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Co-small Module

Research Article

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Abstract. The main goal of the present article is to study basic properties of co-small modules. Let R be a Noetherian ring, For all co-small module B and index I, we get isomorphic $\mathrm{Tor}_n(B, \prod A_i) \cong \prod \mathrm{Tor}_n(B, A_i)$. Finally, we prove that If two modules of the sequence $0 \to A \to B \to C \to 0$ are co-small modules, so is the third.

Keywords. Co-small modules; Strong modules; Semisimple and Noetherian ring

MSC. 13E05; 16P40

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1. Introduction

Throughout the paper, a ring R means an associative ring with unit, and a module means a right R-module over an arbitrary ring R. We will use the letter R for a ring in all claims.

Let R be an artinian ring. A module M is called co-small if the functor $M \otimes_R$-commutes with direct products of arbitrary module. The notion of a compact object of a category, i.e., an object c for which the covariant functor $\mathrm{Hom}(c, -)$ commutes with all direct sums, has appeared as a natural tool in many branches of module theory. Small module, which are precisely compact objects of the category of all modules over a ring, have been useful in study of the structure theory of graded rings and almost free modules. The most recent motivation of this topic comes from the context of representable equivalences of module categories. A self-small module, which can be defined as a compact object c of the category of direct summand of all direct sums of copies of c, was introduced in Arnold and Murley (1975) as a tool for generalization of Baeer's lemma. Nevertheless, self-small modules, similarly to small modules, turn out to be important in study of generalization of Morita equivalence. Let M be a module and N its submodule, then by [5] denote $V_M(N) = \{f \in \mathrm{End}(M) \mid f(N) = 0\}$. Recall that M is a small module if each increasing chain of submodule $N_n \subseteq N_{n+1}$ of M for which $M = \bigcup_{n<w} N_n$, there exists n such that $N_n = M$. Moreover, a self-small module M can be characterized in similar way: if $M = \bigcup_{n<w} N_n$ for a increasing chain of M, then there exists n such that $V_M(N_n) = 0$. A right strongly steady ring as a ring over which every right self-small module is necessary finitely generated.

2. Co-small Module

We begin with the following

Definition 2.1 Let R be an artinian ring. A module M is called co-small if the functor $M\otimes_R$- commutes with direct products of arbitrary module.

Lemma 2.2 *Let R be an artinian ring and $(A_i \mid i \in I)$ be a system of modules with an index set I. If B is a co-small module, then* $\mathrm{Tor}_1(B, \prod A_i) \cong \prod \mathrm{Tor}_1(B, A_i)$.

Proof. For every $i \in I$, there exists an exact sequence

$$0 \longrightarrow K_i \longrightarrow P_i \longrightarrow A_i \longrightarrow 0 \tag{2.1}$$

where P_i is projective, which induces an exact sequence

$$0 \longrightarrow \prod K_i \longrightarrow \prod P_i \longrightarrow \prod A_i \longrightarrow 0 \tag{2.2}$$

and $\prod P_i$ is projective, hence there is a commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Tor}_1(B,\prod P_i) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Tor}_1(B,\prod A_i) \arrow[r] \arrow[d] & B\otimes(\prod K_i) \arrow[r] \arrow[d, "\simeq"] & B\otimes(\prod P_i) \arrow[d, "\simeq"] \\
\prod \mathrm{Tor}_1(B,P_i) \arrow[r] & \prod \mathrm{Tor}_1(B,A_i) \arrow[r] & \prod(B\otimes K_i) \arrow[r] & \prod(B\otimes P_i)
\end{tikzcd}$$

by the five lemmas, $\mathrm{Tor}_1(B, \prod A_i) \cong \prod \mathrm{Tor}_1(B, A_i)$. □

Theorem 2.3 *Let R be an artinian ring and $(A_i \mid i \in I)$ be a system of modules with an index set I. Then* $\mathrm{Tor}_n(B, \prod A_i) \cong \prod \mathrm{Tor}_n(B, A_i)$.

Proof. It follows by induction on n. By Lemma 2.2, it is true for $n = 1$, if $n > 1$, by (2.1) and (2.2) in Lemma 2.2 and long exact sequence lemma, there is a diagram

$$\begin{tikzcd}
\mathrm{Tor}_n(B,\prod P_i) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Tor}_n(B,\prod A_i) \arrow[r, "\delta"] \arrow[d] & \mathrm{Tor}_{n-1}(B,\prod K_i) \arrow[r] \arrow[d, "\psi"] & \mathrm{Tor}_{n-1}(B,\prod P_i) \arrow[d, "\simeq"] \\
\prod \mathrm{Tor}_n(B,P_i) \arrow[r] & \prod \mathrm{Tor}_n(B,A_i) \arrow[r, "\delta'"] & \prod \mathrm{Tor}_{n-1}(B,K_i) \arrow[r] & \prod \mathrm{Tor}_{n-1}(B,\prod P_i)
\end{tikzcd}$$

The first and last terms in each row are 0, so exactness gives δ and δ' isomorphism. By induction, there is an isomorphism ψ, and $(\delta')^{-1}\psi\delta\colon \mathrm{Tor}_n(B, \prod A_i) \cong \prod \mathrm{Tor}_n(B, A_i)$. □

Proposition 2.4 *Let R be an artinian ring, and $0 \to A \to B \to C \to 0$ be a short exact sequence of modules. If A and C are co-small modules, then B is co-small module.*

Proof. For every index set I and $i \in I$, $- \otimes (\prod L_i)$ is right exact, then

$$\begin{array}{ccccccccc}
\mathrm{Tor}_1(C,\prod L_i) & \longrightarrow & A\otimes(\prod L_i) & \longrightarrow & B\otimes(\prod L_i) & \longrightarrow & C\otimes(\prod L_i) & \longrightarrow & 0 \\
\downarrow \simeq \text{Lemma 3.2} & & \downarrow \simeq & & \downarrow & & \downarrow \simeq & & \downarrow \simeq \\
\prod \mathrm{Tor}_1(C,L_i) & \longrightarrow & \prod(A\otimes L_i) & \longrightarrow & \prod(B\otimes L_i) & \longrightarrow & \prod(C\otimes L_i) & \longrightarrow & 0
\end{array}$$

by the five lemmas, $B \otimes (\prod L_i) \cong \prod(B \otimes L_i)$, i.e., B is co-small module. □

Corollary 2.5 *Let R be an artinian ring. If M_1 and M_2 are co-small modules, then $M_1 \oplus M_2$ is co-small. In particular, If every $(M_i \mid i \in I)$ is co-small module for every finite set I, then $\prod M_i$ is co-small module.*

Proposition 2.6 *Let R be an artinian ring, If two modules of the sequence $0 \to A \to B \to C \to 0$ are co-small modules, so is the third.*

Definition 2.7 A module M is called strong if the functor $\mathrm{Hom}_R(M,-)$ change direct sums to direct products, i.e., for every index set I and a system of modules $(A_i \mid i \in I)$, there is an isomorphic $\mathrm{Hom}(B,\coprod A_i) \cong \prod \mathrm{Hom}(B,A_i)$

Proposition 2.8 *Let M be a strong module and $(A_i \mid i \in I)$ be a system of modules with every index set I. Then $\mathrm{Ext}^1(M,\coprod A_i) \cong \prod \mathrm{Ext}^1(M,A_i)$.*

Proof. For every $i \in I$, there exists an exact sequence

$$0 \longrightarrow A_i \longrightarrow E_i \longrightarrow K_i \longrightarrow 0 \tag{2.3}$$

where E_i is injective, which induces an exact sequence

$$0 \longrightarrow \coprod A_i \longrightarrow \coprod E_i \longrightarrow \coprod K_i \longrightarrow 0 \tag{2.4}$$

and $\coprod E_i$ is injective, hence there is a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}(M,\coprod E_i) & \longrightarrow & \mathrm{Hom}(M,\coprod K_i) & \longrightarrow & \mathrm{Ext}^1(M,\coprod A_i) & \longrightarrow & \mathrm{Ext}^1(M,\coprod E_i) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow & & \downarrow \simeq \\
\prod \mathrm{Hom}(M,E_i) & \longrightarrow & \prod \mathrm{Hom}(M,K_i) & \longrightarrow & \prod \mathrm{Ext}^1(M,A_i) & \longrightarrow & \prod \mathrm{Ext}^1(M,E_i)
\end{array}$$

the last terms in each row are 0, by five lemmas, then $\mathrm{Ext}^1(M,\coprod A_i) \cong \prod \mathrm{Ext}^1(M,A_i)$. □

Proposition 2.9 *Let M be a strong module and $(A_i \mid i \in I)$ be a system of modules with every index set I. Then $\mathrm{Ext}^n(M,\coprod A_i) \cong \prod \mathrm{Ext}^n(M,A_i)$.*

Proof. It follows by induction on n. By Proposition 2.8, it is true for $n=1$, if $n>1$, by (2.3) and (2.4) in Proposition 2.8 and long exact sequence lemma, there is a diagram

$$\begin{tikzcd}
\mathrm{Ext}^{n-1}(M,\coprod E_i) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Ext}^{n-1}(M,\coprod K_i) \arrow[r, "\delta"] \arrow[d, "\psi"] & \mathrm{Ext}^{n}(M,\coprod A_i) \arrow[r] \arrow[d] & \mathrm{Ext}^{n}(M,\coprod E_i) \arrow[d, "\simeq"] \\
\prod \mathrm{Ext}^{n-1}(M,E_i) \arrow[r] & \prod \mathrm{Ext}^{n-1}(M,K_i) \arrow[r, "\delta\prime"] & \prod \mathrm{Ext}^{n}(M,A_i \arrow[r] & \prod \mathrm{Ext}^{n}(M,E_i)
\end{tikzcd}$$

The first and last terms in each row are 0, so exactness gives δ and δ' isomorphism. By induction, there is an isomorphism ψ, and $\delta'\psi\delta^{-1}$: $\mathrm{Ext}^n(M,\coprod A_i)\cong\prod\mathrm{Ext}^n(M,A_i)$. □

Proposition 2.10 *Let $0\to A\to B\to C\to 0$ be a short exact sequence of modules. If A and C are strong modules, then B is strong module.*

Proof. Let $(M_i \mid i\in I)$ be a system of modules with index set I. Since $\mathrm{Hom}(-,\coprod M_i)$ and $\mathrm{Hom}(-,M_i)$ are left exact, there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(C,\coprod M_i) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(B,\coprod M_i) \arrow[r] \arrow[d] & \mathrm{Hom}(A,\coprod M_i) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Ext}^1(C,\coprod M_i) \arrow[d, "\simeq"] \\
0 \arrow[r] & \prod\mathrm{Hom}(C,M_i) \arrow[r] & \prod\mathrm{Hom}(B,M_i) \arrow[r] & \prod\mathrm{Hom}(A,M_i) \arrow[r] & \prod\mathrm{Ext}^1(C,M_i)
\end{tikzcd}$$

By five lemma $\mathrm{Hom}(B,\coprod M_i)\cong\prod\mathrm{Hom}(B,M_i)$, then B is strong module. □

Proposition 2.11 *If two modules of the sequence $0\to A\to B\to C\to 0$ are strong modules, so is the third.*

Proposition 2.12 *Let R and S be commutative rings. If A_R and ${}_RB_S$ are strong modules, then $A\otimes_R B$ is strong module.*

Proof. Suppose $\coprod_I L_i$ is right S-module for finite set I, by Adjoint Isomorphism

$$\mathrm{Hom}(A\otimes B,\coprod L_i)\cong\mathrm{Hom}(A,\mathrm{Hom}(B,\coprod L_i)).$$

Since A_R and ${}_RB_S$ are strong modules, we have

$$\mathrm{Hom}(A\otimes_R B,\coprod L_i)\cong\mathrm{Hom}(A,\mathrm{Hom}(B,\coprod L_i))\cong\mathrm{Hom}(A,\prod\mathrm{Hom}(B,L_i))\cong\prod\mathrm{Hom}(A\otimes B,L_i),$$

then $A\otimes_R B$ is strong module. □

Corollary 2.13 *Let R and S be rings. If ${}_RA$ and ${}_SB_R$ are strong modules, then $B\otimes_R A$ is strong module.*

Proposition 2.14 *Let R be a ring. If M_1 and M_2 are strongly small modules, then $M_1\oplus M_2$ is strong module. In particular, if every $(M_i \mid i\in I)$ are strong modules for every finite set I, then $\coprod M_i$ is small module.*

Proof. Let $(L_i \mid i \in I)$ be a system of modules with an index set I. Since M_1 and M_2 are strong module, we obtain $\mathrm{Hom}(M_1 \oplus M_2, \coprod L_i) \cong \mathrm{Hom}(M_1, \coprod L_i) \oplus \mathrm{Hom}(M_2, \coprod L_i) \cong \prod \mathrm{Hom}(M_1, L_i) \oplus \prod \mathrm{Hom}(M_2, L_i) \cong \prod \mathrm{Hom}(M_1 \oplus M_2, L_i)$, then $M_1 \oplus M_2$ is strong module. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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