Journal of Informatics and Mathematical Sciences
Vol. 6, No. 2, pp. 61–76, 2014
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

On the Hypergroups Associated with n-ary Relations

Research Article

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Abstract. The paper deals with hypergroupoids associated with n-ary ($n \geq 3$) relations. We give necessary and sufficient condition for an n-ary relation such that the hypergroupoid associated with it is a hypergroup or a join space. First we analyze for sufficient condition using ternary relation such that the hypergroupoid associated with it is a hypergroup or a joinspace and then we generalize it for an n-ary relation.

Keywords. n-ary relations; Hypergroups; Join spaces

MSC. 20N20; 04A05

Received: October 17, 2014 **Accepted:** November 21, 2014

1. Introduction

Hyper structure theory was born during the 8th Congress of Scandinavian Mathematicians in 1934, when F. Marty [16] defined hypergroups, a natural generalization of the concept of group and began to analyze their properties and applied them to non commutative groups, algebraic functions etc. Since then various connection between hypergroups and other subjects of theoretical and applied mathematics have been established. The most important applications to geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy sets and rough sets, automata theory are found in [6]. The first association between binary relation and hyperstructures appeared in [18] J. Nieminen, who studied hypergroup related to connected simple graphs. In the same direction P. Corsini [2] worked considering different hyperoperations associated with graphs.

I. Rosenberg [21] introduced a hyperoperation obtained by a binary relation; the new hypergroupoid has been investigated by P. Corsini [3], P. Corsini and V. Leoreanu [7] and recently by I. Cristea and M. Ştefănescu [9]. Another approach to the connection between hypergroups and ordered sets is given by M. Ştefănescu [26], and recently hypergroupoids obtained from n-ary ($n \geq 3$) relations has been investigated by I. Cristea and M. Ştefănescu [11] and I. Cristea alone in [10]. In [12], B. Davvaz and T. Vougioklis introduced the concept of n-ary hypergroups as a generalization of hypergroups in the sense of Marty. In [17], V. Leoreanu and B. Davvaz introduced and studied the notion of partial n-ary hypergroupoids associated with a binary relation. The connections between hyperstructures and binary relations have been analyzed by many researchers such as De Salvo and Lo Fro [13, 14], S. Spartalis [23, 24], S. Spartalis and C. Mamaloukas [25]. In [22], S.M. Anvariyah and S. Momeni studied the n-ary hypergroups associated with n-ary relations.

In this paper, we investigate for sufficient condition such that the hypergroupoid obtained in [10] by Irina Cristea associated with an n-ary ($n \geq 3$) relation to be a hypergroup. The reason to obtain sufficient condition is to determine some relationship between hypergroupoids associated with two n-ary relations and the hypergroupoids associated with the union, intersection, join and Cartesian product of two n-ary relations.

The n-ary relations were studied for their applications in theory of dependence space. Moreover, they used in Database Theory, providing a convenient tool for database modeling. In this paper, we consider compatibility relation. Compatibility relations are useful in solving certain minimization problems. It is obvious that the reflexive and symmetric ternary relations are applicable in spherical geometry.

For a non empty set H, we denote by $\mathscr{P}^*(H)$ the set of all non empty subsets of H.

A nonempty set H, endowed with a mapping, called *hyperoperation*
$\circ : H \times H \to \mathscr{P}^*(H)$ is called a hypergroupoid which satisfies the following conditions:

(i) $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in H$

(ii) $x \circ H = H = H \circ x$, for all $x \in H$, (reproduction axiom)

is called a hypergroup.

If, for any $x, y \in H$, $x \circ y = H$, then $(H, \circ)$ is called the total hypergroup.

If A and B are nonempty subsets of H, then we denote the set $A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b$.

If A and B are nonempty subsets of H, then we denote $A/B = \bigcup_{\substack{a \in A \\ b \in B}} a/b$.

A commutative hyper groupoid $(H, \circ)$ is called a join space if the following implication holds: for any $(a, b, c, d) \in H^4$,

$$a/b \cap c/d \neq \phi \quad \Rightarrow \quad a \circ d \cap b \circ c \neq \phi \quad \text{(“transposition axiom”)}$$

For more details on hypergroup theory, see [1] and for H_v-groups (see [27]).

2. Properties of the n-ary Relations

In this section we present some basic notions about the n-ary relations defined on a non-empty set H, $n \in N$ a natural number such that $n \geq 3$, and $\rho \subseteq H^n$ is an n-ary relation on H.

Definition 2.1 ([10], [11])**.** The relation ρ is said to be:

1. *reflexive* if , for any $x \in H$, the n-tuple $(x,\dots,x) \in \rho$
2. n-transitive if it has the following property: if $(x_1,\dots,x_n) \in \rho$, $(y_1,\dots,y_n) \in \rho$ hold if there exist natural numbers $i_0 > j_0$ such that $1 < i_0 \le n$, $1 \le j_0 < n$, $x_{i_0} = y_{j_0}$, then the n-tuple $(x_{i_1},\dots,x_{i_k},y_{j_{k+1}},\dots,y_{j_n}) \in \rho$ for any natural number $1 \le k < n$ and $i_1,\dots,i_k,j_{k+1},\dots,j_n$ such that $1 \le i_1 < \dots < i_k < i_0$, $j_0 < j_{k+1} < \dots < j_n \le n$;
3. strongly symmetric if $(x_1,\dots,x_n) \in \rho$ implies $(x_{\sigma(1)},\dots,x_{\sigma(n)} \in \rho$ for any permutation σ of the set $\{1,\dots,n\}$;
4. n-ary preordering on H if it is reflexive and n-transitive;
5. an n-equivalence on H if it is reflexive, strongly symmetric and n-transitive;
6. compatibility relation if it is reflexive and symmetric.

Remark 2.2. Obviously, all equivalence relations are compatibility relations. We however are concerned with those compatibility relations which are not n-equivalence relations.

Example 2.3 ([10], [11])**.** (1) For $n = 2$, a binary relation is 2 transitive if and only if it is transitive in the usual sense and therefore it is 2-equivalence if and only if it is an equivalence in the usual sense.

(2) A ternary relation ρ is 3-transitive if and only if it satisfies the following conditions:

(i) If $(x,y,z) \in \rho$, $(y,u,v) \in \rho$, then $(x,u,v) \in \rho$.
(ii) If $(x,y,z) \in \rho$, $(z,u,v) \in \rho$, then $(x,y,u) \in \rho$, $(x,y,v) \in \rho$, $(x,u,v) \in \rho$, $(y,u,v) \in \rho$.
(iii) If $(x,y,z) \in \rho$, $(u,z,v) \in \rho$, then $(x,y,v) \in \rho$.

Definition 2.4 ([10])**.** Let ρ be an n-ary relation on a nonempty set H and $k < n$. The $(i_1,\dots,i_k)$-*projection* of ρ, denoted by $\rho_{i_1,\dots,i_k}$, is a k-ary relation on H defined by:

$$\text{if } (a_1,a_2,a_3,\dots,a_n) \in \rho, \text{ then } (a_{i_1},\dots,a_{i_k}) \in \rho_{i_1,\dots,i_k}.$$

Definition 2.5 ([10])**.** Let ρ be an m-ary relation on a nonempty set H, λ an n-ary relation on the same set H. The join of ρ and λ, denoted by $J_p(\rho,\lambda)$, where $1 < p < n$, $1 < p < m$, is an $(m+n-p)$-ary relation on H that consist of all $(m+n-p)$-tuples

$$(a_1,\dots,a_{m-p},c_i,\dots,c_p,b_1,\dots,b_{n-p})$$

such that

$$(a_1,\dots,a_{m-p},c_1,\dots,c_p) \in \rho \quad \text{and} \quad (c_1,\dots,c_p,b_1,\dots,b_{n-p}) \in \lambda.$$

Let ρ be a ternary relation on H. The join relation $J_2(\rho,\rho)$ denoted by α is a 4-ary relation such that

$$(x,y,z,t) \in J_2(\rho,\rho) \Rightarrow (x,y,z),(y,z,t) \in \rho.$$

We denote the join relation $J_2(\rho,\rho)$ by the symbol α.

The projection relations are denoted by $\alpha_{1,2,4}$ and $\alpha_{1,3,4}$.

3. Hypergroups Associated with n-ary Relations

The hyperproduct associated with the ternary relation is generalized to the case of n-ary relation ($n \geq 3$), using the projection $\rho_{1,i,n}$ by Irina Cristea [10] as follows:

For any $i \in \{2,\ldots,n-1\}$,

$$x \circ_i y = \{z \in H \mid (x,z,y) \in \rho_{1,i,n}\} \tag{δ_1}$$

$$\begin{aligned} x \circ_\rho y &= \left\{ z \in H \mid (x,z,y) \in \bigcup_{i=2}^{n-1} \rho_{1,i,n} \right\} \\ &= \left\{ \bigcup_{i=2}^{n-1} x \circ_i y \right\} \end{aligned} \tag{δ_2}$$

The necessary and sufficient conditions for the hypergroupoid $(H,\circ_\rho)$ to be a quasi hypergroup and necessary condition for the hypergroupoid $(H,\circ_\rho)$ to be a semi hypergroup are obtained in [10]. However, we reproduce them for convenient.

Proposition 3.1 ([10, Proposition 11]). *Let ρ be an n-ary relation on H. Then $(H,\circ_\rho)$ is a quasi hypergroup if and only if $\rho_{1,n} = H \times H$, and there exists i, j with $2 \leq i$, $j \leq n-1$, such that $\rho_{1,i} = \rho_{j,n} = H \times H$.*

Proof. The reproducibility law means: for any $x \in H$, $x \circ_\rho H = H \circ_\rho x = H$, that is, for any $x,y \in H$, there exists $t,z \in H$ such that $y \in x \circ_\rho z \cap t \circ_\rho x$.

First we suppose that, $\rho_{1,n} = H \times H$, and there exists i, j with $2 \leq i$, $j \leq n-1$, such that $\rho_{1,i} = \rho_{j,n}, = H \times H$. Then, for any $x,y \in H$, $(x,y,z) \in \bigcup_{i=2}^{n-1} \rho_{1,i,n}$, and for any $t \in H$, $(t,y,x) \in \bigcup_{i=2}^{n-1} \rho_{1,i,n}$, it follows that, for any $x \in H$, $x \circ_\rho H = H \circ_\rho x = H$, so it is a quasi hypergroup.

Now we consider $(H,\circ_\rho)$ is a quasi hypergroup and we suppose that $\nexists$ i with $2 \leq i \leq n-1$ such that $\rho_{1,i} = H \times H$, then, since $\rho_{1,n} = H \times H$, for any $y,z \in H$, $(x,y,z) \notin \bigcup_{i=2}^{n-1} \rho_{1,i,n} \Rightarrow (x,y,H) \notin \rho \Rightarrow y \notin x \circ_\rho H$. Thus $x \circ_\rho H \neq H$, this is a contradiction to the reproducibility law. Similarly, $\nexists$ j with $2 \leq j \leq n-1$ such that $\rho_{1,j} = H \times H$, then $H \circ_\rho x \neq H$ and again, we obtain a contradiction. □

Remark 3.2. If ρ is a symmetric n-ary relation on H such that $\rho_{1,n} = H \times H$. Then $(H,\circ_\rho)$ is a quasi hypergroup implies that for any $x,y \in H$, $(x,y,x) \in \rho_{1,i,n} \Rightarrow (x,y,H) \in \rho_{1,i,n}$ with $2 \leq i$, $j \leq n-1$.

Proposition 3.3 ([10, Proposition 12]). *Let ρ be a an n-ary relation on H such that $\rho_{1,n} = H \times H$. If ρ is preordering, then $(H,\circ_\rho)$ is the total hypergroup.*

Proof. We have to prove that for any $x,y,z \in H$, $z \in x \circ_\rho y$. Set $x,y,z \in H$. Since $\rho_{1,n} = H \times H$, $\exists$ $a_1,a_2,\ldots,a_{n-2} \in H$, such that $(z,a_1,\ldots,a_k,\ldots,a_{n-2},y) \in \rho$. By the reflexivity of ρ, and then by n-transitivity $(z,z,\ldots,z) \in \rho$ and $(z,z,\ldots,y) \in \rho$. Again since $\rho_{1,n} = H \times H$, $\exists$ $b_1,b_2,\ldots,b_{n-2} \in H$, such that $(x,b_1,\ldots,b_{n-2},z) \in \rho$. Using n-transitivity for $(z,z,\ldots,y) \in \rho$ and $(x,b_1,\ldots,b_{n-2},z) \in \rho$, we obtain that $(x,z,\ldots,y) \in \rho$. Therefore, $z \in x \circ_\rho y$. □

Proposition 3.4. *Let ρ be an n-ary relation H such that $\rho_{1,n} = H \times H$. If ρ is n-ary transitive and strongly symmetric, then $(H,\circ_\rho)$ is the total hypergroup.*

Proof. Set x arbitrary in H.

Since $\rho_{1,n} = H \times H$, $\exists$ $a_1, a_2, \dots, a_{n-2} \in H$, such that $(x, a_1, \dots, a_k, \dots, a_{n-2}, x) \in \rho$ and by the strongly symmetry it follows that $(a_1, \dots, a_k, \dots, a_{n-2}, x, x) \in \rho$.

Using the n-transitivity, $(x, a_2 \dots, a_k, \dots, a_{n-2}, x, x) \in \rho$. Again by the symmetry of ρ, we obtain that $(a_2, \dots, a_k, \dots, a_{n-2}, x, x, x) \in \rho$ and therefore, by the n-transitivity, that $(x, a_3, \dots, a_{n-2}, x, x, x) \in \rho$ and so on; finally it results that $(x, x, \dots, x) \in \rho$, for any $x \in H$, and so ρ is reflexive. Also, ρ is n-transitivity, whence ρ is preordering. By the previous proposition, $(H, \circ_\rho)$ is the total hypergroup. $\square$

Let ρ be a ternary relation on H. We denote the join relation $J_2(\rho, \rho)$ by α (see [10]). We recall that $J_2(\rho, \rho)$ is a 4-ary relation such that

$$(x, y, z, t) \in J_2(\rho, \rho) \quad \Rightarrow \quad (x, y, z), (y, z, t) \in \rho.$$

Using the projection of $J_2(\rho, \rho)$, necessary condition for a hypergroupoid $(H, \circ_\rho)$ to be a semi hypergroup is obtained in [10] as follows.

Proposition 3.5 ([10, Proposition 16]). *Let ρ be a reflexive and symmetric ternary relation on H. If $\rho \nsubseteq \alpha_{1,2,4}$ or $\rho \nsubseteq \alpha_{1,3,4}$, then the hyperoperation "$\circ_\rho$" is not associative.*

Corollary 3.6 ([10, Corollary 17]). *Let ρ be a reflexive and symmetry ternary relation on H. If $(H, \circ_\rho)$ is a semihypergroup, then $\rho \subset \alpha_{1,2,4} \cap \alpha_{1,3,4}$.*

Let ρ be a reflexive and symmetric ternary relation on H such that $\rho_{1,3} = H \times H$ with $|H| \geq 3$.

From the following example, we observe that there exists a hypergroup equipped with the product (δ_2) which is different from the total hypergroup or at least one can associate a ternary relation to a given hypergroupoid $(H, \circ)$ such that $(H, \circ_\rho)$ to be a hypergroup.

Example 3.7. Set $H = \{x, y, z\}$. Suppose that ρ be reflexive and symmetric ternary relation on H and (x, y, z), (x, x, y), (x, x, z), (x, z, z), $(y, z, z) \in \rho$. This ρ satisfies the conditions of a quasi hypergroup; i.e. $\rho_{1,n} = \rho_{1,2} = \rho_{2,3} = H \times H$. Moreover, it is a hypergroup but not the total hypergroup.

Proposition 3.8 ([10, Proposition 18]). *Let ρ be a reflexive and symmetric n-ary relation on H which satisfies the condition;*

(S): $(x, a_1, a_2, a_3, \dots, a_{n-2}, y) \in \rho \Leftrightarrow (x, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, \dots, a_{\sigma(n-2)}, y) \in \rho$ for any permutation σ of the set $\{1, 2 \dots, n-2\}$. If, there exists $j \in \{2, \dots, n-1\}$ such that $\rho_{1,j,n} \not\subset \alpha_{1,n,2n-2}$ or $\rho_{1,j,n} \not\subset \alpha_{1,n-12n-2}$. Then the hyperoperation "$\circ_\rho$" is not associative.

Let ρ be reflexive and symmetric n-ary relation on H such that $\rho_{1,n} = H \times H$, and there exists i, j with $2 \leq i$, $j \leq n-1$, such that $\rho_{1,i} = \rho_{j,n} = H \times H$. The condition there exists $j \in \{2, \dots, n-1\}$ such that $\rho_{1,j,n} \subset \alpha_{1,n,2n-2} \cap \alpha_{1,n-1,2n-2}$ is necessary condition, but not Sufficient one such that the hyperoperation is "$\circ_\rho$" is associative, as we can observe from the following examples.

Example 3.9. On the set $H = \{1, 2, 3\}$ we consider the n-ary relation: $\rho = \{(1, 1, \dots, 1), (2, 2, \dots, 2), (3, 3, \dots, 3), (1, 3, , \dots, 3), (1, 1, \dots, 3), (2, 3, \dots, 3), (2, 2, \dots, 3), (1, 1, \dots, 2), (1, 2, \dots, 1), (2, 1, 1, \dots, 2)\}$. Further, we suppose that ρ is symmetry then clearly $\rho_{1,n} = H \times H$ and $j \in \{2, \dots, n-1\}$ exists such that the condition $\rho_{1,j,n} \subset \alpha_{1,n,2n-2} \cap \alpha_{1,n-1,2n-2}$ is satisfied, but $(2 \circ_\rho 1) \circ_\rho 3 = \{1,3\} \neq 1,2,3 = 2 \circ_\rho (1 \circ_\rho 3)$, it follows that "$\circ_\rho$" is not associative.

Example 3.10. On the set $H = \{1,2,3\}$ we consider the reflexive and symmetric n-ary relation ρ that contains $(1,2,\dots,1)$, $(1,1,\dots,2)$, $(1,2,\dots,2)$, $(1,3,\dots,3)$, $(1,2,\dots,3)$, $(1,1,\dots,3)$, $(2,3,\dots,3)$, $(2,2,\dots,3)$, $(2,3,3,\dots,2)$, $(2,2,\dots,3)\}$. Clearly $\rho_{1,n}H \times H$ and for ρ there exists $j \in \{2,\dots,n-1\}$ such that the condition $\rho_{1,j,n} \subset \alpha_{1,n,2n-2} \cap \alpha_{1,n-1,2n-2}$ is satisfied. Moreover, for any $x,y,z \in H$, $(x \circ_\rho y) \circ_\rho z = x \circ_\rho (y \circ_\rho z)$, it follows that "$\circ_\rho$" is associative.

If the following is satisfied in Example 3.9, then $(H, \circ_\rho)$ is a hyper group.

(i) for any $i,j \in \{2,\dots,n-1\}$, $(x,z,y) \in \rho_{1,i,n} \Leftrightarrow (x,z,x) \in \rho_{1,i,n} \wedge (y,z,y) \in \rho_{1,j,n}$ (δ_3)

However, the following example shows that (δ_3) together with the conditions $\rho_{1,n} = H \times H$,

(ii) there exists i, j with $2 \leq i$, $j \leq n-1$, such that $\rho_{1,i} = \rho_{j,n} = H \times H$ are insufficient for a reflexive and symmetric n-ary relation ρ on a nonempty set H such that $(H, \circ_\rho)$ is a hyper group.

Example 3.11. Let $H = \{1,2,3,4\}$, ρ be reflexive and symmetric n-ary relations on H. Let ρ contains the elements $(1,2,1,2,\dots,1,2,1)$, $(1,1,2,3,\dots,1,2,3,2)$, $(1,1,2,3,\dots,1,2,3,3)$, $(1,1,2,4,\dots,4)$, $(2,2,3,\dots,2,3,2)$, $(2,2,3,\dots,2,3,3)$, $(2,2,3,4\dots,2,3,4,3)$, $(2,2,4,2,4,4\dots,2,4,4)$ $(2,3,4,3,3,4,\dots,3,4)$, $(3,3,4,3,3,4,\dots,3,4)$. If the condition (S) is satisfied then ρ satisfies the conditions (i) or (δ_3) and (ii), but $(H, \circ_\rho)$ is not a hypergroup. Since $(1 \circ_\rho 4) \circ_\rho 4 = H \neq 1,2,4 = 1 \circ_\rho (4 \circ_\rho 4)$.

Proposition 3.12. *Let ρ be reflexive and symmetric ternary relation on H such that $\rho_{1,2} = \rho_{1,3} = \rho_{2,3} = H \times H$. If, ρ is not 3-transitive and $\rho \subset \alpha_{1,2,4} \cap \alpha_{1,3,4} \Rightarrow \forall\ (x,y,z) \in \rho$ and $\forall\ x \in H$, $(x,x,y) \in \rho$, $(y,y,x) \in \rho$, then $(H, \circ_\rho)$ is a hypergroup different from total hypergroup.*

Proof. $(H, \circ_\rho)$ is a quasihypergroup easily follows from hypothesis. It is remains to check that "$\circ_\rho$ is associative. We suppose that $\rho \subset \alpha_{1,2,4} \cap \alpha_{1,3,4}$; thus for any $(x,y,z) \in \rho$, we have $(x,y,z) \in \alpha_{1,2,4 \cap} \alpha_{1,3,4}$. This implies that, there exists $a \in H$ such that $(x,y,a,z) \in J_2(\rho,\rho)$ and $(x,a,y,z) \in J_2(\rho,\rho)$. Since

$$(x,y,a,z) \in J_2(\rho,\rho) \Rightarrow (x,y,a),(y,a,z) \in \rho \tag{*}$$

and

$$(x,a,y,z) \in J_2(\rho,\rho) \Rightarrow (x,a,y),(a,y,z) \in \rho, \tag{**}$$

there exists $a \in H$ such that $a \in x_\rho y \cap y \circ_\rho z$.

Now, let $u \in (x \circ_\rho y) \circ_\rho z$, then there exists $v \in x \circ_\rho y$, such that $u \in v \circ_\rho z$.

We suppose that $\rho_{1,2} = \rho_{1,3} = H \times H$; thus for any $u \in H$, there exists $t \in H$ and $x \in H$ such that $(u,x,u) \in \rho$ and $(x,u,t) \in \rho$. Set $t = a$ in $(*)$, then for any $u \in H$, there exists $t = a \in H$ such that $(x,u,a) \in \rho$ with $(y,a,z) \in \rho$.

Hence, $u \in x \circ_\rho (y \circ_\rho z)$, whence $(x \circ_\rho y) \circ_\rho z \subset x \circ_\rho (y \circ_\rho z)$.

Next, $\forall\ (x,y,z) \in H^3$, we shall verify that $(x \circ_\rho y) \circ_\rho z \supseteq x \circ_\rho (y \circ_\rho z)$.

Let $u \in x \circ_\rho (y \circ_\rho z)$. Then, there exists $t \in H$ such that $t \in y \circ_\rho z$ with $u \in x \circ_\rho t$, that is, $(y,t,z) \in \rho$ with $(x,u,t) \in \rho$. Since $\rho_{1,2} = \rho_{1,3} = H \times H$, for any $t \in H$, if there exists $(y,t,z) \in \rho$, then there exists $u_z \in H$ such that $(t,u_z,z) \in \rho$. Set $t = a$ in $(**)$ and $u = u_z$, it follows that $u \in (x \circ_\rho y) \circ_\rho z$, so $(x \circ_\rho y) \circ_\rho z \in x \circ_\rho (y \circ_\rho z)$.

We justify that $u = u_z$ holds good.

Suppose that $u \neq u_z$. Let $(x,y,a) \in \rho$, for $y = a$, we have $(x,y,y) \in \rho$ and from $\rho_{1,2} = \rho_{1,3} = H \times H$, it follows that $(y,x,y) \in \rho$. This is a contradiction to ρ is not 3-transitive. Thus $u \in H$ is the unique element such that $(x,u,u) \in \rho$. □

The following example is an illustration of the above proposition.

Example 3.13. Set $H = \{1,2,3\}$ and let $\rho \subseteq H \times H \times H$ be the ternary relation on H defined by $\rho = \{(1,1,1),\ (2,2,2),\ (3,3,3),\ (1,2,3),(1,1,2),\ (1,2,1),\ (1,1,3),\ (1,3,3),\ (2,3,3),\ (2,1,1),\ (3,1,1),\ (3,3,1),\ (3,2,1),\ (3,2,3)\}$.

This ρ is reflexive and symmetric ternary relation on H which satisfies the conditions $\rho_{1,2} = \rho_{1,3} = H \times H$ and $\rho = \alpha_{1,2,4} \cap \alpha_{1,3,4}$. This ρ is not 3-transitive. Because of

(i) $(1,2,3) \in \rho$, $(2,2,2) \in \rho$, but $(1,2,2) \notin \rho$

(ii) $(1,2,3) \in \rho$, $(3,2,3) \in \rho$, but $(1,2,2) \notin \rho$, $(2,2,2) \notin \rho$

(iii) $(1,2,3) \in \rho$, $(2,3,2) \notin \rho$ such that $(1,2,2) \in \rho$

Further, $(H,\circ_\rho)$ is a quasi hypergroup and "$\circ_\rho$" is associative, hence it is a hypergroup.

Since $x \circ_\rho y = \{z \in H \mid (x,z,y) \in \bigcup_{i=2}^{n-1} \rho_{1,i,n}\}$, to prove "$\circ_\rho$" is associative, it is suffices to prove that for any i, j with $2 \le i, j \le n-1$, "$\circ_{R\cup S}$" where R and S are the ternary relations $R = \rho_{1,i,n}$ and $S = \rho_{1,j,n}$ respectively.

We denote the join relations $J_2(R,R)$, $J_2(S,S)$, $J_2(R,S)$, $J_2(S,R)$ by α, β, λ and π respectively.

Let R, S be ternary relations satisfying the conditions of Proposition 3.1 and Corollary 3.6.Then $R \cup S$ is also satisfies the conditions: $(R\cup S)_{1,2} = (R \cup S)_{1,3} = H \times H$ and $(R \cup S)_{2,3} = H \times H$, so, $H_{R\cup S}$ is a hypergroup, but generally, as the following example shows that, $H_{R\cup S}$ is not a hypergroup even if both H_R and H_S are.

Example 3.14. I. Let $H = \{a,b,c,d\}$, $H' = \{a,c,d\}$, $H'' = \{a,b,d\}$

$$R = \{(H,x,x) \mid x \in H\} \cup \{(x,x,H) \mid x \in H\} \cup \{(a,b,H')\} \cup \{(H',b,a)\},$$

$$S = \{(H,x,x) \mid x \in H\} \cup = \{(x,x,H) \mid x \in H\} \cup \{(b,c,H'')\} \cup \{(H'',c,b)\}.$$

Clearly, H_R and H_S are hypergroups and we have; $R \supset \alpha_{1,2,4} \cap \alpha_{1,3,4}$, $S \subset \beta_{1,2,4} \cap \beta_{1,3,4}$. But $(a \circ_{R\cup S} d) \circ_{R\cup S} d = H \neq \{a,b,d\} = a \circ_{R\cup S} (d \circ_{R\cup S} d)$.

II. If one supposes $(a,c,d) \in R \cup S$ for $(a,b,c) \in R$, $(b,c,d) \in S$, then "$\circ_{R\cup S}$" is associative.

III. If one supposes $(\lambda_{1,2,4} \cap \lambda_{1,3,4}) \cup (\pi_{1,2,4} \cap \pi_{1,3,4}) \cup R \cup S \subset \{(\alpha_{1,2,4} \cap \alpha_{1,3,4}) \cup (\beta_{1,2,4} \cap \beta_{1,3,4})\}$.

Then $H_{R\cup S}$ is the total hypergroup.

Remark 3.15. The condition $(a,c,d) \in R \cup S$ for $(a,b,c) \in R$, $(b,c,d) \in S$ is not necessary for $H_{R\cup S}$ to be a hypergroup but sufficient one as one sees in IV.

IV. Set $H = \{1,2,3\}$, $R = \{(H,x,x) \mid x \in H\} \cup \{(x,x,H) \mid x \in H\} \cup \{(1,2,1),(1,2,3),(3,2,3)\}$
$S = \{(H,x,x) \mid x \in H\} \cup \{(x,x,H) \mid x \in H\} \cup \{(1,3,2),(2,3,1),((2,3,2)\}$.

We have $(1,2,3) \in R$, $(2,3,1) \in S$, $(1,3,1) \notin R \cup S$, but $H_{R\cup S}$ is hypergroup.

Remark 3.16. Neither of $R \subset \alpha_{1,2,4} \cap \alpha_{1,3,4}$, $S \subset \beta_{1,2,4} \cap \beta_{1,3,4}$, R and S are reflexive nor both H_R, H_S be hypergroups is necessary for $H_{R\cup S}$ to be a hypergroup as one sees in V.

V. Set $H = \{1,2,3\}$, $R = \{(1,2,1),(1,2,2),(1,2,3),(1,3,3),(2,1,2),(2,1,3),(2,3,3),(3,3,3)\}$ and R is symmetry, $S = \{(1,1,1),(1,3,2),(1,2,3),(1,3,3),(3,3,3),(2,2,3),(2,3,3),(3,2,3),(2,3,2)\}$ and S is symmetry, so $(1,2,3) \in R \not\subset \alpha_{1,2,4} \cap \alpha_{1,3,4}$ (since $(1,1,2),(2,2,3) \notin R \Rightarrow (1,2,3) \in \alpha_{1,2,4}$, but $(1,2,3) \notin \alpha_{1,3,4}$) and by similarity, $(1,2,3) \in S \not\subset \beta_{1,2,4} \cap \beta_{1,3,4}$. It is obvious that R and S are not reflexive, $(R\cup S)_{1,2} = (R\cup S)_{1,3} = H \times H$ and $(R\cup S)_{2,3} = H \times H$, by verification it shows that "$\circ_{R\cup S}$" is associative, hence $H_{R\cup S}$ is a hypergroup.

Proposition 3.17. *Let ρ be a reflexive and symmetric ternary relation on H with $|H| \geq 3$ such that $\rho_{1,3} = H \times H$ and $\rho_{1,2} = \rho_{2,3} = H \times H$. If the following conditions are satisfied then $(H, \circ_\rho)$ is a hypergroup.*

(1) $\rho \subset \alpha_{1,2,4} \cap \alpha_{1,3,4}$
(2) *if there exist $(x,y,z) \in \rho$ such that $(x,y,z) \notin \alpha_{1,2,4} \cap \alpha_{1,3,4} \Rightarrow (x,x,y) \in \rho$, for all $x \in H$*

Proof. First of all,we shall check the following equality:

$$\forall\ (x,y,z) \in H^3,\ x \circ_\rho (y \circ_\rho z) = x \circ_\rho y \cup y \circ_\rho y \cup y \circ_\rho z$$

Case (1): We suppose that $(x,y,z) \in \rho \Rightarrow (x,y,z) \in \alpha_{1,2,4} \cap \alpha_{1,3,4}$.

Now, we verify: $u \in x \circ_\rho (y \circ_\rho z) \Rightarrow u \in x \circ_\rho y \cup y \circ_\rho y \cup y \circ_\rho z$.

"$\Rightarrow$" There exists $v \in y \circ_\rho z$, such that $u \in x \circ_\rho v$. Hence $(x,u,v) \in \rho$ with $(y,v,z) \in \rho$. Since $(x,y,z) \in \alpha_{1,2,4} \cap \alpha_{1,3,4} \Rightarrow (x,y,t,z) \in J_2(\rho,\rho) \Leftrightarrow (x,y,t),(y,t,z) \in \rho$, for $t = v$, we have $(x,y,v),(y,v,z) \in \rho$, whence $(x,u,v) \in \rho$ and $(x,y,v) \in \rho$. Next we suppose that $\rho_{1,2} = \rho_{2,3} = H \times H$, then,for any $x,u \in H$, there exists, $y,v \in H$, such that $u \in x \circ_\rho v \cap x \circ_\rho y$ and for any $y,x \in H$, there exists, $u,v \in H$, such that $y \in x \circ_\rho v \cap x \circ_\rho u$. Therefore from $(x,u,v) \in \rho$, $(x,y,v) \in \rho$ and $u \neq y$ it results $(x,u,y) \in \rho$ or $(x,y,u) \in \rho$.

If $(x,u,y) \in \rho$, then $u \in x \circ_\rho y$.

If $(x,y,u) \in \rho$, then $(x,y,z) \in \alpha_{1,2,4} \cap \alpha_{1,3,4} \Rightarrow (y,u,z) \in \rho$, whence $u \in y \circ_\rho z$.

Since ρ is reflexive,we assume without loss of generality $(y,x,y) \notin \rho$, for any $x,y \in H$, then it follows that $u = y \in y \circ_\rho y$.

Therefore, $u \in x \circ_\rho y \cup y \circ_\rho y \cup y \circ_\rho z$.

"$\Leftarrow$" Suppose $u \in x \circ_\rho y$. Then $(x,u,y) \in \rho$. So we have $(x,u,y) \in \rho$ and $(x,y,z) \in \rho$. From $(x,y,z) \in \alpha_{1,2,4} \cap \alpha_{1,3,4}$, we obtain $(u,y,z) \in \rho$ or $\exists\ t \in H$ such that $(x,y,t),(y,t,z) \in \rho$.

If $(u,y,z) \notin \rho$, then $(x,u,y) \in \rho \subset \alpha_{1,2,4} \cap \alpha_{1,3,4} \Rightarrow (x,u,y) \in \alpha_{1,2,4} \cap \alpha_{1,3,4} \Rightarrow$ there exist $t \in H$ such that $(x,u,t) \in \rho$ and $(u,t,y) \in \rho$. From $(y,t,z) \in \rho$, it follows $t \in y \circ_\rho z$. Therefore $u \in x \circ_\rho (y \circ_\rho z)$.

Now, suppose $u \in y \circ_\rho y$. Since ρ is reflexive and $(x,y,z) \in \rho$, as $(y,x,y) \in \rho$ it follows that$u = y \in y \circ_\rho y \subset x \circ_\rho (y \circ_\rho z)$.

Let's suppose by absurd that there exists $a \in y \circ_\rho z$, with $y \in x \circ_\rho a$. Then, $(y,a,z) \in \rho$ and $(x,y,a) \in \rho$. Therefore it results that $(x,y,a,z) \notin J_2(\rho,\rho)$ for any $a \in H$, which is a contradiction with $(x,y,z) \in \alpha_{1,2,4} \cap \alpha_{1,3,4}$, so $u = y \in x \circ_\rho (y \circ_\rho z)$.

Finally, suppose $u \in y \circ_\rho z$. Choose $v \in H$ such that $(z,v,u) \in \rho$ (this choice is possible because of $\rho_{1,2} = \rho_{1,3} = H \times H$, for any $v \in H$, $\exists\ u_v, z_u \in H$ such that $(z,v,u) \in \rho$, in particular $u = v$ exist). Again, since $\rho_{1,3} = H \times H$ it follows that $(v,u,x) \in \rho$, and by symmetry of ρ, $(x,u,v) \in \rho$. Now, we have $(z,y,u) \in \rho$, $(z,v,u) \in \rho$ and $\rho \subset \alpha_{1,2,4} \cap \alpha_{1,3,4}$, so $(z,y,v) \in \rho$ or $(z,v,y) \in \rho$ (since $(a,t,b),(a,s,b) \in \rho \subset \alpha_{1,2,4} \cap \alpha_{1,3,4} \Leftrightarrow (a,s,t) \in \rho \wedge (s,t,b) \in \rho$ or $(a,t,s) \in \rho \wedge (t,s,b) \in \rho$). On the other hand, $(v,u,x) \in \rho$ and so $u \in x \circ_\rho (y \circ_\rho z)$.

Therefore, $\forall\ (x,y,z) \in H^3$,

$$x \circ_\rho (y \circ_\rho z) = x \circ_\rho y \cup y \circ_\rho y \cup y \circ_\rho z.$$

Similarly, we prove $\forall\ (x,y,z) \in H^3$,

$$(x \circ_\rho y) \circ_\rho z = x \circ_\rho y \cup y \circ_\rho y \cup y \circ_\rho z.$$

Case (2): We suppose that $(x,y,z) \in \rho$ such that $(x,y,z) \notin \alpha_{1,2,4} \cap \alpha_{1,3,4} \Leftrightarrow (x,x,y) \in \rho$, for all $x \in H$.

If $u = x \in x \circ_\rho (y \circ_\rho z)$, then $u = x \in x \circ_\rho y$ (because $(x,x,y) \in \rho$).

If $u = y \in x \circ_\rho (y \circ_\rho z)$, then $u = y \in y \circ_\rho y$ (because $(y,y,y) \in \rho$ by reflexivity).

If $u = z \in x \circ_\rho (y \circ_\rho z)$, then $u = z \in y \circ_\rho z$ (because $(z,z,y) \in \rho$, by symmetry $(y,z,z) \in \rho$).

Suppose $u \neq x \neq y \neq z$. If $u \in x \circ_\rho y$, then $(x,u,y) \in \rho$ and $\rho_{1,3} = H \times H \Rightarrow (u,y,z) \in \rho$. This shows that $(x,y,z) \in \alpha_{1,3,4}$. Similarly,if $u \in y \circ_\rho z$, then we obtain $(x,y,z) \in \alpha_{1,2,4}$ which is a contradiction to the hypothesis.

Therefore, in this case we have,

$$\begin{aligned} x \circ_\rho (y \circ_\rho z) &= x \circ_\rho y \cup y \circ_\rho y \cup y \circ_\rho \\ &= \{x,y,z\} \\ &= (x \circ_\rho y) \circ_\rho z. \end{aligned}$$

This completes the proof. □

Corollary 3.18. *Let ρ be a reflexive and symmetric ternary relation on H with $|H| \geq 3$ such that $\rho_{1,3} = H \times H$ and $\rho_{1,2} = \rho_{2,3} = H \times H$. If*

(1) $\rho \subset \alpha_{1,2,4} \cap \alpha_{1,3,4}$;

(2) $(x,y,z) \in \rho$, *then x, y, z are distinct;*

(3) $(x,y,x) \notin \rho$, *for any $x,y \in H$;*

then the hyperoperation "$\circ_\rho$" is not associative.

Proof. Since $\rho_{1,3} = H \times H = \rho_{1,2}$, we have (i) $(x,y,z) \in \rho \Rightarrow (y,z,t) \in \rho$, for some $t \in H$, and from $(x,y,z) \in \alpha_{1,2,4} \cap \alpha_{1,3,4}$; we obtain (ii) if $y \neq x$ and $y \neq z$, then there exist $u \in H$ such that $(x,u,y) \in \rho$.

Then the proof follows (from [6, Theorem 27, p. 39]).

Corollary 3.19. *Let ρ be a reflexive and symmetric ternary relation on H with $|H| \geq 3$ such that $\rho_{1,3} = H \times H$ and $\rho_{1,2} = \rho_{2,3} = H \times H$. If*

(1) $\rho \subset \alpha_{1,2,4} \cap \alpha_{1,3,4}$;
(2) $\nexists\ (x,y,z) \in \rho$ *such that* $(x,x,y) \in \rho$, $\forall\ (x,y) \in H^2$;

then $(H, \circ_\rho)$ is the total hypergroup.

Proof. We prove that ρ is 3-transitive and then, by Proposition 3.3, it results the conclusion. Since $\rho_{1,2} = \rho_{1,3} = H \times H$, for any $x \in H$, there exists $a_x, c_x \in H$ such that $(x, a_x, x) \in \rho$ and $(a_x, x, c_x) \in \rho$. Again, $\rho_{1,3} = H \times H$, it follows that there exists $u_x \in H$ such that $(x, u_x, c_x) \in \rho$.
We suppose that, $\nexists\ (x,y,z) \in \rho$ such that $(x,x,y) \in \rho$, that is $(x,y,y) \in \rho$ with $(x,y,z) \in \rho$.
Now, we verify the conditions in Example 2.3. Set $u_x = y = a_x$, $c_x = z$, then we have $(z,y,y) \in \rho$, $(y,x,z) \in \rho$ and $(a_z, z, c_z) \in \rho$, for any $z \in H$.

(i) $(x,y,z) \in \rho$, $(y,x,z) \in \rho \Rightarrow (x,x,z) \in \rho$ (because $(y,x,z) \in \rho \Rightarrow (x,x,z)$).

Since ρ is reflexive and $(x,y,z) \in \alpha_{1,2,4} \cap \alpha_{1,3,4} \Leftrightarrow (x,y,y) \in \rho \wedge (y,y,z) \in \rho$ and by symmetry $(z,y,y) \in \rho$, it follows that

(ii) $(x,y,z) \in \rho$, $(z,y,y) \in \rho \Rightarrow (x,y,y) \in \rho$, $(y,y,y) \in \rho$.

(iii) $(x,y,z) \in \rho$, $(x,x,x) \in \rho \Rightarrow (x,y,x) \in \rho$ (because ρ is reflexive).

This completes the proof. □

Remark 3.20. Obviously, the element, $y \in H$ such that $(y,x,x) \in \rho$ and ρ is symmetric relation implies that y acts as scalar identity. That is $|x \circ_\rho y| = 1$, $\forall\ x \in H$. We denote it by "e".
Let ρ be a ternary relation on H. Let $(H, \circ_\rho)$ be the hypergroupoid defined as follows.

$$\begin{array}{ll} \forall\ x \in H & x \circ_\rho x = \{x\} \\ \forall\ (x,y) \in H^2 & x \circ_\rho y \ni z \Leftrightarrow (x,z,y) \in \rho, \\ & x \circ_\rho e = x,\ \forall\ x \in H, \text{ where } e \text{ is an ideal element } e \notin H \end{array}$$

Let H_ρ denote the hypergroupoid associated as in (δ_1).

Proposition 3.21. *Let ρ be a reflexive and symmetric ternary relation on H with $|H| \geq 3$ such that $\rho_{1,3} = H \times H$. If there exists $(x,z,y) \in \rho$ such that $(x,z,y) \notin \alpha_{1,2,4} \cap \alpha_{1,3,4} \Rightarrow$ the extension of "$\circ_\rho$" defined by setting*

$$\begin{array}{ll} \forall\ x \in H & x \circ_\rho x = \{x\} \\ \forall\ (x,y) \in H^2 & x \circ_\rho y = \{x,y,e\} \Leftrightarrow (x, z = e, y) \in \rho, \\ \forall\ x \in H & x \circ_\rho e = \{x\}, \\ & (\textit{that is, if } \exists\ (x,z,y) \notin \alpha_{1,2,4} \cap \alpha_{1,3,4}, \textit{ then } z = e) \end{array}$$

then $(H, \circ_\rho)$ is a join space.

Proof. By the resetting of the product, for any $(x,y,z) \in \rho$, we have, $y \in H$ exist such that $(y,x,x) \in \rho$, for any, $x \in H$, while $(x,x,z) \in \rho$, $(x,z,z) \in \rho$.

Therefore, for any $(x,y) \in H^2$, we have $(x,y) \in \rho_{1,2}$, and $(x,y) \in \rho_{2,3}$ (by symmetry).

Moreover, ρ is a reflexive relation,hence $(x,x) \in \rho_{1,2} \cap \rho_{2,3}$, whence $\rho_{1,3} = \rho_{1,2} = \rho_{2,3} = H \times H$. It follows from Proposition prop3.1, that $(H,\circ_\rho)$ is a quasihypergroup.

Since $x \circ_\rho y = \{x,y,z\}$ where $(x,z,y) \in \rho$ and by the symmetry of relation ρ, we obtain that $\forall\ (x,y) \in H^2$, $x \circ_\rho y = y \circ_\rho x = \{x,y,z\}$, hence $(H,\circ_\rho)$ is commutative.

Now we prove that the hyper operation $\circ_\rho$ is associative.

Let $u \in (x \circ_\rho y) \circ_\rho z$. Then, there exists $v \in x \circ_\rho y$ such that $u \in v \circ_\rho z$.

Now, $(x,v,y) \in \rho$ with $(v,u,z) \in \rho$, hence $x \circ_\rho y = \{x,v,y\}$ and $v \circ_\rho z = \{v,u,z\}$.

We suppose that $(x,y,z) \in \rho$, then we notice that $y = e$.

Therefore, $\forall\ (x,y,z) \in H^3$,

$$\begin{aligned}(x \circ_\rho y) \circ_\rho z &= \cup\{x,e,y\} \circ_\rho z \\ &= x \circ_\rho z \cup e \circ_\rho z \cup y \circ_\rho z \\ &= \{x,e,z\} \cup \{z\} \cup \{y,z,e\} \\ &= \{x,e,y,z\}.\end{aligned}$$

Similarly we can show that $x \circ_\rho (y \circ_\rho z) = \{x,y,e,z\}$.

It remains to check the condition of the join space. Set $a,b,c,d \in H$, $b = e$, then $a/b = \{a\}$, $a/c = \{a\}$, $a/d = \{a\}$, $b/b = \{b\}$, $c/c = \{c\}$, $c/d = \{c\}$, $a/a = H$, and by symmetry we have $\forall\ (a,b) \in H^2$, $a/b = a \circ_\rho c$, where $c \neq b$ is such that $b \circ_\rho c = \{b,z,e\}$.

So, $a/b \cap c/d \neq \phi \Rightarrow a \circ_\rho c \cap d \circ_\rho c \neq \phi$, where $d \neq c$, whence $\{a\} \cap b \circ_\rho (d \circ_\rho c)$, hence $a/d \cap b \circ_\rho c \neq \phi$, that is $a \circ_\rho d \cap b \circ_\rho c \neq \phi$ (by [6, Theorem 64.2, p. 12]), vide [3, Theorem 157,2]. □

Example 3.22. On the set $H = \{1,2,3\}$, consider the ternary relation ρ.

We suppose that $(1,2,3) \notin \alpha_{1,2,4} \cap \alpha_{1,3,4}$ but $(1,2,3) \in \rho$. From the product, we have $\forall\ x \in H$, $x \circ_\rho x = \{x\} \cap x \in x \circ_\rho x$, whence $(x,x,x) \in \rho$, $\forall\ x \in H$, hence ρ is reflexive $1 \circ_\rho 3 = \{1,2,3\} \Leftrightarrow (1,2 = e,3) \in \rho$, whence $1 \in 1 \circ_\rho 3 = 3 \circ_\rho 1$, hence $(1,1,3) \in \rho$ and $(3,1,1) \in \rho$. Similarly, $(1,2,3) \in \rho$, $(3,2,1) \in \rho$, $(1,3,3) \in \rho$ and $(3,3,1) \in \rho$.

Set $e = 2$, $x \circ_\rho 2 = \{x\} \Rightarrow x \in x \circ_\rho 2 = 2 \circ_\rho x$, whence $(x,x,2) \in \rho$, $(2,x,x) \in \rho$, $\forall\ x \in H$.

Thus the associated ρ is reflexive and symmetry. It is easy to check that $\rho_{1,3} = H \times H$ and $\rho_{1,2} = \rho_{2,3} = H \times H$. The associative axiom is obvious from the product. Also,it is a join space with scalar identity.

Remark 3.23. Given a hypergroupoid $(H,\circ_\rho)$ associated with a ternary relation ρ, we defined the ternary relation ρ such that $\rho_{1,3} = H \times H$ and ρ is reflexive and symmetric by adjoining an ideal element "e". The extended product satisfies the axioms of a hypergroup. That is, if $(H,\circ_\rho)$ is a hypergroup with scalar identity, then a reflexive symmetric relation is associated on H.

Definition 3.24. Let H be hypergroup and K be a nonempty set of H. K is called subhyper-group of H if for every $a \in K$, $a \circ K = K \circ a = K$. A subhyper-group K of H is called closed on the left (on the right) if for every x, y in K and for every a in H, from $x \in a \circ y$ $(x \in y \circ a)$ it results

$a \in K$, K is closed if it is closed on the left and on the right.

Proposition 3.25. *Let H_ρ be a hypergroup with scalar identity associated with an n-ary relation ρ with $|H| \geq 3$. Then, the n-ary relation ρ is reflexive and symmetric such that $\rho_{1,n} = H \times H$ and satisfies the following.*

(1) $\rho_{1,j,n} \subset \alpha_{1,n,2n-2} \cap \alpha_{1,n-1,2n-2}$, $\forall\ j \in \{2,\dots,n-1\}$

(2) *if there exists $j \in \{2,\dots,n-1\}$ such that $\rho_{1,j,n} \not\subset \alpha_{1,n,2n-2}$ or $\rho_{1,j,n} \not\subset \alpha_{1,n-1,2n-2}$, then "$\circ_\rho$" is redefined by setting,*

for any $i \in \{2,\dots,n-1\}$,

$$\begin{aligned}
&\forall\ x \in H, && x \circ_i x = \{x\}\\
&\forall\ (x,y) \in H^2 && x \circ_i y = \{x,y,e\} \Leftrightarrow (x,z=e,y) \in \rho_{1,i,n},\\
&\forall\ x \in H && x \circ_i e = \{x\}
\end{aligned}$$

and

$$\begin{aligned}
&\text{(i)}\quad \forall\ x \in H, && x \circ_\rho x = \left\{x \in H \mid (x,x,x) \in \bigcup_{i=2}^{n-1} \rho_{1,i,n}\right\} = \bigcup_{i=2}^{n-1} x \circ_i x = \{x\}\\
&\text{(ii)}\quad \forall\ (x,y) \in H^2, && x \circ_\rho y = \left\{z \in H \mid (x,z,y) \in \bigcup_{i=2}^{n-1} \rho_{1,i,n}\right\} \cup \{x,y,e\}\\
& && \qquad = \bigcup_{i=2}^{n-1} x \circ_i y \cup \{x,y,e\}\\
&\text{(iii)}\quad \forall\ x \in H && x \circ_i e = \{x\} = \bigcup_{i=2}^{n-1} x \circ_i e
\end{aligned}$$

Proof. Reflexivity and symmetric nature of the n-ary relation ρ such that $\rho_{1,n} = H \times H$ follows from Proposition prop3.21. Next, we prove (1). Since $(H,\circ_i)$, $\forall\ i \in \{2,\dots,n-1\}$ is a subhypergroup of $(H,\circ_\rho)$, by Corollary 3.6,the ternary relations $\lambda = \rho_(1,i,n) \subset \beta_{1,2,4} \cap \beta_{1,3,4}$, where β denote the join relation $J_2(\lambda,\lambda)$.

Let $(a,b,d) \in \lambda = \rho_{1,i,n}$; then $\exists\ (a,b,c) \in \lambda = \rho_{1,i,n}$ and $(b,c,d) \in \lambda = \rho_{1,i,n}$ such that $(a,b,c,d) \in J_2(\lambda,\lambda)$.

Similarly, $(a,c,b,d) \in J_2(\lambda,\lambda)$ or $\exists\ u \in H$ such that $(a,u,b,d) \in J_2(\lambda,\lambda)$. It is not severe restriction that $(a,c,b,d) \in J_2(\lambda,\lambda)$ and $(a,c,d) \in \lambda$, since $(H,\circ_i)$ is satisfies the associative axiom.

As the above observation for $\rho_{1,i,n}$ is true, for any $i \in \{2,\dots,n-1\}$; without loss of generality, we assume that $(a,b,d) \in \lambda = \rho_{1,i,n}$, $\forall\ i \in \{2,\dots,n-1\}$. Since $\rho_{1,n} = H \times H$, for $(a,b,d) \in \lambda = \rho_{1,i,n}$, there exists $(a_2,\dots,a_i = b,\dots,a_{n-1}) \in H^{n-2}$ such that $(a,a_2,\dots,\underbrace{a_i = b}_{i\text{-th place}},\dots,a_{n-1},d) \in \rho$.

Similarly, for $(a,b,c) \in \lambda = \rho_{1,i,n} = \lambda \ni (b,c,d)$, there exists $(b_2,\dots,b_i,\dots,b_{n-1}) \in H^{n-2} \ni (c_2,\dots,c_i,\dots,c_{n-1})$ such that $(a,b_2,\dots,b_i,\dots,b_{n-1} = b,c) \in \rho$ and $(b,c = c_2,\dots,c_i,\dots,c_{n-1},d) \in \rho$. If $(a,b,d) \in \lambda = \rho_{1,i,n}$, $\forall\ i \in \{2,\dots,n-1\}$, then the condition (S): $(x,a_1,a_2,a_3,\dots,a_{n-2},y) \in \rho \Leftrightarrow$

$(x, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, \ldots, a_{\sigma(n-2)}, y) \in \rho$ for any permutation ρ to the set $\{1, 2 \ldots, n-2\}$ is satisfied for $(a, a_2, \ldots, \underbrace{a_i = b}_{i\text{-th place}}, \ldots, a_{n-1}, d) \in \rho$.

We keep (b, c) as $\rho_{n-1,n} \ni (b, c) \in \rho_{1,2}$ while doing this permutation on other elements of ρ, otherwise $(a, b, c, d) \notin J_2(\lambda, \lambda)$. Now, using the definition of join relation, we obtain that

$$(a, b_2, \ldots, b_i, \ldots, b_{n-1} = b, c = c_2, \ldots, c_i, \ldots, c_{n-1}, d) \in J_{2n-2}(\rho, \rho)$$

and by similar argument

$$(a, b_2, \ldots, b_i, \ldots, b_{n-1} = c, b = c_2, \ldots, c_i, \ldots, c_{n-1}, d) \in J_{2n-2}(\rho, \rho),$$

whence

$$\rho_{1,j,n} \subset \rho_{1,n,2n-2} \cap \alpha_{1,n-1,2n-2}, \quad \forall\ j \in \{2, \ldots, n-1\}.$$

This proves (1).

Next, we prove (2). Let $(a, b, c) \in \rho_{1,i,n}$, where $i \in \{2, \ldots, n-1\}$.

We suppose that $\exists$, $j \in \{2, \ldots, n-1\}$ such that $(a, b, c) \in \rho_{1,j,n}$ and $(a, b, c) \in \rho_{1,n,2n-2} \cap \rho_{1,n-1,2n-2}$, that is, $\rho_{1,j,n} \not\subset \alpha_{1,n,2n-2} \cap \alpha_{1,n-1,2n-2}$. With out loss of generality we suppose that $\rho_{1,j,n} \not\subset \alpha_{1,n,2n-2} \cap \cap_{1,n-1,2n-2}$, $\forall\ j \in \{2, \ldots, n-1\}$.

Now,we consider the following cases.

(i) If $a \neq b$, $a \neq c$, then $(a, b, a) \in \rho_{1,j,n}$, $\forall\ j$, hence $b \in a \circ_j a$, $\forall\ j$, but by Proposition 3.21 $a \in a \circ_j a$ for atleast one $j \in \{2, \ldots, n-1\}$, a contradiction to $a \neq b$.

Therefore $a \in a \circ_j a$, $\forall\ j \in \{2, \ldots, n-1\}$.

If $a = b$, $b \neq c$, $c \neq e$, then $(a, a, c) \in \rho_{1,j,n}$ $\forall\ j$, hence $a \in a \circ_j c$, $\forall\ j$. On the other hand $(H, \circ_j)$ $\forall\ j$, is a join space with scalar identity "e", (since each $(H, \circ_j) \supset \langle a \rangle$ the least closed subhypergroup; H_ρ is generated by the set $X = \{a, b, e\}$, where b is the inverse of a), so it is a canonical hypergroup, $\exists$ inverse of "a" say 'a'. From $a \in a \circ_j c$, we obtain $c \in a \circ_j a' = \{a, a', e\}$, it follows that $a = c$, $a' = c$, $c = e$, which are contradiction, whence $a \in a \circ_j a$, for all $2 \leq j \leq n-1$. Therefore, $\bigcup_{i=2}^{n-1} x \circ_i x = \{x\}$, $\forall\ x \in H$.

(ii) First of all ,we notice that $\forall\ (x, y) \in H^2$, $\{x, y, e\} \subset H_i \subset H_j$ for any $i < j$.

Now, let $x \neq y$, $x \neq z$ and $(x, z, y) \in \rho_{1,i,n}$ for any $i \in \{2, \ldots, n-1\}$. Then $z \in x \circ_i y$.

We suppose that $\exists\ t \in H$ such that $(x, t, y) \in \rho_{1,j,n}$ for any $j \in \{2, \ldots, n-1\}$, $i < j$.

Then

$$x \circ_{i \cup j} y = x \circ_i y \cup x \circ_j y = \{x, y, e\} \cup \{z \in H \mid (x, z, y) \in \rho_{1,i,n}\}$$

or

$$x \circ_{i \cup j} y = x \circ_i y \cup x \circ_j y = \{x, y, e\} \cup \{t \in H \mid (x, t, y) \in \rho_{1,j,n}\}.$$

$\Leftrightarrow$ $t = e$ or $z = e$, for any $i < j$.

Therefore it is suffices to prove that $t = e$ or $z = e$, for any $i < j$ and $t \circ_{i \cup j} z = t, z$.

Suppose that $t \neq e$ and $z \neq e$, then $t \circ_k z = \{t, z, e\}$, for $i < j < k$, hence $e \in t \circ_k z$, $t \in t \circ_k z$, $z \in t \circ_k z$. This implies that t is inverse of z, but $x \circ_i y = \{x, y, e\}$, so $z = x$ or $z = y$ as $z \neq e$, it follows that y is inverse of z, a contradiction to t is inverse of z, hence $z = e$. Similarly, we show that $t = e$, if $z \neq e$.

It is remains to check that $t \circ_{i \cup j} z = \{t, z\}$. Suppose to the contrary $t \circ_{i \cup j} z = \{t, e, z\}$. Then by the previous paragraph, we obtain that $t = e$, if $z \neq e$ and $z = e$ if $t \neq e$ whenever $z \neq t$ and $j = i + 1$. By induction on j, $\forall$ $(x, y) \in H^2$,

$$x \circ_\rho y = \left\{ z \in H \mid (x, z, y) \in \bigcup_{i=2}^{n-1} \rho_{1,i,n} \right\} \cup \{x, y, e\} = x \circ_i y \cup \{x, y, e\}.$$

(iii) Finally, we show that $\forall$ $x \in H$, $x \circ_j e = \{x\}$, $\forall$ $j \in \{2, \ldots, n-1\}$, that is the identity e is scalar one, and thereby proving that $\{x\} = \bigcup_{i=2}^{n-1} x \circ_i e$. Let $x \neq y \neq z$ and y is inverse of x. Indeed, we have $x \circ_j e \supset \{x, y, e\}$ for any $2 \leq j \leq n-1$ which is a subhypergroup with identity e. Suppose to the contrary $e \in x \circ_j e$. Then $y \in x \circ_j e$, otherwise $x \circ_j e = \{e\}$, a contradiction to Proposition 3.21. Hence $y \in x \circ_j e$, whence $x \in e/y \cap y/e$.

Since $(H, \circ_j)$ is join space, it follows that $e = y$ which is a contradiction.

Hence $(x, x, e) \in \rho_{1,j,n}$, $\forall$ $j \in \{2, \ldots, n-1\}$, therefore $\forall$ $x \in H x \circ_i e = \{x\} = x \circ_i e$.

This completes the proof of the proposition. □

Finally, we give an example of an infinite join space.

Example 3.26. Set $H = \{x_n, y_n, z_n, \ldots \mid n \in Z^+\}$ and let $\rho \supseteq H \times H \times H$ be the ternary relation on H defined by $\rho = \{(x_0, y_0, z_0), (x_1, y_0, z_0), (x_0, y_1, z_0), (x_1, y_1, z_1), (x_2, y_0, z_1), (x_0, y_2, z_1), (x_2, y_2, z_2), \ldots, (x_n, y_n, z_n), (x_{n+1}, y_0, z_n), (x_0, y_{n+1}, z_n), (x_{n+1}, y_{n+1}, z_{n+1}), \ldots\}$.

Now, we define u, v, w are functions such that $u(x_0) = 0$, $v(y_0) = 0$ and $w(z_0) = 1$ and $u(x_n) = -2^{n-1} = v(y_n)$, $w(z_n) = 2^n$, for $n > 0$. Then, we define the hyperproduct f as $f(x, y, z) = \{u(x) + v(y), 0, w(z)\}$, $\forall$ $(x, y, z) \in \rho$, where $u(x) + v(y)$ is inverse of $w(z)$.

Also, we remark that the product is defined in the solution space of the equation

$$f(x, y, z) = 1 \text{ at } (x_0, y_0, z_0).$$

It verifies the Proposition 3.21.

4. Conclusion and Future Work

Many connection between hypergroups and ternary relation have been considered and investigated.In this paper we considered,a joinspace and we associated it to a particular ternary relation.We generalised it to the case of n-ary relation.

The operations on databases (such union, intersection, cartesian product, projection, join) can be extended to similar operations on n-ary relation. In a future work we try to obtain some relations ships between the hypergroupoids associated with the two n-ary relations and the hypergroupoids associated with their union intersection, join, cartesian product.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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