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Characterization of Special Curves

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Abstract In this study, the new characterizations of special curves are investigated without using the curvatures of these special curves: general helices, slant helices, Bertrand curves, Mannheim curves. The curvatures are given by the help of the norms of the derivatives of Frenet vectors.

1. Introduction

As is well known fundamental structure of differential geometry is the curves. Within the process, most of classical differential geometry topics have been extended to space curves. There are many studies which implies different characterizations of these curves. Kula and Yaylı [3] have investigated spherical images the tangent indicatrix, binormal indicatrix of a slant helix and obtained that the spherical images are spherical helices. By defining slant helices and conical geodesic curves, Izumiya and Takeuchi [1] have considered geometric invariants of space curves.

In this study, using some approaches in [1] and [3], we give the new characterizations of special curves. In these characterizations, the curvatures of these special curves are not used. We show that the curvatures can be given by the help of the norms of the derivatives of Frenet vectors. In this connection, some different theorems are presented.

2. Preliminaries

Now, we recall some basic concepts of the differential geometry of curves:

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Definition 2.1. We assume that the curve α is parametrized by arclength. Then, $\alpha'(s)$ is the unit tangent vector to the curve, which we denote by $T(s)$ Since t has constant length, $T'(s)$ will be orthogonal to $T(s)$. If $T'(s) \neq 0$ then we define principal normal vector

$$N(s) = \frac{T'(s)}{\|T'(s)\|} \tag{2.1}$$

and the curvature

$$\kappa(s) = \|T'(s)\|.$$

So far, we have

$$T'(s) = \kappa(s) \cdot N(s). \tag{2.2}$$

If $\kappa(s) = 0$, the principal normal vector is not defined. If $\kappa(s) \neq 0$ then the binormal vector b(s) is given by

$$B(s) = T(s) \times N(s)$$

Then $\{T(s), N(s), B(s)\}$ form a right-handed orthonormal basis for $\mathbb{R}^3$. In summary, for the derivatives of Frenet frame, the Frenet-Serret formulae can be given as [5]:

$$T'(s) = \kappa(s) \cdot N(s), \tag{2.3}$$

$$N'(s) = -\kappa(s)T(s) + \tau(s)B(s), \tag{2.4}$$

$$B'(s) = -\tau(s)N(s). \tag{2.5}$$

Here we denote the curvature of the curve α by $\kappa(s)$ and the torsion of the curve α by $\tau(s)$.

Definition 2.2. Let α be a unit speed regular curve in Euclidean 3-space with Frenet vectors T, N and B. The unit tangent vectors along the curve α generate a curve $\widetilde{T}$ on the sphere of radius 1 about the origin. The curve $\widetilde{T}$ is called the spherical indicatrix of T or more commonly, $\widetilde{T}$ is called the tangent indicatrix of the curve α. If $\alpha = \alpha(s)$ is a natural representation of α, then $\widetilde{T}(s) = T(s)$ will be a representation of $\widetilde{T}$. Similarly one considers the principal normal indicatrix $\widetilde{N} = N(s)$ and binormal indicatrix $\widetilde{B} = B(s)$ [3, 6].

Theorem 2.3. *The curve α is a general helix if and only if $\frac{\tau}{\kappa}(s) = constant$. If $\kappa(s) \neq 0$ and $\tau(s)$ are constant, it is called as circular helix.*

Theorem 2.4. *Let α be a unit speed space curve with $\kappa(s) \neq 0$. Then α is a slant helix if and only if*

$$\sigma(s) = \left(\frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)' \right)(s)$$

is a constant function [1, 3, 4].

Theorem 2.5. *For a curve α in E^3, there is a curve α^* so that (α, α^*) is a Mannheim pair* [2].

Theorem 2.6. *Let $\{\alpha, \alpha^*\}$ be a Mannheim pair in E^3. The torsion of the curve α^* is $\tau^* = \dfrac{\kappa}{\lambda\tau}$. Here, λ is the distance between corresponding points of the Mannheim partner curves* [2].

Theorem 2.7. *The curve α is Bertrand curve if and only if $\lambda\kappa + \eta\tau = 1$.*

3. Curves and their Characterizations

Let

$$\alpha : I \to E^3 \tag{3.1}$$

$$s \mapsto \alpha(s)$$

be unit speed curve with Frenet vectors T, N, B and with non-zero curvatures κ and τ in $\mathbb{R}^3$.

In this section, using tangent indicatrix, principal normal indicatrix and binormal indicatrix of the curve α, some characterizations have been given as follows:

Theorem 3.1. *The curve α is general helix if and only if $\dfrac{\|B'\|}{\|T'\|}$ is constant.*

Proof. It can be easily seen that if $T' = \kappa N$ then $\|T'\| = \kappa$ and if $B' = \tau N$ then $\|B'\| = \tau$. The ratio

$$\frac{\|B'\|}{\|T'\|} = \frac{\tau}{\kappa}$$

is constant. This completes the proof. □

Theorem 3.2. *Let the Frenet frame of the spherical tangent indicatrix $\widetilde{T}$ of the curve α be $\{T, N, B\}$. The curve α is slant helix if and only if*

$$\frac{\|D_T B\|}{\|D_T T\|} = \textit{constant}.$$

Theorem 3.3. *Let the Frenet frame of the spherical binormal indicatrix $\widetilde{B}$ of the curve α be $\{\overset{*}{T}, \overset{*}{N}, \overset{*}{B}\}$. The curve α is slant helix if and only if*

$$\frac{\|D_T \overset{*}{B}\|}{\|D_T \overset{*}{T}\|} = \textit{constant}.$$

Theorem 3.4. *The curve α is Bertrand curve if and only if*

$$\lambda\|T'\| + \eta\|B'\| = 1\,.$$

Theorem 3.5. *The curve α is Mannheim curve if and only if*

$$\frac{\|T'\|}{\|W\|^2} = \frac{\|T'\|}{\|N'\|^2} = \lambda = constant$$

where the Darboux vector is $W = \tau T + \kappa B$.

Theorem 3.6. *Let α be a geodesic curve on the surface M. The curve α is helix on E^3if and only if*

$$\frac{\|D_T Y\|}{\|D_T T\|} = constant.$$

Proof. Let κ_g, κ_n, t_r be the geodesic curvature, asymptotic and curvature line respectively. Here it can be easily given that

$$\|D_T T\| = \kappa_n^2 + \kappa_g^2,$$

$$\|D_T Y\| = \kappa_n^2 + t_r^2.$$

Thus,

$$\frac{\|D_T Y\|}{\|D_T T\|} = \frac{1 + \left(\dfrac{t_r}{\kappa_n}\right)^2}{1 + \left(\dfrac{\kappa_g}{\kappa_n}\right)^2}.$$

If the curve α is geodesic then $\kappa_g = 0$. In that case,

$$\frac{\|D_T Y\|}{\|D_T T\|} = \text{constant}. \qquad \square$$

Theorem 3.7. *Let α be a asymptotic curve on the surface M.The curve α is helix on E^3if and only if*

$$\frac{\|D_T N\|}{\|D_T T\|} = constant.$$

Proof. Similarly in Proof 13, it can be easily seen that

$$\|D_T T\| = \kappa_n^2 + \kappa_g^2,$$

$$\|D_T N\| = \kappa_g^2 + t_r^2.$$

In that case,

$$\frac{\|D_T N\|}{\|D_T T\|} = \frac{\kappa_g^2 + t_r^2}{\kappa_n^2 + \kappa_g^2}$$

$$= \frac{1 + \left(\dfrac{t_r}{\kappa_g}\right)^2}{1 + \left(\dfrac{\kappa_n}{\kappa_g}\right)^2}.$$

If the curve α is asymptotic, then $\kappa_n = 0$. Thus,

$$\frac{\|D_T N\|}{\|D_T T\|} = \text{constant}. \qquad \square$$

Theorem 3.8. *Let $\{\alpha, \alpha^*\}$ be a Mannheim pair. The torsion of the curve α^* is*

$$\tau^* = \frac{\|D_T \overset{*}{T}\|}{\lambda \|D_T \overset{*}{B}\|} = \frac{\kappa}{\lambda \tau}.$$

Theorem 3.9. *Let $\{\alpha, \alpha^*\}$ be a Mannheim pair. The curve α^* is anti-Salkowski curve if and only if α is a general helix.*

Proof. As it is seen from the equation

$$\tau^* = \frac{\kappa}{\lambda \tau},$$

if the curve α^* is anti-Salkowski curve then τ^* is constant. Then $\frac{\kappa}{\tau}$ is constant. If $\frac{\kappa}{\tau}$ is constant then the curve α is general helix. This completes the proof. $\square$

Theorem 3.10. *The axis of the accompanying screw-motion at a point $c(0)$ is the line in the direction of the Darboux vector*

$$W(0) = \tau(0)T(0) + \kappa(0)B(0)$$

through the point

$$P(0) = c(0) + \frac{\kappa(0)}{\kappa^2(0) + \tau^2(0)} N(0).$$

It can be shown that under these circumstances the tangent to the curve which passes through all of these points, namely

$$P(s) = c(s) + \frac{\kappa(s)}{\kappa^2(s) + \tau^2(s)} N(s)$$

is proportional to $W(s)$ if and only if $\frac{\kappa}{\kappa^2 + \tau^2}$ is constant [8].

Proof. It can be easily seen that

$$P'(s) = \frac{\tau^2}{\kappa^2 + \tau^2} T + \frac{\kappa \tau}{\kappa^2 + \tau^2} B + \left(\frac{\kappa}{\kappa^2 + \tau^2}\right)' N$$

and

$$W(s) = \tau T + \kappa B.$$

Here, if $\dfrac{\kappa}{\kappa^2+\tau^2}$ is constant then

$$\begin{aligned} P'(s) &= \frac{\tau^2}{\kappa^2+\tau^2}T + \frac{\kappa\tau}{\kappa^2+\tau^2}B \\ &= \frac{\tau}{\kappa^2+\tau^2}(\tau T + \kappa B) \\ &= \frac{\tau}{\kappa^2+\tau^2}W(s). \end{aligned}$$

Thus,

$$P'(s) = \lambda(s)W(s). \qquad \square$$

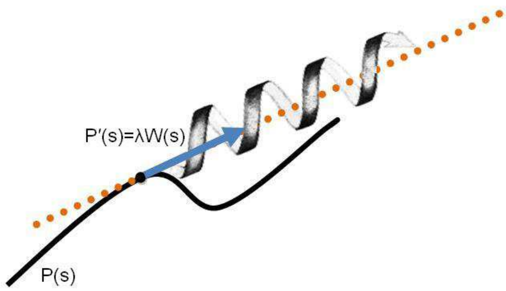

Figure 1. The tangent to the curve

Result. *Under these circumstances the tangent to the curve which passes through all of these points, namely*

$$P(s) = c(s) + \frac{\kappa(s)}{\kappa^2(s)+\tau^2(s)}N(s)$$

is proportional to $W(s)$ if and only if $c(s)$ is a Mannheim curve.

4. Conclusions

The starting point of this study is to develop some important characterizations of special curves by using the curvatures that given by the help of the norms of the derivatives of Frenet vectors. At this time, it is obtained that the tangent to the curve is proportional to the Darboux vector if and only if the curve is Mannheim curve. Additionally, different theorems have showed that there is a relation between the curvatures and the norms of the derivatives of Frenet vectors.

We hope that this study will gain different interpretation to the other studies in this field.

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