$, $v^* \in By$, where $D(B)$ is the domain of B. A monotone operator B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. By [13], we known that if B be a maximal monotone operator of E into 2^{E^*}, then $R(I + rJ^{-1}B) = E$. From [14] if E be a uniformly convex and smooth Banach space, then B is *maximal monotone* if and only if $R(J + rB) = E^*$ for $r > 0$, where $R(J + rB)$ is the range of $J + rB$. This mean that $R(I + rJ^{-1}B) = E$.

For a maximal monotone operator B, we can define a nonexpansive single-valued mapping $Q_r^B : R(I + rJ^{-1}B) \to D(B)$ by $Q_r^B = (I + rJ^{-1}B)^{-1}$ for each $r > 0$, which is called the metric resolvent of B. It is known that $0 \in B(x) \Leftrightarrow x \in Fix(Q_r^B)$. The set of null point of B is generated by $B^{-1}(0) = \{x \in B :\ 0 \in Bx\}$. From Takahashi [15], we known that $B^{-1}(0)$ are closed and convex.

In Hilbert space, we known that $H = H^*$. For a monotone operator $A: H \to 2^H$, we define a nonexpansive single-valued mapping $J_r^A : R(I + rA) \to D(A)$ by $J_r^A = (I + rA)^{-1}$ for each $r > 0$, which is called the resolvent of A. It is known that $0 \in A(x) \Leftrightarrow x \in Fix(J_r^A)$.

Let C be a nonempty closed convex subset of a Banach space E and $D \subset C$, then a mapping $Q : C \to D$ is said to be *sunny* if $Q(x + t(x - Q(x))) = Q(x)$ whenever $Qx + t(x - Q(x)) \in C$ for all $x \in C$ and $t \ge 0$.

A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. Note that if a mapping Q is a retraction, then $Qz = z$ for all $z \in R(Q)$ where $R(Q)$ is the range of Q. A subset D of Q is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

Lemma 2.2 ([16]). *Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q: E \to C$ be a retraction and let J be the normalized duality mapping on E. Then, the followings are equivalent:*

(i) *Q is sunny and nonexpansive;*

(ii) $\|Qx - Qy\|^2 \le \langle x - y, J(Qx - Qy)\rangle$, $\forall\ x, y \in E$;

(iii) $\|(x - y) - (Qx - Qy)\|^2 \le \|x - y\|^2 - \|Qx - Qy\|^2$;

(iv) $\langle x - Qx, J(y - Qx)\rangle \le 0$, $\forall\ x \in E,\ y \in C$.

Lemma 2.3 ([17]). *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let S be a nonexpansive mapping of C into itself with $Fix(S) \neq \emptyset$. Then, the set $Fix(S)$ is a sunny nonexpansive retract of C.*

In 2004, Xu [9] studied the continuous scheme $x_t = tf(x_t) + (1-t)Sx_t$, where f is a k-contraction and S is nonexpansive mapping. On a uniformly smooth Banach space, Xu proved that the sequence $x_t \in C$ be the unique fixed point of the contraction $x \to tf(x) + (1-t)Sx$, that is $x_t = tf(x_t) + (1-t)Sx_t$.

Lemma 2.4 ([9]). *Let E be a uniformly smooth Banach space, C be a closed convex subset of E, $S: C \to C$ be a nonexpansive mapping with $Fix(S) \neq \emptyset$, and and $f \in \Xi_c$. Then x_t defined by $x_t = tf(x_t) + (1-t)Sx_t$ converges strongly to a point in $Fix(S)$. If we defines mapping $Q : \Xi_c \to Fix(S)$, where Ξ_c denote the set of k-contraction, by $Q(f) := \lim_{t\to 0} x_t$, then $Q(f)$ solves the following variational inequality:*

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad \forall\, f \in \Xi_c,\ \forall\, p \in Fix(S).$$

Namely, if $\bar{x} = Q_C(f)$, then

$$\langle \bar{x} - f(\bar{x}), J(\bar{x}-p)\rangle \leq 0, \quad \forall\, f \in \Xi_c,\ \forall\, p \in Fix(S).$$

It well known that if $E = H$ is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection P_C from E onto C, that is $Q_C = P_C$.

In the sequel to give our main results, we need the following lemmas.

Lemma 2.5 ([18]). *Let E be a real Banach space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad j(x+y) \in J(x+y),$$

where j denote a single-value.

Lemma 2.6 ([12]). *Let H be a Hilbert space. Then*

$$\|x+y\|^2 = \|x\|^2 + 2\langle y, x\rangle + \|y\|^2, \quad \forall\, x, y \in H.$$

Lemma 2.7 ([20]). *Let E be a real uniformly convex Banach space and $B_r = \{x \in E : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0,\infty] \to [0,\infty]$, $g(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta g(\|x-y\|), \quad \forall\, x, y, z \in [0,1],$$

with $\alpha + \beta + \gamma = 1$.

Lemma 2.8 ([21], The Resolvent Identity). *For all $r > 0$, $s > 0$. Let $x \in E$ and B is maximal monotone then*

$$J_r^B x = J_s^B\left(\frac{s}{r}x + \left(1 - \frac{s}{r}\right)J_r^B x\right).$$

Lemma 2.9 ([24], Demiclosed Principle). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and S be a nonexpansive mapping. Then $I - S$ is demiclosed at zero, i.e., $x_n \rightharpoonup x$ and $x_n - Sx_n \to 0$ imply $x = Sx$.*

Lemma 2.10 ([25]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \le 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.11 ([26]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the condition*

$$a_{n+1} \le (1-t_n)a_n + t_n b_n + c_n, \quad \forall\ n \ge 0,$$

where $\{t_n\}$ is a sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} t_n = \infty$, $\{b_n\}$ is a sequence such that $\limsup_{n\to\infty} b_n \le 0$ and $c_n \ge 0$, $\forall n \ge 0$ such that $\sum_{n\to\infty}^{\infty} c_n < \infty$. Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.12 ([6]). *Let E_1 and E_2 be strictly convex, reflexive and smooth Banach space and let J_{E_1} and J_{E_2} be the duality mapping on E_1 and E, respectively. Let $A : E_1 \to 2^{E_1^*}$ and $B : E_2 \to 2^{E_2^*}$ be maximal monotone operators such that $A^{-1}(0) \neq \emptyset$ and $B^{-1}(0) \neq \emptyset$, respectively. Let J_λ^A and Q_μ^B be the metric resolvent of be the resolvent of A for $\lambda > 0$ and B for $\mu > 0$, respectively. Let $T : E_1 \to E_2$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $\Omega := A^{-1}(0) \cap T^{-1}(B^{-1}(0)) \neq \emptyset$. Let $\lambda,\ \mu,\ r > 0$ and $p \in E$. Then the following are equivalent:*

(i) $p = J_\lambda^A(p - rJ_{E_1}^{-1}T^*J_{E_2}(Tp - Q_\mu^B Tp))$;

(ii) $p \in A^{-1}(0) \cap T^{-1}(B^{-1}(0))$.

3. Main Result

Proposition 3.1. *Let H be a Hilbert space and E be a real 2-uniformly smooth Banach space with the constant $c_1 \in \left(0, \frac{1}{\|T\|^2}\right)$, where c_1 is positive constant such that $\|J(x) - J(y)\| \le c_1\|x - y\|$ for all $x,\ y \in E$. Let J_E be the duality mapping on E. Let $B : E \to 2^{E^*}$ be an maximal monotone operator such that $B^{-1}(0) \neq \emptyset$. Let $Q_\mu^B = (I + \mu J^{-1}B)^{-1}$ be the metric resolvent B. Let $T : H \to E$ be a bounded linear operator such that $T \neq \emptyset$ and $T^* : E^* \to H$ be an adjoint operator of T. Assume that $T^{-1}(B^{-1}0) \neq \emptyset$. Let $M := T^*J_E(T - Q_\mu^B T)$, then $M := T^*J_E(T - Q_\mu^B T)$ is nonexpansive.*

Proof. Since I and Q_μ^B are nonexpansive mappings, and we know that $\|J_E(x) - J_E(y)\| \le c_1\|x - y\|$. We compute

$$\begin{aligned}
\|Mx - My\| &= \|T^*J_E(I - Q_\mu^B)Tx - T^*J_E(I - Q_\mu^B)Ty\| = \|T^*(J_E(I - Q_\mu^B)Tx - J_E(I - Q_\mu^B)Ty)\| \\
&\le \|T\|\ \|J_E(T - Q_\mu^B T)x - J_E(I - Q_\mu^B)Ty\| \le c_1\|T\|\ \|(I - Q_\mu^B)Tx - (I - Q_\mu^B)Tx\|^2 \\
&\le c_1\|T\|\ \|Tx - Ty\| = c_1\|T\|\ \|T(x - y)\| \\
&\le c_1\|T\|^2\ \|x - y\|.
\end{aligned}$$

Since $c_1 \in \left(0, \frac{1}{\|T\|^2}\right)$, therefore M is a nonexpansive mapping. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a Hilbert spaces H, and let E be a real uniformly convex and 2-uniformly smooth of Banach space with $c_1 \in \left(0, \frac{1}{\|T\|^2}\right)$. Let J_E be the duality mapping on E. Let $A : H \to 2^H$ be maximal monotone operator such that $A^{-1}(0) \neq \emptyset$ and let $B : E \to 2^{E^*}$ be an maximal monotone operator such that $B^{-1}(0) \neq \emptyset$. Let $J^A_{\lambda_n} = (I + \lambda_n A)^{-1}$ be the resolvent of A for $\lambda_n > 0$ and $Q^B_\mu = (I + \mu J^{-1}B)^{-1}$ be the metric resolvent of B for $\mu > 0$. Let $f : C \to C$ be a $k-$contraction mapping with $k \in (0,1)$. Let $T : H \to E$ be a bounded linear operator such that $T \neq \emptyset$ and $T^* : E^* \to H$ be an adjoint operator of T. Assume that $\Omega := A^{-1}(0) \cap T^{-1}(B^{-1}(0)) \neq \emptyset$. Let $x_1 \in H$, and let $\{x_n\}$ be a sequence generate by the following*

$$\begin{cases} y_n = \sigma_n x_n + (1-\sigma_n) J^A_{\lambda_n}(x_n - \lambda_n T^* J_E(Tx_n - Q^B_\mu T x_n)), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \quad \forall\ n \geq 0, \end{cases} \tag{3.1}$$

where $\{\sigma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ and $\alpha_n + \beta_n + \gamma_n = 1$. Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(c) $0 < \liminf_{n\to\infty} \lambda_n \leq \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{\|T\|^2}$, *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

(d) $0 < \liminf_{n\to\infty} \sigma_n \leq \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$, *and* $\lim_{n\to\infty} |\sigma_{n+1} - \sigma_n| = 0$;

(e) $0 < \liminf_{n\to\infty} \gamma_n \leq \gamma_n$.

Then $\{x_n\}$ converges strongly to a point $\bar{x} \in \Omega$, where $\bar{x} = Q_\Omega f(\bar{x})$.

Proof. We have divide the proof into five steps.

Step 1. We prove that the sequence $\{x_n\}$ is bounded.

Fixed $p \in \Omega := A^{-1}(0) \cap T^{-1}(B^{-1}(0)) \neq \emptyset$, then we have $p \in (A)^{-1}(0) = Fix(J^A_{\lambda_n})$, i.e., $J^A_{\lambda_n} p = p$ and $Tp = Q^B_\mu(Tp)$, and by Lemma 2.12 we have $p = J^A_{\lambda_n}(p - \lambda_n T^* J_E(Tp - Q^B_\mu Tp))$.

Set $M := T^* J_E(T - Q^B_\mu T)$, we see that

$$\begin{aligned} \|y_n - p\|^2 &= \|\sigma_n x_n + (1-\sigma_n) J^A_{\lambda_n}(I - \lambda_n M)x_n - p\|^2 \\ &\leq \sigma_n \|(x_n - p)\|^2 + (1-\sigma_n)\|[J^A_{\lambda_n}(I - \lambda_n M)x_n - p]\|^2 \\ &\quad - \sigma_n(1-\sigma_n) g(\|x_n - J^A_{\lambda_n}(I - \lambda_n M)x_n\|) \\ &\leq \sigma_n \|x_n - p\|^2 + (1-\sigma_n)\|x_n - p\|^2 - \sigma_n(1-\sigma_n) g(\|x_n - J^A_{\lambda_n}(I - \lambda_n M)x_n\|) \\ &= \|x_n - p\|^2 - \sigma_n(1-\sigma_n) g(\|x_n - J^A_{\lambda_n}(I - \lambda_n M)x_n\|). \end{aligned} \tag{3.2}$$

Therefore $\|y_n - p\|^2 \leq \|x_n - p\|^2$, this implies that

$$\|y_n - p\| \leq \|x_n - p\|.$$

Consider

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - p\| \\ &\leq \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|y_n - p\| \\ &\leq \alpha_n k \|x_n - p\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| + \gamma_n \|y_n - p\| \end{aligned}$$

$$
\begin{aligned}
&\le [\alpha_n k + \beta_n + \gamma_n]\|x_n - p\| + \alpha_n \|f(p) - p\| \\
&= [\alpha_n k + (1-\alpha_n)]\|x_n - p\| + \alpha_n \|f(p) - p\|\| \\
&\le (1 + \alpha_n k - \alpha_n)\|x_n - p\| + \alpha_n \|f(p) - p\|\| \\
&\le (1 - \alpha_n(1-k))\|x_n - p\| + \alpha_n \|f(p) - p\| \\
&= (1 - \alpha_n(1-k))\|x_n - p\| + \alpha_n(1-k)\frac{\|f(p) - p\|}{(1-k)}.
\end{aligned}
$$

It follows by mathematical induction, we get that

$$
\|x_{n+1} - p\| \le \max\left\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-k}\right\}.
$$

Therefore, this show that the sequence $\{x_n\}$ is bounded. Furthermore, since $\|y_n - p\| \le \|x_n - p\|$ then we obtain that $\{y_n\}$ is bounded too. Also, we have $\{f(x)\}$, $\{J^A_{\lambda_n}(I - \lambda_n M)\}$ are bounded sequence.

Step 2. We prove that the sequence $\{x_n\}$ is asymptotically regular, i.e., $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Step 2.1. By use Lemma 2.10, then we set $x_{n+1} := \beta_n x_n + (1-\beta_n) z_n$ and we let $z_n := \frac{\alpha_n f(x_n) + \gamma_n y_n}{(1-\beta_n)}$.

We compute

$$
\begin{aligned}
z_{n+1} - z_n &= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} y_{n+1}}{(1-\beta_{n+1})} - \frac{\alpha_n f(x_n) + \gamma_n y_n}{(1-\beta_n)} \\
&= \frac{\alpha_{n+1} f(x_{n+1}) + y_{n+1} - (\alpha_{n+1} + \beta_{n+1}) y_{n+1}}{(1-\beta_{n+1})} - \frac{\alpha_n f(x_n) + y_n - (\alpha_n + \beta_n) y_n}{(1-\beta_n)} \\
&= \frac{\alpha_{n+1} f(x_{n+1}) + (1-\beta_{n+1}) y_{n+1} - \alpha_{n+1} y_{n+1}}{(1-\beta_{n+1})} - \frac{\alpha_n f(x_n) + (1-\beta_n) y_n - \alpha_n y_n}{(1-\beta_n)} \\
&= \frac{\alpha_{n+1}(f(x_{n+1}) - y_{n+1}) + (1-\beta_{n+1}) y_{n+1}}{(1-\beta_{n+1})} - \frac{\alpha_n [f(x_n) - y_n] + (1-\beta_n) y_n}{(1-\beta_n)} \\
&= \frac{\alpha_{n+1}[f(x_{n+1}) - y_{n+1}]}{(1-\beta_{n+1})} - \frac{\alpha_n [f(x_n) - y_n]}{(1-\beta_n)} + y_{n+1} - y_n, \qquad (3.3)
\end{aligned}
$$

which implies that

$$
\|z_{n+1} - z_n\| \le \frac{\alpha_{n+1}}{(1-\beta_{n+1})}\|f(x_{n+1}) + y_{n+1}\| + \frac{\alpha_n}{(1-\beta_n)}\|f(x_n) - y_n\| + \|y_{n+1} - y_n\|. \qquad (3.4)
$$

Step 2.2 Next, we compute $\|y_{n+1} - y_n\|$.

Observe that

$$
\begin{aligned}
&\|J^A_{\lambda_{n+1}}(I - \lambda_{n+1} M) x_{n+1} - J^A_{\lambda_n}(I - \lambda_n M) x_n\| \\
&\quad \le \|J^A_{\lambda_{n+1}}(I - \lambda_{n+1} M) x_{n+1} - J^A_{\lambda_{n+1}}(I - \lambda_{n+1} M) x_n\| + \|J^A_{\lambda_{n+1}}(I - \lambda_{n+1} M) x_n - J^A_{\lambda_n}(I - \lambda_n M) x_n\| \\
&\quad \le \|x_{n+1} - x_n\| + \left\| J^A_{\lambda_{n+1}}(I - \lambda_{n+1} M) x_n - J^A_{\lambda_{n+1}}\left[\frac{\lambda_{n+1}}{\lambda_n}(I - \lambda_n M x_n) - \left(1 - \frac{\lambda_{n+1}}{\lambda_n}\right) J^A_{\lambda_n}(I - \lambda_n M) x_n\right]\right\| \\
&\quad \le \|x_{n+1} - x_n\| + \left\|(x_n - \lambda_{n+1} M) x_n - \frac{\lambda_{n+1}}{\lambda_n} x_n + \lambda_{n+1} M x_n + \left(1 - \frac{\lambda_{n+1}}{\lambda_n}\right) J^A_{\lambda_n}(I - \lambda_n M) x_n\right\| \\
&\quad \le \|x_{n+1} - x_n\| + \left\|\left(1 - \frac{\lambda_{n+1}}{\lambda_n}\right) x_n + (\lambda_{n+1} - \lambda_n) M x_n + \left(1 - \frac{\lambda_{n+1}}{\lambda_n}\right) J^A_{\lambda_n}(I - \lambda_n M) x_n\right\|
\end{aligned}
$$

$$\leq \|x_{n+1}-x_n\| + \left|\frac{\lambda_n-\lambda_{n+1}}{\lambda_n}\right| \|x_n\| + |\lambda_{n+1}-\lambda_n|\|Mx_n\| + \left|\frac{\lambda_n-\lambda_{n+1}}{\lambda_n}\right| \|J^A_{\lambda_n}(x_n-\lambda_n Mx_n)\|.$$

Consider that

$$y_{n+1}-y_n = \sigma_{n+1}(x_{n+1}-x_n) + (1-\sigma_{n+1})\big(J^A_{\lambda_{n+1}}(I-\lambda_{n+1}M)x_{n+1} - J^A_{\lambda_n}(I-\lambda_n M)x_n\big)$$
$$+ (\sigma_{n+1}-\sigma_n)(x_n - J^A_{\lambda_n}(x_n-\lambda_n Mx_n).$$

Then, we have

$$\begin{aligned}
&\|y_{n+1}-y_n\| \\
&\leq \sigma_{n+1}\|x_{n+1}-x_n\| + (1-\sigma_{n+1})\|J^A_{\lambda_{n+1}}(I-\lambda_{n+1}M)x_{n+1} - J^A_{\lambda_n}(I-\lambda_n M)x_n\| \\
&\quad + (\sigma_{n+1}-\sigma_n)\|x_n - J^A_{\lambda_n}(x_n-\lambda_n Mx_n)\| \\
&\leq \sigma_{n+1}\|x_{n+1}-x_n\| + |\sigma_{n+1}-\sigma_n|\|x_n - J^A_{\lambda_n}(x_n-\lambda_n Mx_n)\| + (1-\sigma_{n+1}) \\
&\quad \times \left(\|x_{n+1}-x_n\| + \left|\frac{\lambda_n-\lambda_{n+1}}{\lambda_n}\right| \|x_n\| + |\lambda_{n+1}-\lambda_n|\|Mx_n\| + \left|\frac{\lambda_n-\lambda_{n+1}}{\lambda_n}\right| \|J^A_{\lambda_n}(x_n-\lambda_n Mx_n)\|\right) \\
&\leq \|x_{n+1}-x_n\| + |\sigma_{n+1}-\sigma_n|\|x_n - J^A_{\lambda_n}(x_n-\lambda_n Mx_n)\| \\
&\quad + (1-\sigma_{n+1})\left(\left|\frac{\lambda_n-\lambda_{n+1}}{\lambda_n}\right| \|x_n\| + |\lambda_{n+1}-\lambda_n|\|Mx_n\| + \left|\frac{\lambda_n-\lambda_{n+1}}{\lambda_n}\right| \|J^A_{\lambda_n}(x_n-\lambda_n Mx_n)\|\right). \qquad (3.5)
\end{aligned}$$

Step 2.3. To show that $\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \leq 0$. From (3.4) and (3.5) we derive that

$$\begin{aligned}
&\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \\
&\quad \leq \frac{\alpha_{n+1}}{(1-\beta_{n+1})}\|f(x_{n+1})+y_{n+1}\| - \frac{\alpha_n}{(1-\beta_n)}\|f(x_n)-y_n\| + |\sigma_{n+1}-\sigma_n|\|x_n - J^A_{\lambda_n}(x_n-\lambda_n Mx_n)\| \\
&\qquad + (1-\sigma_{n+1})\left[\left|\frac{\lambda_n-\lambda_{n+1}}{\lambda_n}\right| \|x_n\| + |\lambda_{n+1}-\lambda_n|\|Mx_n\| + \left|\frac{\lambda_n-\lambda_{n+1}}{\lambda_n}\right| \|J^A_{\lambda_n}(x_n-\lambda_n Mx_n)\|\right]. \qquad (3.6)
\end{aligned}$$

It follows from condition (a)-(d), we obtain

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \leq 0. \qquad (3.7)$$

By using Lemma 2.10, we obtain that

$$\limsup_{n\to\infty}\|z_n-x_n\| = 0. \qquad (3.8)$$

Since $x_{n+1}-x_n = (1-\beta_n)(z_n-x_n)$, therefore

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \qquad (3.9)$$

Next, we prove that $\|y_n-x_n\| \to 0$, as $n\to\infty$.

Since $z_n = \frac{\alpha_n f(x_n)+\gamma_n y_n}{(1-\beta_n)} = \frac{\alpha_n}{(1-\beta_n)}f(x_n) + (1-\frac{\alpha_n}{(1-\beta_n)})y_n$, we have

$$\begin{aligned}
\|z_n-y_n\| &= \left\|\frac{\alpha_n}{(1-\beta_n)}f(x_n) + \left(1-\frac{\alpha_n}{(1-\beta_n)}\right)y_n - y_n\right\| \\
&= \left\|\frac{\alpha_n}{(1-\beta_n)}f(x_n) - \frac{\alpha_n}{(1-\beta_n)}y_n\right\| \\
&\leq \frac{\alpha_n}{(1-\beta_n)}\|f(x_n)-y_n\|. \qquad (3.10)
\end{aligned}$$

By condition (a) we obtain that

$$\lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{3.11}$$

From (3.8) and (3.10), we get that

$$\|y_n - x_n\| \le \|y_n - z_n\| + \|z_n - x_n\| \to 0, \ as\ n \to \infty. \tag{3.12}$$

Therefore

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.13}$$

Step 3. To show that $\lim_{n\to\infty} \|x_n - J^A_{\lambda_n}(I - \lambda_n M)x_n\| = 0$ and $\lim_{n\to\infty} \|Tx_n - Q^B_u Tx_n\| = 0$.

Step 3.1. We want to show that $\lim_{n\to\infty} \|x_n - J^A_{\lambda_n}(I - \lambda_n M)x_n\| = 0$. Consider

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - p\|^2 \\
&= \|\alpha_n(f(x_n) - f(p)) + \beta_n(x_n - p) + \gamma_n(y_n - p) + \alpha_n(f(p) - p)\|^2 \\
&\le \alpha_n \|f(x_n) - f(p)\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|y_n - p\|^2 + 2\alpha_n \langle f(p) - p, x_{n+1} - p\rangle \\
&\le \alpha_n k \|x_n - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|y_n - p\|^2 + 2\alpha_n \langle f(p) - p, x_{n+1} - p\rangle \\
&= \alpha_n k \|x_n - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|x_n - p\|^2 \\
&\quad - \gamma_n \sigma_n (1 - \sigma_n) g(\|x_n - J^A_{\lambda_n}(I - \lambda_n M)x_n\|) + 2\alpha_n \langle f(p) - p, x_{n+1} - p\rangle \\
&= [\alpha_n k + \beta_n + \gamma_n]\|x_n - p\|^2 - \gamma_n \sigma_n (1 - \sigma_n) g(\|x_n - J^A_{\lambda_n}(I - \lambda_n M)x_n\|) \\
&\quad + 2\alpha_n \langle f(p) - p, x_{n+1} - p\rangle \\
&\le \|x_n - p\|^2 - \gamma_n \sigma_n (1 - \sigma_n) g(\|x_n - J^A_{\lambda_n}(I - \lambda_n M)x_n\|) + 2\alpha_n \langle f(p) - p, x_{n+1} - p\rangle.
\end{aligned}$$

It follow that

$$\begin{aligned}
&\gamma_n \sigma_n (1 - \sigma_n) g(\|x_n - J^A_{\lambda_n}(I - \lambda_n M)x_n\|) \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n \langle f(p) - p, x_{n+1} - p\rangle \\
&\quad \le \|x_{n+1} - x_n\| (\|x_n - p\| + \|x_{n+1} - p\|) + 2\alpha_n \langle f(p) - p, x_{n+1} - p\rangle.
\end{aligned}$$

By condition (a), (d), (e), (3.9) and by property of g, we have

$$\lim_{n\to\infty} \|x_n - J^A_{\lambda_n}(I - \lambda_n M)x_n\| = 0. \tag{3.14}$$

Step 3.2 We want to show that $\lim_{n\to\infty} \|Tx_n - Q^B_u Tx_n\| = 0$. We consider

$$\begin{aligned}
\|y_n - p\|^2 &= \|\sigma_n x_n + (1 - \sigma_n) J^A_{\lambda_n}(I - \lambda_n M)x_n - p\|^2 \\
&\le \sigma_n \|x_n - p\|^2 + (1 - \sigma_n)\|J^A_{\lambda_n}(I - \lambda_n M)x_n - p\|^2 \\
&\le \sigma_n \|x_n - p\|^2 + (1 - \sigma_n)\|(I - \lambda_n M)x_n - p\|^2 \\
&\le \sigma_n \|x_n - p\|^2 + (1 - \sigma_n)\|(x_n - p) - \lambda_n M x_n\|^2 \\
&\le \sigma_n \|x_n - p\|^2 + (1 - \sigma_n)\{\|x_n - p\|^2 - 2\lambda_n \langle x_n - p, T^* J_E(Tx_n - Q^B_u Tx_n)\rangle \\
&\quad + \lambda^2 \|T^* J_E(Tx_n - Q^B_u Tx)\|^2\} \\
&= \sigma_n \|x_n - p\|^2 + (1 - \sigma_n)\{\|x_n - p\|^2 - 2\lambda_n \langle Tx_n - Tp, J_E(Tx_n - Q^B_u Tx_n)\rangle \\
&\quad + \lambda^2 \|T^* J_E(Tx_n - Q^B_u Tx_n)\|^2\}
\end{aligned}$$

$$
\begin{aligned}
&= \sigma_n \|x_n - p\|^2 + (1-\sigma_n)\{\|x_n - p\|^2 - 2\lambda_n \langle Tx_n - Q_u^B Tx_n + Q_u^B Tx_n - Tp, J_E(Tx_n - Q_u^B Tx_n)\rangle \\
&\quad + \lambda^2 \|T^* J_E(Tx_n - Q_u^B Tx_n)\|^2\} \\
&= \sigma_n \|x_n - p\|^2 + (1-\sigma_n)\{\|x_n - p\|^2 - 2\lambda_n \langle Tx_n - Q_u^B Tx_n, J_E(Tx_n - Q_u^B Tx_n)\rangle \\
&\quad - 2\lambda_n Q_u^B Tx_n - Tp, J_E(Tx_n - Q_u^B Tx_n)\rangle + \lambda^2 \|T\|^2 \|Tx_n - Q_u^B Tx_n\|^2\} \\
&\le \sigma_n \|x_n - p\|^2 + (1-\sigma_n) \\
&\quad \{\|x_n - p\|^2 - 2\lambda_n \langle Tx_n - Q_u^B Tx_n, J_E(Tx_n - Q_u^B Tx_n)\rangle + \lambda^2 \|T\|^2 \|Tx_n - Q_u^B Tx_n\|^2\} \\
&\le \|x_n - p\|^2 - 2\lambda_n (1-\sigma_n) \|Tx_n - Q_u^B Tx_n\|^2 + \lambda_n^2 (1-\sigma_n) \|T\|^2 \|Tx_n - Q_u^B Tx_n\|^2 \\
&\le \|x_n - p\|^2 - (1-\sigma_n)(2\lambda_n - \lambda^2 \|T\|^2) \|Tx_n - Q_u^B Tx\|^2.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - p\|^2 \\
&\le \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|y_n - p\|^2 \\
&= \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n (\|x_n - p\|^2 - (1-\sigma_n)(2\lambda_n - \lambda_n^2 \|T\|^2) \|Tx_n - Q_\mu^B Tx_n\|^2) \\
&= \alpha_n \|f(x_n) - p\|\| + (1-\alpha_n) \|x_n - p\|^2 - \gamma_n (1-\sigma_n)(2\lambda_n - \lambda_n^2 \|T\|^2) \|Tx_n - Q_\mu^B Tx_n\|^2 \\
&\le \alpha_n \|f(x_n) - p\|\| + \|x_n - p\|^2 - \gamma_n (1-\sigma_n)(2\lambda_n - \lambda_n^2 \|T\|^2) \|Tx_n - Q_\mu^B Tx_n\|^2.
\end{aligned}
$$

It follow that

$$
\begin{aligned}
\gamma_n (1-\sigma_n)(2\lambda_n - \lambda_n^2 \|T\|^2) \|Tx_n - Q_\mu^B Tx_n\|^2 &\le \alpha_n \|f(x_n) - p\|\| + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&= \alpha_n \|f(x_n) - p\|\| + \|x_n - x_{n+1}\| \|(x_n - p) + (x_{n+1} - p)\|.
\end{aligned}
$$

Since $\sigma_n \in [0,1)$, and by condition (a), (c) and (3.9), then we have

$$
\lim_{n\to\infty} \|Tx_n - Q_u^B Tx_n\| = 0. \tag{3.15}
$$

Step 3.3. To show that $\lim_{n\to\infty} \|T^* J_E(Tx_n - Q_u^B x_n)\| = 0$. Since T is bounded linear and T^* is adjoint operator of T, we have

$$
\begin{aligned}
\|T^* J_E(Tx_n - Q_u^B x_n)\|^2 &\le \|T\|^2 \|J_E(Tx_n - Q_u^B Tx_n)\|^2 \\
&= \|T\|^2 \|Tx_n - Q_u^B Tx_n\|^2.
\end{aligned} \tag{3.16}
$$

By (3.15) then

$$
\lim_{n\to\infty} \|T^* J_E(Tx_n - Q_u^B Tx_n)\| = 0. \tag{3.17}
$$

Step 4. We want to show that$\limsup_{n\to\infty} \langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle \le 0$, $\forall\ x_n \in \Omega$, where $\bar{x} = Q_\Omega f(\bar{x})$.

Step 4.1. Set $W := \left[I + \frac{\gamma_n(1-\sigma_n)}{(1-\alpha_n)} J_{\lambda_n}^A (I - \lambda_n M)\right]$, where $M := T^* J_E(T - Q_\mu^B T)$. Then W is nonexpansive mapping. We see that $x_{n+1} := \alpha_n f(x_n) + (\alpha_n) W x_n$.

By apply Lemma 2.4, then we let x_t be a unique solution of equation $x_t = t f(x_t) + (1-t) W x_t$, $\forall\ t \in (0,1)$. Namely, x_t be a fixed point of contraction mapping which is unique fixed point. Putting $\bar{x} = Q_\Omega f(\bar{x}) = \lim_{t\to 0} x_t$, where $Q_\Omega f(\bar{x})$ is an unique sunny nonexpansive retraction from C onto $Fix(W)$, as $t \to 0$.

Since we know that $Fix(W) = Fix(J_{\lambda_n}^A) \cap Fix(Q_\mu^B(T)) = \Omega$. Thus, we know that $x_t \to \bar{x} = Q_\Omega f(\bar{x})$. By Lemma 2.4, then there exist $\langle f(\bar{x}) - \bar{x}, q - \bar{x}\rangle \le 0$, $\forall\ f \in \Xi_c$, $\forall\ q \in \Omega$.

Step 4.2 To show that $x_n \to q \in A^{-1}(0)$.

Next, we show that there exist q such that $0 \in Aq$. Let $s_n := J^A_{\lambda_n}(I - \lambda_n T^* J_E(T - Q^B_\mu T)x_n$. We obtain

$$\begin{aligned}
&(I - \lambda_n T^* J_E(T - Q^B_\mu T)x_n \in (I + \lambda_n A)s_n \\
&x_n - \lambda_n T^* J_E(Tx_n - Q^B_\mu Tx_n) \in s_n + \lambda_n As_n \\
&x_n - \lambda_n T^* J_E(Tx_n - Q^B_\mu Tx_n) \in s_n + \lambda_n As_n \\
&x_n - s_n - \lambda_n T^* J_E(Tx_n - Q^B_\mu Tx_n) \in \lambda_n As_n \\
&\frac{1}{\lambda_n}(x_n - s_n - \lambda_n T^* J_E(Tx_n - Q^B_\mu Tx_n)) \in As_n. \qquad (3.18)
\end{aligned}$$

Next, we show that $\lim_{n\to\infty} \|x_n - s_n - \lambda_n T^* J_E(Tx_n - Q^B_\mu Tx_n)\| = 0$, we observe that

$$\begin{aligned}
\|x_n - s_n - \lambda_n T^* J_E(Tx_n - Q^B_\mu Tx_n)\| &= \|(x_n - s_n) - \lambda_n T^* J_E(Tx_n - Q^B_\mu Tx_n)\| \\
&\le \|x_n - s_n\| + \frac{1}{\lambda_n}\|T^* J_E(Tx_n - Q^B_\mu Tx_n)\|.
\end{aligned}$$

From (3.12),we get $\|x_n - s_n\| \to 0$, as $n \to \infty$ and (3.14), we derive that

$$\lim_{n\to\infty} \|x_n - s_n - \lambda_n T^* J_E(Tx_n - Q^B_\mu Tx_n)\| = 0. \qquad (3.19)$$

By (3.18) we obtain that $0 \in As_n$.

Since $\lim_{n\to\infty} \|x_n - s_n\| = 0$, and the boundedness of $\{x_n\}$ such that $\{x_n\}$ has a weakly convergence subsequence, i.e., $x_{n_i} \rightharpoonup q$. From $0 \in As_n$, there exist q solve $0 \in Aq$.

Step 4.3. To show that $x_n \to q \in T^{-1}(B^{-1}(0))$. Namely, we show that for some q such that Tq solves $0 \in B(Tq)$. Since Q^B_μ is nonexpansive, then we apply the demiclose principle, i.e., $(I - Q^B_\mu)$ is demiclose at zero. Since T is linear and bounded, then we have $Tx_{n_i} \rightharpoonup Tq$ and from $\|Tx_n - Q^B_\mu Tx_n\| \to 0, as\ n \to \infty$, this implies that $Tq = Q^B_\mu Tq$. Therefore $q \in \Omega$.

Next, we show that $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup q'$. That is $q' \in Fix(W) = Fix(J^A_r) \cap Fix(Q^B_\mu(T)) = \Omega$. We want to show that $q = q'$. Assume that $q \neq q'$, by Opial's condition we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - q\| = \liminf_{i\to\infty}\|x_{n_i} - q\| &< \liminf_{i\to\infty}\|x_{n_i} - q'\| \\
&= \lim_{n\to\infty} \|x_n - q'\| = \lim_{j\to\infty} \|x_{n_j} - q'\| \\
&\le \liminf_{j\to\infty}\|x_{n_j} - q\| = \lim_{n\to\infty} \|x_n - q\|.
\end{aligned}$$

This contradiction. Thus, we have $q = q'$. This implies that $x_n \rightharpoonup q \in \Omega$.

Step 4.4. From above step we have $q \in \Omega$. From step 4.1 there exist $\langle f(\bar{x}) - \bar{x}, q - \bar{x}\rangle \le 0$, $\forall q \in \Omega$, where $\bar{x} = Q_\Omega f(\bar{x})$.

Next, we show that $\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle \le 0$, $\forall\ x_n \in \Omega$, where $\bar{x} = Q_\Omega f(\bar{x})$.

From step 4.3 we have $\langle f(\bar{x}) - \bar{x}, q - \bar{x}\rangle \le 0$, $\forall f \in \Xi_c$, $\forall q \in \Omega$. To show this, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle &= \lim_{i\to\infty} \langle f(\bar{x}) - \bar{x}, x_{n_i} - \bar{x}\rangle \qquad (3.20) \\
&= \langle f(\bar{x}) - \bar{x}, q - \bar{x}\rangle \le 0. \qquad (3.21)
\end{aligned}$$

This implies that $\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},x_{n+1}-\bar{x})\rangle = \limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},x_n-\bar{x})\rangle \le 0$ too.

Step 5. Finally, we prove that $\|x_n-\bar{x}\|\to 0$. We consider

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &= \|\alpha_n f(x_n)+\beta_n x_n+\gamma_n y_n-\bar{x}\|^2\\
&= \|\alpha_n(f(x_n)-f(\bar{x}))+\beta_n(x_n-\bar{x})+\gamma_n(y_n-\bar{x})+\alpha_n(f(\bar{x})-\bar{x})\|^2\\
&\le \|\alpha_n(f(x_n)-f(\bar{x}))+\beta_n(x_n-\bar{x})+\gamma_n(y_n-\bar{x})\|^2+2\alpha_n\langle(f(\bar{x})-\bar{x}),x_{n+1}-\bar{x}\rangle\\
&\le \alpha_n\|f(x_n)-f(\bar{x})\|^2+\beta_n\|x_n-\bar{x}\|^2+\gamma_n\|y_n-\bar{x}\|^2+2\alpha_n\langle f(x_n)-\bar{x},x_{n+1}-\bar{x}\rangle\\
&\le \alpha_n k\|x_n-\bar{x}\|^2+\beta_n\|x_n-\bar{x}\|^2+\gamma_n\|x_n-\bar{x}\|^2+2\alpha_n\langle f(x_n)-\bar{x},x_{n+1}-\bar{x}\rangle\\
&= [\alpha_n k+(1-\alpha_n)]\|x_n-\bar{x}\|^2+2\alpha_n\langle f(x_n)-\bar{x},x_{n+1}-\bar{x}\rangle\\
&= (1-\alpha_n(1-k))\|x_n-\bar{x}\|^2+\alpha_n(1-k)\frac{2}{(1-k)}\langle f(x_n)-\bar{x},x_{n+1}-\bar{x}\rangle.
\end{aligned}$$

This implies that

$$\|x_{n+1}-\bar{x}\|^2 \le (1-\alpha_n(1-k))\|x_n-\bar{x}\|^2+\alpha_n(1-k)\frac{2}{(1-k)}\langle f(\bar{x})-\bar{x},x_{n+1}-\bar{x}\rangle.$$

By condition (a) and (3.20), then we apply Lemma 2.11 to conclude that

$$\lim_{n\to\infty}\|x_n-\bar{x}\|=0.$$

This completes the proof. □

If the space $E=H$ a Hilbert space. In a Hilbert space, we know that we $J_E=I$ is nonexpansive and linear operator, and $c_1=1$. Then we obtain the following corollary.

Corollary 3.3. *Let H_1, H_2 be two Hilbert spaces and let C C be a nonempty closed convex subset of H_1. Let $A:H_1\to 2^{H_1}$ be a maximal monotone operator such that $A^{-1}(0)\neq\emptyset$ and let $B:A:H_2\to 2^{H_2}$ be a maximal monotone operator such that $B^{-1}(0)\neq\emptyset$. Let $J_{\lambda_n}^A=(I+\lambda_n A)^{-1}$ be the resolvent of A for $\lambda_n>0$ and $J_\mu^B=(I+\mu B)^{-1}$ be the resolvent of B for $\mu>0$. Let $f:C\to C$ be a $k-$contraction mapping with $k\in(0,1)$. Let $T:H_1\to H_2$ be a bounded linear operator such that $T\neq\emptyset$ and $T^*:H_2\to H_1$ be an adjoint operator of T. Assume that $\Omega:=A^{-1}(0)\cap T^{-1}(B^{-1}(0))\neq\emptyset$. Let $x_1\in H_1$, and let $\{x_n\}$ be a sequence generate by the following*

$$\begin{cases} y_n=\sigma_n x_n+(1-\sigma_n)J_{\lambda_n}^A(x_n-\lambda_n T^*(Tx_n-J_\mu^B Tx_n)),\\ x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+\gamma_n y_n;\ \ \forall\ n\ge 0,\end{cases} \tag{3.22}$$

where $\{\sigma_n\},\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}\subset(0,1)$ and $\alpha_n+\beta_n+\gamma_n=1$. Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(b) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$;

(c) $0<\liminf_{n\to\infty}\lambda_n\le\lambda_n\le\limsup_{n\to\infty}\lambda_n<\frac{2}{\|T\|^2}$, *and* $\lim_{n\to\infty}|\lambda_{n+1}-\lambda_n|=0$;

(d) $0<\liminf_{n\to\infty}\sigma_n\le\sigma_n\le\limsup_{n\to\infty}\sigma_n<1$, *and* $\lim_{n\to\infty}|\sigma_{n+1}-\sigma_n|=0$;

(e) $0<\liminf_{n\to\infty}\gamma_n\le\gamma_n$.

Then $\{x_n\}$ converges strongly to a point $\bar{x}\in\Omega$, where $\bar{x}=Q_\Omega f(\bar{x})$.

Proof. Since, an operator $M := T^*(T - Q_\mu^B T)$ is inverse strongly monotone in Hilbert spaces. Moreover $(I - \lambda_n M)$ is nonexpansive. The proof is same the our main theorem. □

3.1 Split Minimization Problems

In this part, we consider our result for solve the split minimize problem.

3.1.1 Split Minimization Problem Between Banach spaces and Hilbert spaces

We consider our result for solve the split minimize problem between Banach spaces and Hilbert spaces. The split minimization Problem is formulated as follows:

$$\begin{aligned} &\text{find a point } \hat{x} \in H \text{ that solves } \hat{x} = \operatorname*{argmin}_{x \in H} \phi(x), \\ &\text{and } \hat{y} = T\hat{x} \in E \text{ that solves } \hat{y} = \operatorname*{argmin}_{x \in E} \varphi(y), \end{aligned} \tag{3.23}$$

where $\phi : H \to \mathbb{R}$ and $\varphi : E \to \mathbb{R}$ be two proper convex and lower semicontinuous function. The subdifferential $\partial\phi$ of ϕ is generated by

$$\partial\phi(x) = \{z \in H : \phi(y) \geq \langle y - x, z\rangle + \phi(x);\ \forall\ y \in H\}.$$

The subdifferential of φ at x, for $x \in E$ is generated by

$$\partial\varphi(x) = \{x^* \in E^* : \varphi(y) \geq \langle y - x, x^*\rangle + \varphi(x);\ \forall\ y \in E\}$$

We known that the subdifferential operator $\partial\varphi(x) : E \to 2^{E^*}$ is maximal monotone [22, 23]. Then we can see that $(\partial\varphi)^{-1}(0) = \operatorname{argmin}\{\varphi(x) : x \in E\}$. We set $B = \partial\varphi$ and Q_μ^B is metric resolvent of $\partial\varphi$, for $\mu > 0$, then we know that

$$Q_\mu^B(x) = \operatorname*{Prox}_{\mu\varphi}(x) = \operatorname{argmin}\left\{\varphi(y) + \frac{1}{2\mu}\|y - x\|^2\right\}, \quad y \in E.$$

Also, if we take $A = \partial\varphi$ and $J_{\lambda_n}^A$ is a resolvent of $\partial\phi$, $\lambda_n > 0$, then we know that

$$J_{\lambda_n}^A(x) = \operatorname*{Prox}_{\lambda_n\phi}(x) = \operatorname{argmin}\left\{\phi(y) + \frac{1}{2\lambda_n}\|y - x\|^2\right\}, \quad y \in H.$$

We take $A = \partial\phi$ and $B = \partial\varphi$ in our main theorem, then we obtain the new iterative scheme (3.1) becomes that

$$\begin{cases} y_n = \sigma_n x_n + (1 - \sigma_n)\operatorname{Prox}_{\lambda_n\phi}(x_n - \lambda_n T^* J_E(Tx_n - \operatorname{Prox}_{\mu\varphi} Tx_n)), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n; \quad \forall\ n \geq 0. \end{cases} \tag{3.24}$$

3.1.2 Split Minimization Problem Between in Hilbert spaces

Next, we consider our Corollary 3.3 for solve the split minimize problem in two Hilbert spaces. The split minimization problem is formulated as follows:

$$\begin{aligned} &\text{find a point } \hat{x} \in H_1 \text{ that solves } \hat{x} = \operatorname*{argmin}_{x \in H_1} \phi(x), \\ &\text{and } \hat{y} = T\hat{x} \in H_2 \text{ that solves } \hat{y} = \operatorname*{argmin}_{x \in H_2} \varphi(y), \end{aligned} \tag{3.25}$$

where ϕ and φ be two proper convex and lower semicontinuous function. For $\phi : H_1 \to \mathbb{R}$, the subdifferential $\partial\phi$ of ϕ is defined by

$$\partial\phi(x) = \{z \in H_1 : \phi(y) \geq \langle y - x, z\rangle + \phi(x);\ \forall\ y \in H_1\}.$$

For $\varphi : H_2 \to \mathbb{R}$ be a proper convex and lower semicontinuous function, the subdifferential $\partial\varphi$ of φ defined by

$$\partial\varphi(x) = \{z \in H_2 : \varphi(y) \geq \langle y - x, z\rangle + \varphi(x);\ \forall\ y \in H_2\}.$$

In Hilbert spaces, from [27] we know that $0 \in \partial\phi(x) \Leftrightarrow \phi(x) = \min_{y\in H_1} \phi(y)$ and also $0 \in \partial\varphi(x) \Leftrightarrow \varphi(x) = \min_{y\in H_2} \varphi(y)$. The set of minimizers of ϕ defined by

$$\operatorname{argmin} \phi(y) = \{x \in H : f(x) = \min \phi(y),\ y \in H_1\},$$

and the set of minimizers of φ defined by

$$\operatorname{argmin} \varphi(y) = \{x \in H : f(x) = \min \varphi(y),\ y \in H_2\}.$$

We know that

$$J^A_{\lambda_n}(x) = \operatorname*{Prox}_{\lambda_n\phi}(x) = \operatorname{argmin}\left\{\phi(y) + \frac{1}{2\lambda_n}\|y - x\|^2,\ y \in H_1\right\},$$

and

$$J^A_{\mu}(x) = \operatorname*{Prox}_{\mu\phi}(x) = \operatorname{argmin}\left\{\phi(y) + \frac{1}{2\mu}\|y - x\|^2,\ y \in H_2\right\}.$$

Also, if we take $A = \partial\phi$ and $B = \partial\varphi$ in our Corollary 3.3, then we obtain the new iterative scheme (3.22) change to the iterative scheme (3.26).

$$\begin{cases} y_n = \sigma_n x_n + (1 - \sigma_n)\operatorname{Prox}_{\lambda_n\phi}(x_n - \lambda_n T^*(Tx_n - \operatorname{Prox}_{\mu\varphi} Tx_n)), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n;\ \forall\ n \geq 0. \end{cases} \tag{3.26}$$

4. Conclusion

In this paper, we defined a new iterative scheme for approximation the split common null point problems for set-valued maximal monotone operators by using a viscosity method and some fixed point technically proving method between Banach spaces and Hilbert spaces. We obtained the strong convergence theorem for set-valued maximal monotone operators. We also applied that our result can be solves the split minimization problems.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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