



## Bridging Disks and Shells Methods: A Probabilistic Approach

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**Abstract.** Consideration of a function shaping the surface of a solid of revolution as a random variable allows for extension of the proof that the disks and the shells methods give the same result from monotonic continuous functions to a subclass of Riemann-integrable functions.

### 1. Introduction

Publications [1, 2, 4, 6] raised a problem of equivalence of the disks and the shells methods of calculation of the volumes of solids of revolution. They claim that the equality of the obtained results should be proved. To prove, authors compared the integrals of a function and of its inverse function using integration by parts. They proved that if a function  $f(x)$ ,  $a \leq x \leq b$  is continuous and monotonic then the shells and the disk methods give the same result. For example, for increasing function  $f(x)$  the volume of a solid of revolution around the  $x$ -axis is [4]:

$$(1.1) \quad V_x = \pi \int_a^b [f(x)]^2 dx = \pi \left( bf(b)^2 - af(a)^2 - \int_c^d 2yf^{-1}(y)dy \right),$$

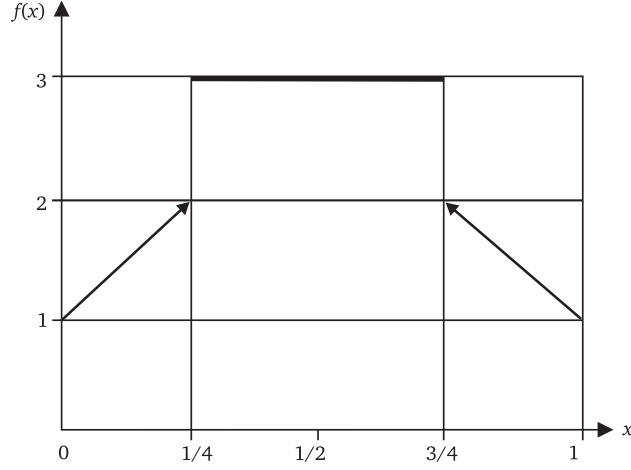
where  $c = f(a)$ ,  $d = f(b)$ ,  $V_x$  stands for the volume of a solid of revolution around the  $x$ -axis, and  $f^{-1}(y)$ , for an inverse function. The first expression corresponds to the disks method, the second, to the shells one. Monotonicity and continuity guarantee existence of the inverse function and the integrals.

The obtained result does not cover most of practically important cases because it is not applicable, for example, to piecewise functions that describe shapes typical for engineering applications. Thus, a function

$$(1.2) \quad f(x) = \begin{cases} 4x + 1, & 0 \leq x < \frac{1}{4} \\ 3, & \frac{1}{4} \leq x \leq \frac{3}{4} \\ 5 - 4x, & \frac{3}{4} < x \leq 1, \end{cases}$$

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**Figure 1.** Piecewise function  $f(x)$ .

is neither monotonic nor continuous, see Figure 1. Only the disks method can be applied. The shells method, as given by formula (1.1), cannot be applied because the inverse function  $f^{-1}(y)$  does not exist. In this paper below we show that the shells method can be modified appropriately to be applicable to a broader class of functions that includes, in particular, piecewise functions. The equality of the disks method and the modified shells method is also proved.

Without loss of generality, we can consider functions  $f(x)$  defined on a closed interval  $[0, 1]$ , because all integrals that appear in this paper below change proportionally if  $a \leq x \leq b$ . Assuming  $0 \leq x \leq 1$ , an approach suggested in [5] allows for consideration of a function  $f(x)$ ,  $0 \leq x \leq 1$ , as a random variable. This random variable is defined on a probability space comprising sample space  $[0, 1]$ ,  $\sigma$ -algebra generated by open subintervals  $(\alpha, \beta)$ ,  $0 \leq \alpha < \beta \leq 1$ , and a measure equal to the lengths of subintervals:  $\text{mes}[(\alpha, \beta)] = \beta - \alpha$ . The measure is a “probability” assigned to subinterval  $(\alpha, \beta)$ :

$$(1.3) \quad P(\alpha, \beta) = \text{mes}[(\alpha, \beta)] = \beta - \alpha.$$

In these settings, the disks method may be considered as calculation of  $\pi$  times the second moment of a random variable  $f(x)$  by integration over the sample space:

$$(1.4) \quad V_{x,D} = \pi \int_0^1 [f(x)]^2 dx = \pi E[(f(x))^2],$$

where  $E$  is the expected value of a random variable.

To obtain a formula for the shells method, we restrict ourselves in this paper below with Riemann-integrable functions  $f(x)$  that have no more than a finite number of discontinuity points of the jump type and are constant on no more than a finite number of subintervals  $(\alpha_i, \beta_i)$ . These restrictions allow us to cover most of calculus applications while eventually remaining in the framework of the Riemann

integral. The formula for the shells method given below for has a clear geometric interpretation and can be used without referring to the measure or probability theory. Proof for a more general class of functions may be obtained following [7].

As known, a Riemann-integrable function  $f(x)$  is bounded:  $c \leq f(x) \leq d$ . As the function  $f(x)$  is not assumed continuous, it may or may not reach its minimum or maximum values. We define  $c = \inf\{f(x), a \leq x \leq b\}$  and  $d = \sup\{f(x), a \leq x \leq b\}$ , the greatest lower bound and the least upper bound of the range of the function  $f(x)$ , respectively. The values of  $c$  and  $d$  serve as limits of integration for the shells method. A Riemann integral of  $f(x)^2$  exists because it is bounded together with  $f(x)$  and has no more discontinuity points than the function  $f(x)$ .

As known from the probability theory [5], the second moment may be calculated alternatively by using a cumulative distribution function  $F_f(y)$ , where  $F_f(y) = P[f(x) < y]$ .

Geometrically,  $F_f(y)$  is the sum of the lengths of the subintervals  $(\alpha_i, \beta_i)$  on which  $f(x) < y$ . Recall that the  $F_f(y)$  is continuous from the left. A formula for calculation of the second moment using  $F_f(y)$  involves the Riemann-Stieltjes or Lebesgue-Stieltjes integral [7] and is this:

$$(1.5) \quad E[(f(x))^2] = \int_{-\infty}^{\infty} y^2 dF_f(y) = \int_c^d y^2 dF_f(y).$$

We use in this paper below more general Lebesgue-Stieltjes integral because it has better properties with regards to the integration by parts [3] that are essential for the objectives of this paper. Recall that the final formula obtained in this paper contains the Riemann integral only.

The integration by parts formula for the Lebesgue-Stieltjes integral is this [3, 7]:

$$(1.6) \quad \int_c^d U dV = U(d+)V(d+) - U(c-)V(c-) - \int_c^d V dU,$$

provided that either  $U(t)$  or  $V(t)$  is continuous. In our case,  $U(y) = y^2$  is continuous,  $V(y) = F_f(y)$ , so that from formula (1.5) it follows that

$$(1.7) \quad \begin{aligned} \int_c^d y^2 dF_f(y) &= (d+)^2 F_f(d+) - (c-)^2 F_f(c-) - \int_c^d F_f(y) d(y^2) \\ &= d^2 \cdot 1 - (c-)^2 \cdot 0 - \int_c^d F_f(y) d(y^2) \\ &= d^2 - \int_c^d 2y F_f(y) dy. \end{aligned}$$

Now we have two equal expressions: (1.5) and (1.7) – for the second moment of the random variable  $f(x)$ . After multiplication by  $\pi$  they give us two equal expressions for the volume of a solid of revolution:

$$(1.8) \quad V_{x,D} = \pi \int_0^1 [f(x)]^2 dx$$

and

$$(1.9) \quad V_{x,S} = \pi \left( d^2 - \int_c^d 2yF_f(y) dy \right).$$

The former stands for the disks method, the latter, for the shells one. The integral in the shells method formula (1.9) exists as the Riemann integral because a finite number of the subintervals where function  $f(x)$  is constant guarantees that the cumulative distribution function  $F_f(y)$  has no more than a finite number of the jump-type discontinuity points.

For continuous monotonically increasing functions  $y = f(x)$ ,  $a = 0 \leq x \leq 1 = b$ , the inverse function  $f^{-1}(y)$  equals the cumulative distribution function  $F_f(y)$  for  $c \leq y \leq d$ , and formula (1.9) for  $V_{x,S}$  in the shells method simplifies to formula (1.1) obtained in [4].

In case  $a \leq x \leq b$ , the cumulative distribution function  $F_f(y)$  should be changed for a more general function that maps  $y$  to  $F_f(y) = \text{mes}\{x | f(x) < y\}$  that is not the cumulative probability because it is not bounded by 1 from the above. But geometrically  $F_f(y)$  remains the sum of the lengths of the subintervals  $(\alpha_i, \beta_i)$  on which  $f(x) < y$ . In this case,  $F_f(c) = 0$ ,  $F_f(d) = b - a$ , and from formula (1.7) we get

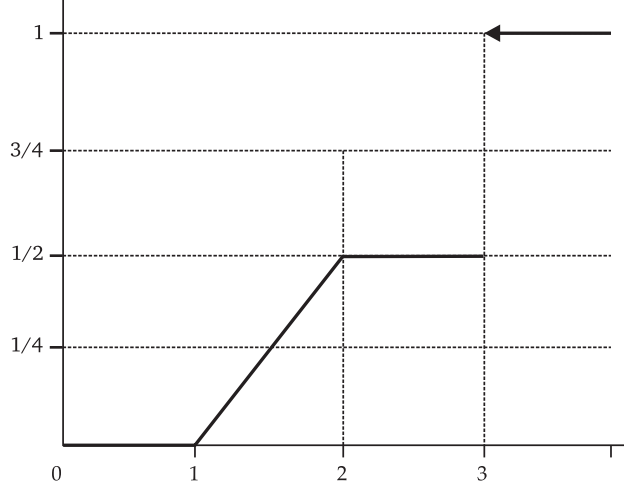
$$\begin{aligned} V_{x,S} &= \pi \int_c^d y^2 dF_f(y) \\ &= \pi \left( (d+)^2 F_f(d+) - (c-)^2 F_f(c-) - \int_c^d F_f(y) d(y^2) \right) \\ &= \pi \left( d^2 \cdot (b - a) - (c-)^2 \cdot 0 - \int_c^d F_f(y) d(y^2) \right) \\ (1.10) \quad &= \pi \left( d^2(b - a) - \int_c^d 2yF_f(y) dy \right). \end{aligned}$$

The last expression generalizes formula (1.1) for the shells method for Riemann-integrable functions.

## 2. Result and Example

Considerations presented above prove the following theorem:

**Theorem.** *The disks and the shells methods of calculation of the volume of a solid of revolution give the same result for Riemann-integrable functions  $f(x)$ ,  $a \leq x \leq b$ , that have no more than a finite number of discontinuity points of the jump type and no more than a finite number of subintervals where they get constant values. As compared with the formula for monotonic continuous functions, the inverse function  $f^{-1}(y)$  in the modified shells method should be changed for the function  $F_f(y)$  equal to the sum of the lengths of the subintervals on which  $f(x) < y$ ; minimum and maximum values should be changed for  $c = \inf\{f(x), a \leq x \leq b\}$*



**Figure 2.** Cumulative distribution function  $F_f(y)$ .

and  $d = \sup\{f(x), a \leq x \leq b\}$ , the greatest lower bound and the least upper bound of the range of the function  $f(x)$ , respectively. The formula for the shells method is this

$$(2.1) \quad V_{x,S} = \pi \left( d^2(b-a) - \int_c^d 2yF_f(y)dy \right).$$

**Example.** For the piecewise function  $f(x)$  given above by formula (1.2) and in Figure 1 we have:  $a = 0$ ,  $b = 1$ ,  $c = 1$ ,  $d = 3$ ,

$$(2.2) \quad \begin{aligned} V_{x,D} &= \pi \int_0^1 [f(x)]^2 dx \\ &= \pi \left( \int_0^{1/4} (4x)^2 dx + \int_{1/4}^{3/4} 3^2 dx + \int_{3/4}^1 (5-4x)^2 dx \right) \\ &= \frac{17\pi}{3}. \end{aligned}$$

The cumulative distribution function  $F_f(y)$  is this (see Figure 2):

$$(2.3) \quad F_f(y) = \begin{cases} 0, & y \leq 1 \\ \frac{y-1}{2}, & 1 < y \leq 2 \\ \frac{1}{2}, & 2 < y \leq 3 \\ 1, & y > 3 \end{cases}$$

Using formula (1.8) for  $V_{x,S}$ , we get

$$V_{x,S} = \pi \left( d^2 - \int_c^d 2yF_f(y)dy \right)$$

$$\begin{aligned}
 &= \pi \left( 3^2 - 2 \left[ \int_1^2 y \frac{y-1}{2} dy + \int_2^3 y \frac{1}{2} dy \right] \right) \\
 (2.4) \quad &= \frac{17\pi}{3}.
 \end{aligned}$$

The results are equal:  $V_{x,D} = V_{x,S}$  as expected.

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