Journal of Informatics and Mathematical Sciences
Vol. 8, No. 3, pp. 139–150, 2016
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

On Some Homomorphic Extensions of Duo Binary Systems

Research Article

V.D. Madan*

Indira Gandhi National Open University, New Delhi 110068, India
prof.vdmadan@adamsmith.tv

Abstract. This paper explores the following problem: Let $A = (A, +, \cdot)$ and $E = (E, +, \cdot)$ be duo binary systems. Let $S = (S, +, \cdot)$ be a subsystem of A. Under what conditions can a homomorphism $p : S \to E$ be extended to a homomorphism $P : A \to E$ and how?

Keywords. Mono Binary System; Duo Binary System; Ringoid; Ringnoid; Semiring; Subsemiring; Subringoid; Subringnoid; Homomorphism; Ring; Field

MSC. 16D99

Received: February 2, 2015 **Accepted:** August 4, 2016

1. Introduction

This paper needs the following definitions and results:

Definition 1. A duo binary system (termed "bigroupoid" in [6]) is a set A together with two binary operations defined on it, conventionally denoted as addition $(+)$ and multiplication $(\cdot)$. It is written as $A = (A, +, \cdot)$ where

$$a + b \in A, \quad a \cdot b \in A \tag{1.1}$$

for all $a, b \in A$.

Rings and fields are some of the well-known examples of Duo Binary systems contrasting the Mono Binary systems [4] like semigroups, monoids and groups.

A *distributive duo binary system is called a ringoid*. Its formal definition is as follows:

**Mailing address*: J-278, Saket, New Delhi 110017, India.

Definition 2 ([6]). A duo binary system $A = (A,+,\cdot)$ is said to be a Ringoid if the binary operation '$\cdot$' is distributive over the binary operation '+' in the sense that

$$a(b+c) = ab + ac, \quad (b+c)a = ba + ca \tag{1.2}$$

for all $a,b,c \in A$. A ringoid $A = (A,+,\cdot)$ will be abelian if the binary operation '$\cdot$' is commutative in A and it will be known as a ringoid with unity if there exists in A, the neutral element with respect to the second binary operation '$\cdot$'. Accordingly an Abelian ringoid with unity can also be defined. Note that a ringoid can be empty also.

Definition 3 ([6]). A duo binary system $A = (A,+,\cdot)$ is said to be semiring if $(A,+)$ is an Abelian semigroup and $(A,\cdot)$ is a semigroup while the binary operation '$\cdot$' is distributive over the binary operation '+' in the sense of (1.2). Further, a semiring $A = (A,+,\cdot)$ is termed as Abelian if the second binary operation '$\cdot$' is also commutative in A. Moreover, the semiring $A = (A,+,\cdot)$ is said to be a semiring with unity if there exists in A, the identity (neutral) element in A with respect to the second binary operation '$\cdot$'. Accordingly, an Abelian semiring with unity can be defined.

Note that a semiring can be empty also, and a semiring is essentially a ringoid but a ringoid may or may not be a semiring.

Definition 4. A duo binary system $A = (A,+,\cdot)$ will be termed as a Ringnoid if $(A,+)$ is an Abelian Monoid and $(A,\cdot)$ is just a Monoid while the binary operation '$\cdot$' is distributive over the binary operation '+' in the sense of (1.2). Further a ringnoid $A = (A,+,\cdot)$ will be termed as an Abelian ringnoid if the binary operation '$\cdot$' is also commutative in A.

Some Examples

(i) Let N denote the set of all natural numbers (positive integers). Let '+' and '$\cdot$' denote the usual addition and multiplication of natural numbers. Then $N = (N,+,\cdot)$ forms a duo binary system which happens to be an Abelian semiring with unity. The set $W = N \cup \{0\}$ forms an *Abelian Ringnoid*. Note that $(N,+,\cdot)$ is not a ringnoid. It is a ringoid.

(ii) Let $A = \{(a,b,c) : a \in W, b \in W, c \in W\}$ be the set of all ordered triples where $W = N \cup \{0\}$. In other words the elements a, b, c are non-negative integers.

Define, for a,b,c,d,e,f in W,

$$(a,b,c) = (d,e,f) \text{ if } a = d, b = e, c = f. \tag{1.3}$$

Define binary operations $\oplus$ and $\odot$ on A [5] as under:

$$(a,b,c) \oplus (d,e,f) = (a+d, b+e, c+f), \tag{1.4}$$

$$(a,b,c) \odot (d,e,f) = (ad, bd+ce, cf) \tag{1.5}$$

where the symbols '+' and '$\cdot$' denote respectively the usual addition and multiplication of integers. It can be verified that $A = (A,\oplus,\odot)$ forms a duo binary system which happens to be a *non-Abelian ringnoid* with $(1,0,0)$ as it unity element for the simple reason that for elements $(0,1,0)$ and $(1,0,0)$ of A, we see that

$$(0,1,0) \odot (1,0,0) = (0,1,0) \text{ while } (1,0,0) \odot (0,1,0) = (0,0,0)$$

that is

$$(0,1,0)\odot(1,0,0)\neq(1,0,0)\odot(0,1,0).$$

(iii) Let A be the set of all squared matrices each of order 2 described as

$$A=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}: a\in N, b\in N, c\in N, d\in N\right\},$$

where N denotes the set of all natural numbers. Let the binary operations $\oplus$ and $\odot$ respectively denote the binary operations of matrix-addition and matrix multiplication. Then $A=(A,\oplus,\odot)$ forms a *non-Abelian semiring*. In case, the elements $a,b,c\in W=N\cup\{0\}$, then $(A,\oplus,\odot)$ forms a duo binary system which happens to be a *non-Abelian Ringnoid* with unity $\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$.

Result 1 ([2, p. 265])**.** *Let S be a subsemigroup of an Abelian group G such that G is generated by S in the form $G=SS^{-1}=\{xy^{-1}: x\in S, y\in S\}$. Let f be a homomorphism of S into an Abelian group H. Then there exists a unique extension $g: G\to H$ of the form*

$$g(xy^{-1})=f(x)f(y)^{-1} \tag{1.6a}$$

for all $x\in S$, $y\in S$.

Result 2 ([2, p. 266])**.** *Let G, H be groups, S be a subsemigroup of G such that*

$$G=SS^{-1}=\{xy^{-1}: x\in S, y\in S\}$$

and let f be a homomorphism of S into H. Then f can be extended to a homomorphism g of G into H in a unique way. Such an extension is of the form

$$g(xy^{-1})=f(x)f(y)^{-1} \tag{1.6b}$$

for all $x,y\in S$.

The paper deals with the following precise problem:

Let $A=(A,+,\cdot)$ and $E=(E,+,\cdot)$ be duo binary systems. Let $S=(S,+,\cdot)$ be a sub-system of A and let $p: S\to E$ be a homomorphism in the sense that the mapping $p: S\to E$ satisfies the pair of equations [1]

$$p(x+y)=p(x)+p(y), \tag{1.7}$$

$$p(xy)=p(x)p(y), \tag{1.8}$$

for all $x\in S$, $y\in S$. Can the mapping $p: S\to E$ be extended to a mapping $P: A\to E$ such that

$$P(x+y)=P(x)+P(y), \tag{1.9}$$

$$P(xy)=P(x)P(y), \tag{1.10}$$

for all $x,y\in A$, and

$$P(x)=p(x), \tag{1.11}$$

for $x\in S$?

If yes, under what conditions? What, then, are the corresponding forms of such extensions? Are such extensions uniquely determined?

The problem has been explored under the following three conditions:

1. The duo binary system $A = (A, +, \cdot)$ is a ring and the subsystem $S = (S, +, \cdot)$ is a subsemiring of A such that A is generated by S in the form
$$A = S - S = \{a - b : a \in S, b \in S\}$$
where $-b$ denotes the negative of b under the binary operation '+'.
2. The duo binary system $A = (A, +, \cdot)$ is a field and its subsystem is a subring $B = (B, +, \cdot)$ such that A is generated by B in the form
$$A = BB^{-1} = \{ab^{-1} : a \in B, b \in B\}, \quad b \neq 0_A$$
where b^{-1} denotes the inverse of b under the binary operation '$\cdot$', and 0_A denotes the zero of A.
3. The duo binary system $A = (A, +, \cdot)$ is a field and its subsystem $S = (S, +, \cdot)$ is its subsemiring. Suppose there exists a subring $B = (B, +, \cdot)$ of A which contains S such that A is generated by B in the form
$$A = BB^{-1} = \{ab^{-1} : a \in B, b \in B\}$$
while B is itself generated by S in the form
$$A = S - S = \{a - b : a \in S, b \in S\}.$$

2. Extension of Subsemiring Homomorphisms to Homomorphisms of Rings

Let $(A, +, \cdot)$ and $(E, +, \cdot)$ be rings. Let $(S, +, \cdot)$ be a subsemiring of A such that A is generated by S in the form
$$A = S - S = \{a - b : a \in S, b \in S\}.$$
Let $p : S \to E$ be a subsemiring homomorphism given by (1.7) and (1.8).

Define a mapping $P : A \to E$ as
$$P(x) = P(a - b) = p(a) - p(b), \quad a \in S, b \in S \tag{2.1}$$
for every $x \in A$ where $x = a - b$ for $a \in S$, $b \in S$.

We claim that $P : A \to E$ defined by (2.1) is well defined, and it satisfies (1.9), (1.10) and (1.11) for all x, y in A.

Indeed, for $x = a - b$, $y = c - d$, $a, b, c, d \in S$,
$$\begin{aligned} x = y &\Rightarrow a - b = c - d \\ &\Rightarrow a + d = b + c \\ &\Rightarrow p(a + d) = p(b + c) \\ &\Rightarrow p(a) + p(d) = p(b) + p(c) \\ &\Rightarrow p(a) - p(b) = p(c) - p(d) \end{aligned}$$

$$\Rightarrow P(a-b) = P(c-d)$$
$$\Rightarrow P(x) = P(y).$$

Further,

$$\begin{aligned} P(x+y) = P[(a-b)+(c-d)] &= P[(a+c)-(b+d)] \\ &= p(a+c)-p(b+d) \Rightarrow p(a)-p(b)+p(c)-p(d) \\ &\Rightarrow P(a-b)+P(c-d) \\ &\Rightarrow P(x)+P(y). \end{aligned}$$

$$\begin{aligned} P(xy) &= P[(a-b)(c-d)] \\ &= P[(ac+bd)-(ad+bc)] \\ &= p(ac+bd)-p(ad+bc) \\ &= p(a)p(c)+p(b)p(d)-p(a)p(d)-p(b)p(c) \\ &= [p(a)-p(b)][p(c)-p(d)] \\ &= P(a-b)P(c-d) \\ &= P(x)P(y). \end{aligned}$$

Moreover, for an $x \in S$, we have

$$\begin{aligned} P(x) &= P(x+x-x) \\ &= p(x+x)-p(x) = p(x)+p(x)-p(x) \\ &= p(x). \end{aligned}$$

Thus, the mapping $P : A \to E$ is well-defined by (2.1) and extends $p : S \to E$ in the sense of (1.9), (1.10) and (1.11).

Now, we claim that the extension $P : A \to E$ is uniquely determined in the sense that every homomorphism of A into E which extends $p : S \to E$ in the sense of (1.9), (1.10) and (1.11), is always of the form (2.1).

Indeed, let $P' : A \to E$ be some arbitrary homomorphism which extends the subsemiring homomorphism $p : S \to E$ in the sense of (1.9), (1.10) and (1.11) i.e.

$$P'(x+y) = P'(x)+P'(y), \tag{2.2}$$

$$P'(xy) = P'(x)P'(y) \tag{2.3}$$

for $x, y \in A$, and that

$$P'(x) = p(x) \tag{2.4}$$

for $x \in S$. Then for an, $x \in A$, $x = a-b$ for some $a \in S$, $b \in S$,

$$P'(x) = P'(a-b) = P'(a)+P'(-b). \tag{2.5}$$

From (2.2), it follows that

$$P'(0) = P'(b-b) = P'(b)+P'(-b). \tag{2.6}$$

while $x=y=0$ in (2.2) gives

$$P'(0)=P'(0)+P'(0),$$

$$P'(0)=0.$$

Therefore (2.6) implies that

$$P'(-b)=-P'(b).$$

Eventually, (2.5) becomes

$$\begin{aligned} P'(x) &= P'(a-b) \\ &= P'(a)-P'(b) \\ &= p(a)-p(b) \\ &= P(x). \end{aligned}$$

This proves the uniqueness of the extension $P:A\to E$.

The whole discussion may be summed in the form of the following:

Theorem 1. *Let $A=(A,+,\cdot)$ and $E=(E,+,\cdot)$ be two rings. Let $S=(S,+,\cdot)$ be a subsemiring of A such that A is generated by S in the form*

$$A=S-S=\{a-b:a\in S,b\in S\}.$$

Then every subsemiring homomorphism $p:S\to E$ given by the equations (1.7) *and* (1.8) *can be extended to a ring homomorphism $P:A\to E$ in the sense of* (1.9), (1.10) *and* (1.11). *Moreover, such an extension is uniquely determined and is of the form* (2.1).

Examples (i), (ii) and (iii) of section 1, can be used to illustrate Theorem 1.

Also, Theorem 1 holds for the extension of a subringnoid homomorphism to a ring homomorphism.

3. Extension of Subring Homomorphisms to Homomorphisms of Fields

Let $A=(A,+,\cdot)$ and $E=(E,+,\cdot)$ be fields. Let $B=(B,+,\cdot)$ be a subring of A such that A is generated by B in the form

$$A=BB^{-1}=\{ab^{-1}:a\in B,b\in B,b\neq 0_A\}$$

where 0_A is the zero element of A.

Then every $x\in A$ can be expressed as $x=ab^{-1}$ for some $a,b\in B$, $b\neq 0_A$.

Let $x\in A$ and $y\in A$. Then $x=ab^{-1}$, $y=cd^{-1}$ for some $a,b,c,d\in B$, $b\neq 0_A$, $d\neq 0_A$.

Then we say that

$$ab^{-1}=cd^{-1} \quad \text{if} \quad ad=bc. \tag{3.1}$$

Define

$$x+y=ab^{-1}+cd^{-1}=(ad+bc)(bd)^{-1}, \tag{3.2}$$

$$x \cdot y = ab^{-1} \cdot cd^{-1} = (ac)(bd)^{-1}. \tag{3.3}$$

Let $p : B \to E$ be a subring homomorphism given by the equation (1.7) and (1.8).

Define a mapping $P : A \to E$ as

$$P(x) = P(ab^{-1}) = p(a)p(b)^{-1} \tag{3.4}$$

for $a, b \in B$, $b \neq 0_A$. We claim that $P : A \to E$ defined by (3.4) is well defined, and that it satisfies (1.9), (1.10) and (1.11).

Indeed, for $x \in A$, $y \in A$, $x = ab^{-1}$, $y = cd^{-1}$, $a, b, c, d \in B$, $b \neq 0_A$, $d \neq 0_A$,

$$\begin{aligned} x = y &\Rightarrow ab^{-1} = cd^{-1} \\ &\Rightarrow ad = bc \\ &\Rightarrow p(ad) = p(bc) \\ &\Rightarrow p(a)p(d) = p(b)p(c) \\ &\Rightarrow p(a)p(b)^{-1} = p(c)p(d)^{-1} \\ &\Rightarrow P(ab^{-1}) = P(cd^{-1}) \\ &\Rightarrow P(x) = P(y). \end{aligned}$$

Also

$$\begin{aligned} P(x+y) &= P(ab^{-1} + cd^{-1}) \\ &= P((ad+bc)(bd)^{-1}) \\ &= p(ad+bc)p(bd)^{-1} \\ &= [p(ad) + p(bc)]p(bd)^{-1} \\ &= [p(a)p(d) + p(b)p(c)]p(b)^{-1}p(d)^{-1} \\ &= p(a)p(b)^{-1} + p(c)p(d)^{-1} \\ &= P(ab^{-1}) + P(cd^{-1}) \\ &= P(x) + P(y). \end{aligned}$$

and

$$\begin{aligned} P(x \cdot y) &= P(ab^{-1} \cdot cd^{-1}) \\ &= P((ac)(bd)^{-1}) \\ &= p(ac)p(bd)^{-1} \\ &= p(a)p(c)p(b)^{-1}p(d)^{-1} \\ &= p(a)p(b)^{-1}p(c)p(d)^{-1} \\ &= P(ab^{-1})P(cd^{-1}) \\ &= P(x)P(y). \end{aligned}$$

Moreover, for an element $t \in B$

$$P(t) = P(ttt^{-1})$$

$$= p(tt)p(t)^{-1}$$
$$= p(t)p(t)p(t)^{-1}$$
$$= p(t).$$

Thus, $P : A \to E$ is well defined by (3.4) and its satisfies (1.9), (1.10) and (1.11) i.e. $P : A \to E$ extends $p : B \to E$ in the sense of (1.9), (1.10) and (1.11). We claim that this extension is uniquely determined. Indeed, let $P' : A \to E$ be an arbitrary mapping which extends $p : B \to E$ in the sense of (1.9), (1.10) and (1.11). This means

$$P'(x+y) = P'(x) + P'(y) \tag{3.5}$$

$$P'(xy) = P'(x)P'(y) \tag{3.6}$$

for all $x, y \in A$, and that

$$P'(x) = p(x) \tag{3.7}$$

for $x \in B$.

Let 1_A and 1_E denote the unit elements of A and E respectively.

Putting $x = y = 1_A$ in (3.6), we get

$$P'(1_A) = P'(1_A)P'(1_A)$$

$$P'(1_A) = 1_E. \tag{3.8}$$

Then for $x \in A$, we have

$$P'(x) = P'(ab^{-1}) = P'(a)P'(b^{-1}). \tag{3.9}$$

But

$$P'(b^{-1})P'(b) = P'(b^{-1}b) = P'(1_A) = 1_E.$$

So

$$P'(b^{-1}) = P'(b)^{-1}. \tag{3.10}$$

Then (3.9) becomes

$$\begin{aligned} P'(x) &= P'(ab^{-1}) \\ &= P'(a)P'(b^{-1}) \\ &= p(a)p(b)^{-1} \\ &= P(ab^{-1}) \\ &= P(x) \end{aligned}$$

which proves the uniqueness of the extension $P : A \to E$. Thus, we have proved the following:

Theorem 2. *Let $A = (A, +, \cdot)$ and $E = (E, +, \cdot)$ be fields. Let $B = (B, +, \cdot)$ be a subring of A such that A is generated by B in the form*

$$A = BB^{-1} = \{ab^{-1} : a \in B, b \in B, b \neq 0_A\}$$

where 0_A is the zero element of A.

Then every subring homomorphism $p: B \to E$ given by the equations (1.7) *and* (1.8) *can be extended to a field homomorphism $P: A \to E$ in the sense of* (1.9), (1.10) *and* (1.11). *Moreover, such an extension is uniquely determined and is of the form*

$$P(x) = P(ab^{-1}) = p(a)p(b)^{-1}$$

for $a, b \in B$, $b \neq 0_A$.

Examples.

(i) Let $Q = (Q, +, \cdot)$ be the field of rational numbers and $R = (R, +, \cdot)$ be the field of real numbers where '+' and '·' denote the usual addition and multiplication of real numbers. Let $Z = (Z, +, \cdot)$ be the ring of integers. Then obviously Z is a subring of Q such that Q is generated by Z in the form

$$Q = ZZ^{-1} = \{mn^{-1} : m \in Z, n \in Z, n \neq 0\}.$$

Now, Theorem 2 can be used to show that every subring homomorphism $p: Z \to E$ can be extended to a field homomorphisms $P: Q \to R$ in the sense of (1.9), (1.10) and (1.11). Moreover, this extension is uniquely determined.

(ii) Let $T = \{(a, b) : a \in Z, b \in Z, b \neq 0\}$ where Z stands for the set of integers. Define a relation $\sim$ on T as under:

For any two elements (a, b) and (c, d) of T,

$$(a, b) \sim (c, d) \text{ means } ad = bc$$

for $a, b, c, d \in Z$, $b \neq 0$, $d \neq 0$. Then as proved in [5], the relation $\sim$ is an equivalence relation on T, and it partitions the set T into mutually disjoint equivalence classes each denoted by $[a, b]$ where

$$[a, b] = \{(x, y) : (x, y) \sim (a, b) \Rightarrow xb = ya\}.$$

Let A be the set of all equivalence classes of the form $[a, b]$ i.e.

$$A = \{[a, b] : a \in Z, b \in Z, b \neq 0\}.$$

Define the binary operations $\oplus$ and $\odot$ as under:

$$[a, b] \oplus [c, d] = [ad + bc, bd]$$

$$[a, b] \odot [c, d] = [ac, bd].$$

It has been shown in [5] that $A = (A, \oplus, \odot)$ forms a field with $[0, 1] = [0, d]$ as its zero-element, and $[1, 1] = [d, d]$ as its unit element.

Let $B = \{[a, 1] : a \in Z\}$. Then $B = (B, \oplus, \odot)$ forms a subring of the field $A = (A, \oplus, \odot)$ such that A is generated by B in the form

$$A = B \odot B^{-1} = \{[a, 1] \odot [b, 1]^{-1} : a \in Z, b \in Z\}, \quad b \neq 0.$$

Let $R = (R, +, \cdot)$ be the field of real numbers under usual addition and multiplication of real numbers. Then using Theorem 2, it follows that every subring homomorphism $p: B \to R$ given by (1.7) and (1.8) can be extended to a field homomorphism $P: A \to R$ in the sense of

(1.9), (1.10) and (1.11). Moreover, such an extension is uniquely determined, and is of the form

$$P(t) = P([a,b]) = p[a,1][b,1]^{-1}$$

for all $a \in Z$, $b \in Z$, $b \neq 0$.

4. Extension of Subsemiring Homomorphisms to Homomorphisms of Fields

Let $A = (A,+,\cdot)$ and $E = (E,+,\cdot)$ be fields. Let $S = (S,+,\cdot)$ be a subsemiring of A and let $p : S \to E$ be a subsemiring homomorphism in the sense of (1.7) and (1.8). To extend $p : S \to E$ to a field homomorphism $P : A \to E$, we proceed as follows:

Suppose these exists a subring $B = (B,+,\cdot)$ of the field A which contains the subsemiring $S = (S,+,\cdot)$ such that B is generated by S in the form $B = S - S$ while at the same time B generates A in the form $A = BB^{-1}$.

Now to extend subsemiring homomorphism $p : B \to E$ to the field homomorphism $A \to E$, in the sense of (1.9), (1.10) and (1.11), we first use Theorem 1 to extend $p : S \to E$ to a subring homomorphism $\bar{P} : B \to E$ in the sense that

$$\bar{P}(x+y) = \bar{P}(x) + \bar{P}(y) \tag{4.1}$$

$$\bar{P}(xy) = \bar{P}(x)\bar{P}(y) \tag{4.2}$$

for $x, y \in B$, and that

$$\bar{P}(x) = p(x) \tag{4.3}$$

for $x \in S$. Moreover, this extension in uniquely determined.

Now since A is generated by B in the form $A = BB^{-1}$, therefore, we use Theorem 2 to extend $\bar{P} : B \to E$ to the mapping $P : A \to E$ in the sense that

$$P(x+y) = P(x) + P(y) \tag{4.4}$$

$$P(xy) = P(x)P(y) \tag{4.5}$$

for $x, y \in A$, and that

$$P(x) = \bar{P}(x) \tag{4.6}$$

for $x \in B$. Moreover, this extension is also uniquely determined.

Now combining the two extensions, it follows that the mapping $P : A \to E$ extends $\bar{P} : B \to E$ in the sense of (4.4) and (4.5) while $\bar{P} : B \to E$ extends $p : S \to E$ in the sense of (4.1) and (4.2). In other words, the mapping $P : A \to E$ extends $p : S \to E$ in the sense of (1.9), (1.10) and (1.11). Moreover, this extension is uniquely determined.

Thus, we have proved the following:

Theorem 3. *Let $A = (A,+,\cdot)$ and $E = (E,+,\cdot)$ be fields. Let $S = (S,+,\cdot)$ be a subsemiring of A., Suppose these exists a subring B of A which contains the subsemiring S such that S generates B in the form $B = S - S$ while B generates A in the form $A = BB^{-1}$. Then every subsemiring homomorphism $p : S \to E$ can be extended to a field homomorphism $P : A \to E$ in the sense of* (1.9)*,* (1.10) *and* (1.11)*. Moreover, such an extension is uniquely determined.*

Examples.

(i) Let $B=(N,+,\cdot)$ be a subsemiring of the field $Q=(Q,+,\cdot)$ where N denotes the set of all positive integers and Q stands for the set of all rational numbers. Let $Z=(Z,+,\cdot)$ be the ring of all integers. Then it can be seen that Z forms a subring of A which contains the subsemiring N such that N generates Z in the form $Z=N-N=\{m-n:m\in N,n\in N\}$ while Z generates Q in the form $Q=ZZ^{-1}=\{aa^{-1}:a\in Z,b\in Z,a\neq 0\}$. Let $R=(R,+,\cdot)$ be the field of real numbers.

Therefore, using Theorem 3, it follows that every subsemiring homomorphism $p:B\to R$ can be extended to a field homomorphism $P:Q\to R$ in the sense of (1.9), (1.10) and (1.11). Moreover, this extension is uniquely determined.

(ii) Let $S=\{m+n\sqrt{2}:m\in N,n\in N\}$, where N is the set of all natural numbers. Let $A=\{a+b\sqrt{2}:a\in Q,b\in Q\}$ where Q stands for all rational numbers. Let the usual operations of addition and multiplication, denoted as '+' and '$\cdot$' respectively, be defined on both A and S. Then $S=(S,+,\cdot)$ forms a subsemiring of the field $A=(A,+,\cdot)$.

Construct a set $B=\{\alpha+\beta\sqrt{2}:\alpha\in Z,\beta\in Z\}$, where Z denotes the set of all integers. Then $B=(B,+,\cdot)$ forms a subring of the field A which contains the subsemiring S such that it is generated by S in the form $B=S-S$ while it generates A in the form $A=BB^{-1}$. Let $E=(R,+,\cdot)$ be the field of all real numbers. Then by using Theorem 3, it follows that every subsemiring homomorphism $p:S\to E$ given by (1.7) and (1.8) can be extended to a field homomorphism $P:A\to E$ in the sense of (1.9), (1.10) and (1.11). Moreover, such an extension is uniquely determined.

5. Conclusion

Finally, it may be noted that the conditions mentioned in Theorems 1, 2 and 3, are not necessary but sufficient for obtaining the desired extensions. For example, consider the following case:

Let $R^2=R\times R=\{(a,b):a\in R,b\in R\}$
where R denotes the set of all real numbers.

Define binary operations $\oplus$ and $\odot$ on R^2 as under:

For $x\in R^2$, $y\in R^2$, $x=(a,b)$, $y=(c,d)$, $a,b,c,d\in R$,

$$x\oplus y=(a,b)\oplus(c,d)=(a+c,b+d)$$

$$x\odot y=(a,b)\odot(c,d)=(ac+bd,ad+bc).$$

Then it can be verified [3] that $(R^2,\oplus,\odot)$ forms a commutative ring with $(1,0)$ as its unit element.

Consider a subset B of R^2 given as

$$B=\{(a,0):a\in R\}.$$

Then $(B,\oplus,\odot)$ forms a subring of $R^2=(R^2,\oplus,\odot)$ which also contains the unit element $(1,0)$. Let $R=(R,+,\cdot)$ be the field of real numbers under the usual addition and multiplication of real numbers. Then it can be proved that every subring homomorphism $p:B\to R$ given by (1.7) and (1.8) can be extended to a ring homomorphism $P:R^2\to R$ in the sense of (1.9), (1.10) and (1.11).

Moreover, such an extension is uniquely determined and is always of the form $P : R^2 \to R$

$$P(x) = P(a,b) = p(a,0) + p(b,0)$$

for $a, b \in R$. Note that no-where any of the conditions stated in Theorems 1, 2 and 3 is needed in this case.

Acknowledgement

The author is extremely grateful to Prof. P. Nath, formerly Professor in the Department of Mathematics, University of Delhi, Delhi (India), for going through the manuscript of the paper and giving valuable suggestions.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] J. Aczél, *Lectures on Functional Equations and Their Applcations*, Academic Press, New York (1966).

[2] J. Aczél, J.A. Baker, D.Ž. Djoković, P.L. Kanappan and F. Radó, Extensions of certain homomoprhisms of subsemigroups to homomorphisms of groups, *Aequationes Mathematicae* **6** (2-3) (1971), 263–271.

[3] T.S. Blyth and E.F. Robertson, *Algebra Through Practice, Book 3, Groups, Rings and Fields*, Cambridge University Press, London, England (1984).

[4] R.H. Bruck, *A Survey of Binary Systems*, Springer Verlag, Berlin-Heidelberg, Germany (1971).

[5] Neal H. McCoy, *Introduction to Modern Algebra*, Allyn and Bacon, Inc., Boston, USA (1965).

[6] A. Rosenfield, *An Introduction to Algebraic Structures*, Holden Day, New York, USA (1968).