Journal of Informatics and Mathematical Sciences
Volume 5 (2013), Number 2, pp. 101–112
RGN Publications
http://www.rgnpublications.com

Laplacian Spectrum of Some Classes of Self Complementary Perfect Graphs

S.A.K. Kirmani, Merajuddin, and Madhukar Sharma

Abstract In this paper we study Laplacian spectrum of self-complementary (sc), sc comparability and sc chordal graphs. Some results on Laplacian cospectrality and Laplacian energy are obtained. We also propose two conjectures.

1. Introduction

Let G is an undirected graph with n vertices and m edges. G is self complementary (sc) if it is isomorphic to its complement. G is comparability if it can be oriented into transitive directed graph. G is chordal if it has no chordless cycle of length greater than or equal to 4. A sc graph which is also comparability (chordal) is called a sc comparability graph (sc chordal graph). An sc graph exists on $4k$ or $4k+1$ vertices, where k is a positive integer.

Let A be the adjacency matrix of G. Suppose $D = \text{diag}(d_1, d_2, \ldots, d_n)$ is the diagonal matrix with the degrees $d_1, d_2, \ldots, d_n$ of vertices $v_1, v_2, \ldots, v_n$ of G on the diagonal (with the same vertex ordering as in A). Then $L(G) = D - A$ is the Laplacian matrix of G. $L(G)$ is a real symmetric matrix. Its eigenvalues are real and non negative and form Laplacian spectrum, $\lambda_1, \lambda_2, \ldots, \lambda_n$ of graph G. We call graphs G_1 and G_2 L-cospectral if their Laplacian spectrums are same. For more on Laplacian spectrum see [1], [4], [5] and [6].

Sridharan and Balaji [7] studied the problem of cospectrality for sc chordal graphs. Merajuddin *et al.* [3] studied cospectral and hyperenergetic sc comparability graphs. In this paper we study these classes for Laplacian spectrum. The characteristic polynomials and eigenvalues of Laplacian matrices of the graphs are computed by using the software 'MATHEMATICA 5'.

2010 *Mathematics Subject Classification.* 05C17; 05C50.
Key words and phrases. L-Cospectral; Laplacian energy.

2. L-cospectrality of sc, sc comparability and sc chordal graphs

There exists only one sc graph (P_4) on 4 vertices, no L-cospectral sc graph on $n = 4$ vertices exist. Moreover, Laplacian characteristic polynomials (LCP) of the two non-isomorphic sc graphs (C_5, Bull) with 5 vertices are given below

$$-15x + 40x^2 - 33x^3 + 10x^4 - x^5 \text{ and } -25x + 50x^2 - 35x^3 + 10x^4 - x^5.$$

We note that none of the above polynomials can be obtained from the other polynomial by multiplying by a real number. Hence, no pair of non-isomorphic sc graphs with 5 vertices has the same spectrum.

Theorem 1. *No two non isomorphic sc graphs with* 8 *vertices are* L*-cospectral.*

Proof. All the 10 non-isomorphic sc graphs with 8 vertices and their LCPs are given below.

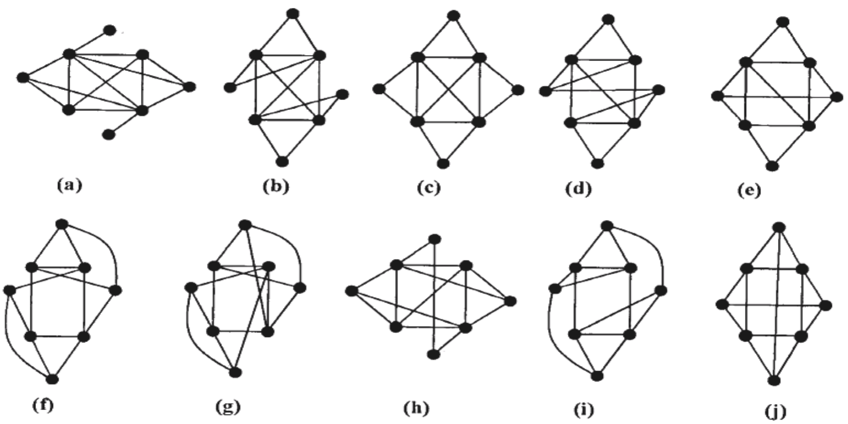

Figure 1

a. $-2688x + 8480x^2 - 10096x^3 + 5876x^4 - 1840x^5 + 316x^6 - 28x^7 + x^8$
b. $-4608x + 11904x^2 - 12224x^3 + 6480x^4 - 1920x^5 + 320x^6 - 28x^7 + x^8$
c. $-4800x + 12080x^2 - 12272x^3 + 6484x^4 - 1920x^5 + 320x^6 - 28x^7 + x^8$
d. $-6336x + 15024x^2 - 14256x^3 + 7076x^4 - 2000x^5 + 324x^6 - 28x^7 + x^8$
e. $-6326x + 15024x^2 - 14256x^3 + 7076x^4 - 2000x^5 + 324x^6 - 28x^7 + x^8$
f. $-6528x + 15200x^2 - 14304x^3 + 7080x^4 - 2000x^5 + 324x^6 - 28x^7 + x^8$
g. $-8704x + 18816x^2 - 16480x^3 + 7688x^4 - 2080x^5 + 328x^6 - 28x^7 + x^8$
h. $-8192x + 18432x^2 - 16384x^3 + 7680x^4 - 2080x^5 + 328x^6 - 28x^7 + x^8$
i. $-9216x + 19200x^2 - 16576x^3 + 7696x^4 - 2080x^5 + 328x^6 - 28x^7 + x^8$
j. $-9408x + 19376x^2 - 16624x^3 + 7700x^4 - 2080x^5 + 328x^6 - 28x^7 + x^8$

Clearly, no pair of non-isomorphic sc graphs with 8 vertices has the same spectrum. □

Theorem 2. *There exist non-isomorphic* L*-cospectral sc graphs with* 9 *vertices.*

Proof. Consider the following sc graphs shown in Figure 2. The $LCPs$ associated with these sc graphs are given below

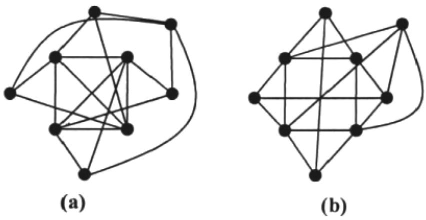

Figure 2

$$-81648x + 174960x^2 - 159408x^3 + 80676x^4 - 24813x^5 + 4752x^6 - 554x^7 + 36x^8 - x^9$$
$$-81648x + 174960x^2 - 159408x^3 + 80676x^4 - 24813x^5 + 4752x^6 - 554x^7 + 36x^8 - x^9.$$

Since spectrums are same for these graphs, they are L-cospectral. Hence the result. □

Thus "*The smallest positive integer for which there exist non isomorphic L-cospectral sc graphs is* 9".

On L-cospectrality of sc comparability graphs

There exists only one sc comparability graph on 4 and 5 vertices, obviously, no L-cospectral sc comparability graph on $n = 4$ or $n = 5$ vertices exist. Following result is obvious.

Corollary 1. *No two non-isomorphic sc comparability graphs with* 8 *vertices are* L*-cospectral.*

Theorem 3. *No two non-isomorphic sc comparability graphs with* 9 *vertices are* L*-cospectral.*

Proof. $LCPs$ of the 4 non-isomorphic sc comparability graphs with 9 vertices, shown in Figure 3, are

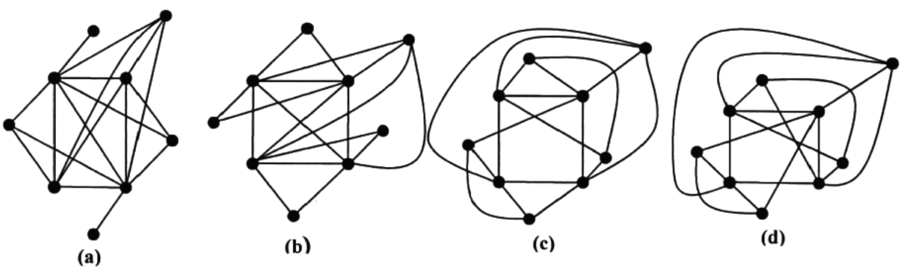

Figure 3

a. $-20349x + 66924x^2 - 85358x^3 + 55224x^4 - 20159x^5 + 4320x^6 - 538x^7 + 36x^8 - x^9$

b. $-35280x + 94752x^2 - 104488x^3 + 61704x^4 - 21329x^5 + 4428x^6 - 542x^7 + 36x^8 - x^9$

c. $-76320x + 169704x^2 - 157528x^3 + 80388x^4 - 24797x^5 + 4752x^6 - 554x^7 + 36x^8 - x^9$

d. $-72000x + 165600x^2 - 156100x^3 + 80172x^4 - 24785x^5 + 4752x^6 - 554x^7 + 36x^8 - x^9$.

Clearly no pair of non-isomorphic sc graphs with 9 vertices has the same spectrum. □

Theorem 4. *There exist non-isomorphic L-cospectral sc comparability graphs with* 12 *vertices.*

Proof. Consider the two non isomorphic sc comparability graphs shown in Figure 4. The characteristic polynomials associated with these sc comparability graphs are given below

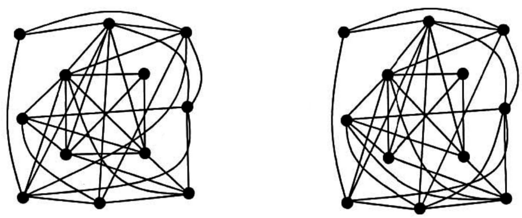

Figure 4

$$-128198700x+292465530x^2-293133618x^3+171180817x^4-64944000x^5+16853520x^6-3059196x^7+389062x^8-34020x^9+1950x^{10}-66x^{11}+x^{12}$$

$$-128198700x+292465530x^2-293133618x^3+171180817x^4-64944000x^5+16853520x^6-3059196x^7+389062x^8-34020x^9+1950x^{10}-66x^{11}+x^{12}.$$

Since spectrums are same for these graphs, they are L-cospectral. □

"*The smallest positive integer for which there exist non-isomorphic L-cospectral sc comparability graphs is* 12."

On L-cospectrality of sc chordal graphs

P_4 and bull are the only one sc chordal graph on 4 and 5 vertices, so no L-cospectral sc chordal graph on $n = 4$ or $n = 5$ vertices exist. Moreover from Theorem 1 we have.

Corollary 2. *No two non-isomorphic sc chordal graphs with 8 vertices are L-cospectral.*

Theorem 5. *No two non-isomorphic sc chordal graphs with 9 vertices are L-cospectral.*

Proof. All the 3 non-isomorphic sc chordal graphs with 9 vertices and their LCPs are given below.

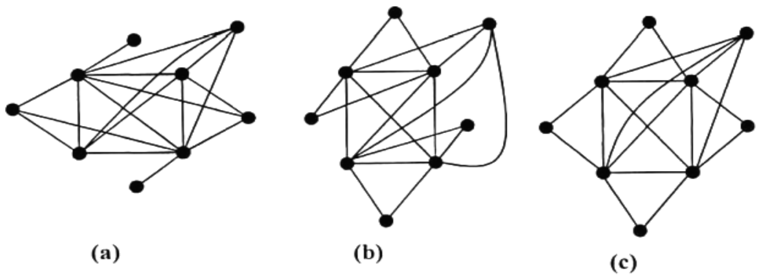

Figure 5

a. $-20349x + 66924x^2 - 85358x^3 + 55224x^4 - 20159x^5 + 4320x^6 - 538x^7 + 36x^8 - x^9$
b. $-35280x + 94752x^2 - 104488x^3 + 61704x^4 - 21329x^5 + 4428x^6 - 542x^7 + 36x^8 - x^9$
c. $-36288x + 95904x^2 - 104940x^3 + 61776x^4 - 21333x^5 + 4428x^6 - 542x^7 + 36x^8 - x^9$.

Obviously, no pair of non-isomorphic sc graphs with 9 vertices has the same spectrum. □

Theorem 6. *No two non-isomorphic sc chordal graphs with 12 vertices are L-cospectral.*

Proof. All the 16 non-isomorphic sc chordal graphs with 12 vertices and their LCPs are mentioned in Figure 6.

a. $-17821440x + 64330368x^2 - 94586112x^3 + 75167744x^4 - 36395760x^5 + 11421352x^6 - 2395512x^7 + 338700x^8 - 31860x^9 + 1910x^{10} - 66x^{11} + x^{12}$
b. $-20643840x + 73187328x^2 - 105274368x^3 + 8167321x^4 - 38660544x^5 + 11900448x^6 - 2458176x^7 + 343648x^8 - 32076x^9 + 1914x^{10} - 66x^{11} + x^{12}$
c. $-20736000x + 73440000x^2 - 105520320x^3 + 81780192x^4 - 38684304x^5 + 11903256x^6 - 2458344x^7 + 343652x^8 - 32076x^9 + 1914x^{10} - 66x^{11} + x^{12}$
d. $-46807488x + 127886880x^2 - 151852320x^3 + 103407888x^4 - 44879712x^5 + 13043984x^6 - 2594256x^7 + 353800x^8 - 32508x^9 + 1922x^{10} - 66x^{11} + x^{12}$
e. $-47159424x + 128467392x^2 - 152238528x^3 + 103541856x^4 - 44905872x^5 + 13046872x^6 - 2594424x^7 + 353804x^8 - 32508x^9 + 1922x^{10} - 66x^{11} + x^{12}$

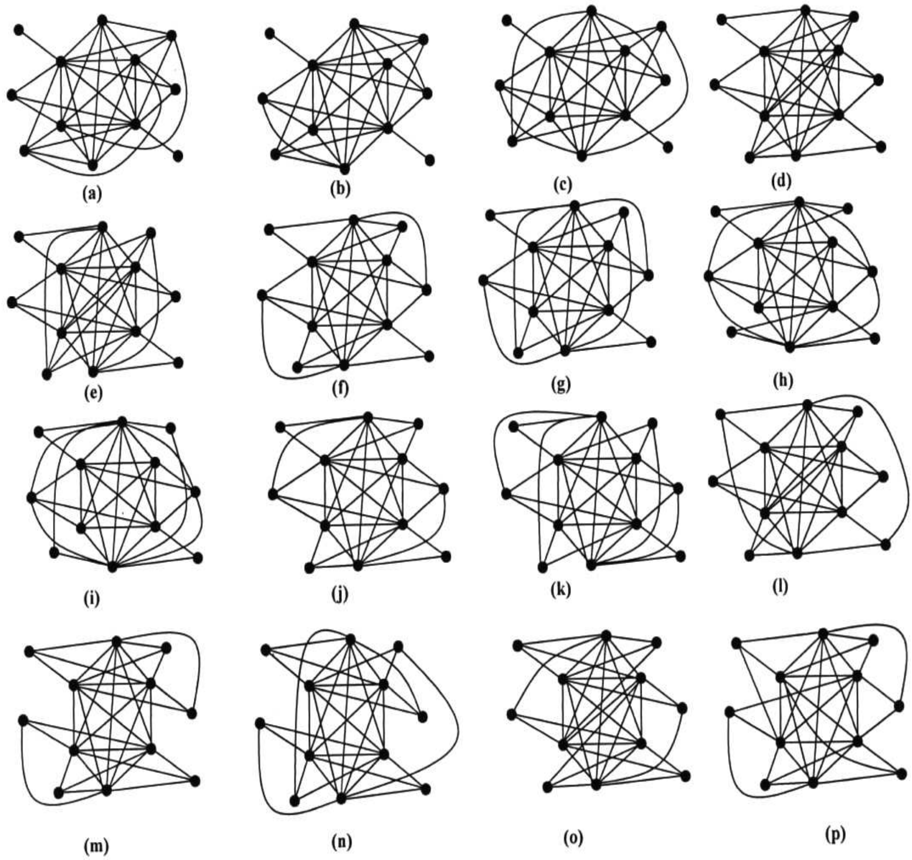

Figure 6

f. $-46369260x+127120698x^2-151316154x^3+103215093x^4-44841192x^5+13039676x^6-2594004x^7+353794x^8-32508x^9+1922x^{10}-66x^{11}+x^{12}$

g. $-47100060x+128313282x^2-152102250x^3+103485709x^4-44893752x^5+13045460x^6-2594340x^7+353802x^8-32508x^9+1922x^{10}-66x^{11}+x^{12}$

h. $-31334400x+93050880x^2-119084544x^3+86513920x^4-39605184x^5+12004512x^6-2464224x^7+343792x^8-32076x^9+1914x^{10}-66x^{11}+x^{12}$

i. $-31726080x+93685248x^2-119496384x^3+86653152x^4-39631824x^5+12007416x^6-2464392x^7+343796x^8-32076x^9+1914x^{10}-66x^{11}+x^{12}$

j. $-46661580x+127632810x^2-151673922x^3+103343641x^4-44866872x^5+13042548x^6-2594172x^7+353798x^8-32508x^9+1922x^{10}-66x^{11}+x^{12}$

k. $-46836972x+127834842x^2-151758594x^3+103359865x^4-44868312x^5+13042596x^6-2594172x^7+353798x^8-32508x^9+1922x^{10}-66x^{11}+x^{12}$

l. $-60742656x+154971648x^2-174320640x^3+113864064x^4-47895216x^5+13604552x^6-2661624x^7+358860x^8-32724x^9+1926x^{10}-66x^{11}+x^{12}$

m. $-57395628x+149866362x^2-171124002x^3+112803273x^4-47694096x^5+13582728x^6-2660364x^7+358830x^8-32724x^9+1926x^{10}-66x^{11}+x^{12}$

n. $-59285196x+152700714x^2-172873602x^3+113377401x^4-47802096x^5+13594392x^6-2661036x^7+358846x^8-32724x^9+1926x^{10}-66x^{11}+x^{12}$

o. $-60159996x+154044018x^2-173719026x^3+113658993x^4-47855616x^5+13600208x^6-2661372x^7+358854x^8-32724x^9+1926x^{10}-66x^{11}+x^{12}$

p. $-60547500x+154661850x^2-174119994x^3+113795701x^4-47882016x^5+13603104x^6-2661540x^7+358858x^8-32724x^9+1926x^{10}-66x^{11}+x^{12}$.

Clearly no pair of non-isomorphic sc chordal graphs with 12 vertices has the same Laplacian spectrum. □

Theorem 7. *No two non-isomorphic sc chordal graphs with* 13 *vertices are* L*-cospectral.*

Proof. All the 16 non-isomorphic sc chordal graphs with 13 vertices and their LCPs are shown in Figure 7.

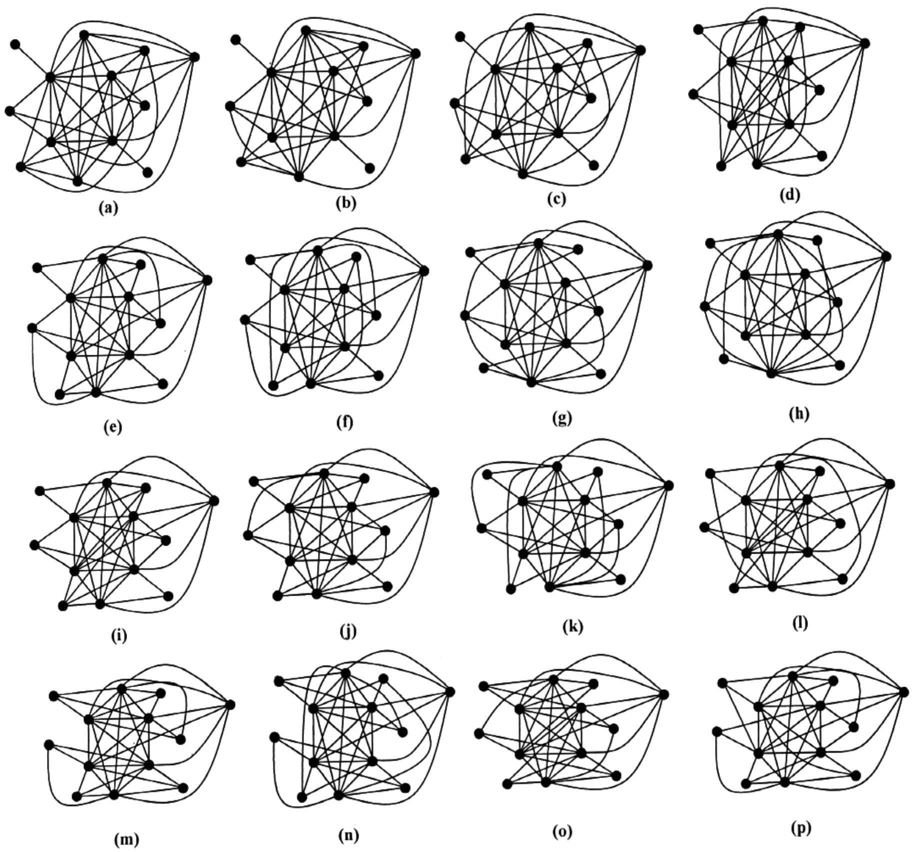

Figure 7

a. $-205562643x+760552052x^2-1158652599x^3+966834934x^4-499054340x^5+169969488x^6-39562075x^7+6392490x^8-716376x^9+54600x^{10}-2699x^{11}+78x^{12}-x^{13}$

b. $-237219840x+862363008x^2-1286496688x^3+1049608976x^4-530542415x^5+177499972x^6-40728379x^7+6509386x^8-723694x^9+54860x^{10}-2703x^{11}+78x^{12}-x^{13}$

c. $-238043520x+864725264x^2-1288981532x^3+1050842104x^4-530873087x^5+177550724x^6-40732835x^7+6509594x^8-723698x^9+54860x^{10}-2703x^{11}+78x^{12}-x^{13}$

d. $-537089280x+1513727280x^2-1872121396x^3+1343145908x^4-622651815x^5+196592032x^6-43394639x^7+6758362x^8-738622x^9+55380x^{10}-2711x^{11}+78x^{12}-x^{13}$

e. $-540286240x+1519397880x^2-1876280328x^3+1344790772x^4-623035503x^5+196646216x^6-43399183x^7+6758570x^8-738626x^9+55380x^{10}-2711x^{11}+78x^{12}-x^{13}$

f. $-533096811x+1506244272x^2-1866372183x^3+1340794494x^4-622090880x^5+196511692x^6-43387847x^7+6758050x^8-738616x^9+55380x^{10}-2711x^{11}+78x^{12}-x^{13}$

g. $-539709963x+1517873032x^2-1874836551x^3+1344120934x^4-622863048x^5+196620372x^6-43396943x^7+6758466x^8-738624x^9+55380x^{10}-2711x^{11}+78x^{12}-x^{13}$

h. $-362192688x+1106298336x^2-1470133288x^3+1121849144x^4-547133591x^5+179808460x^6-40920067x^7+6518122x^8-723862x^9+54860x^{10}-2703x^{11}+78x^{12}-x^{13}$

i. $-365851200x+1112668960x^2-1474686772x^3+1123599152x^4-547531463x^5+179863580x^6-40924635x^7+6518330x^8-723866x^9+54860x^{10}-2703x^{11}+78x^{12}-x^{13}$

j. $-535694835x+1511141684x^2-1870148511x^3+1342345706x^4-622462516x^5+196565096x^6-43392371x^7+6758258x^8-738620x^9+55380x^{10}-2711x^{11}+78x^{12}-x^{13}$

k. $-537348123x+1513281276x^2-1871212175x^3+1342607058x^4-622496228x^5+196567280x^6-43392427x^7+6758258x^8-738620x^9+55380x^{10}-2711x^{11}+78x^{12}-x^{13}$

l. $-693403659x+1831364964x^2-2151024279x^3+1482680342x^4-666801996x^5+205852608x^6-44703099x^7+6881706x^8-746064x^9+55640x^{10}-2715x^{11}+78x^{12}-x^{13}$

m. $-663390000x+1781676000x^2-2116481400x^3+1469552760x^4-663824619x^5+205439832x^6-44668855x^7+6880146x^8-746034x^9+55640x^{10}-2715x^{11}+78x^{12}-x^{13}$

n. $-680238000x+1809194400x^2-2135390040x^3+1476670728x^4-665427403x^5+205660936x^6-44687143x^7+6880978x^8-746050x^9+55640x^{10}-2715x^{11}+78x^{12}-x^{13}$

o. $-688100400x+1822279680x^2-2144524824x^3+1480153376x^4-666219099x^5+205770864x^6-44696271x^7+6881394x^8-746058x^9+55640x^{10}-2715x^{11}+78x^{12}-x^{13}$

p. $-691629120x+1828331232x^2-2148856372x^3+1481837864x^4-666607691x^5+205825360x^6-44700823x^7+6881602x^8-746062x^9+55640x^{10}-2715x^{11}+78x^{12}-x^{13}$.

We note that none of the above polynomials can be obtained from the other polynomials by multiplying by a real number. Hence no pair of non-isomorphic sc chordal graphs with 13 vertices has the same Laplacian spectrum. □

In view of above results, we report the following conjecture.

Conjecture 1. *There do not exist L-cospectral sc chordal graphs on $n = 4k$ or $n = 4k+1$ vertices.*

3. Laplacian Energy of sc, sc comparability and sc chordal graphs

Let G be a graph on n vertices and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of Laplacian matrix of G. Gutman [2] defined the Laplacian energy of G as $LE(G) = \sum_{i=1}^{n} \left|\lambda_i - \frac{2m}{n}\right|$. Let G_1 and G_2, be two graphs such that $LE(G_1) = LE(G_2)$, then we call G_1 and G_2, L-equienergetic.

On L-equi-energetic sc graphs

Laplacian energies of Bull and C_5 are 7.84162 and 6.47212. Clearly they are not L-equienergetic.

Theorem 8. *No two non isomorphic sc graphs with 8 vertices are L-equi-energetic.*

Proof. Laplacian energies of all 10 non isomorphic sc graphs shown in Figure 1 are 18.8098, 17.6569, 17.798, 15.3137, 14.9282, 15.391, 12.391, 11.6569, 9.5, and 13.6568. We note that no two sc graphs have equal energy. Thus, no two non-isomorphic sc graphs with 8 vertices are L-equi-energetic. □

Theorem 9. *There exist non-isomorphic non L-cospectral, L-equi-energetic sc graphs with 9 vertices.*

Proof. Consider the following two sc graphs in Figure 8.

These two graphs are non-isomorphic. They are non L-cospectral as the LCPs of both graphs are different as given below.

$$-90720x+189864x^2-169164x^3+83934x^4-25399x^5+4806x^6-556x^7+36x^8-x^9$$

and

$$-104976x+209952x^2-180792x^3+87480x^4-26001x^5+4860x^6-558x^7+36x^8-x^9$$

The Laplacian energies of graphs are 16. Thus, these graphs are non-isomorphic non cospectral and L-equi-energetic. □

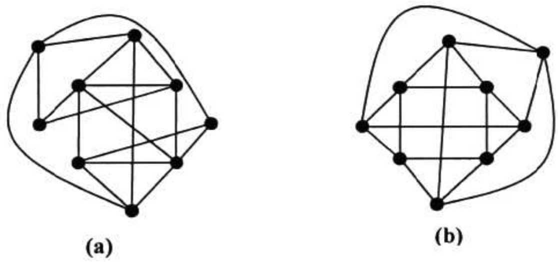

Figure 8

Thus, we have the following result.

"The smallest positive integer for which there exists non-isomorphic non L-cospectral L-equi-energetic sc graph is 9."

On L-equi-energetic sc comparability and sc chordal graphs

The following result is obvious from Theorem 8.

Corollary 3. *No two non isomorphic sc comparability graphs and sc chordal graphs with 8 vertices are L-equi-energetic.*

Theorem 10. *No two non isomorphic sc comparability graphs and sc chordal graphs with 9 vertices are L-equi-energetic.*

Proof. Laplacian energies of the four non-isomorphic sc comparability graphs with 9 vertices, shown in Figure 3 are 23.8299, 23.4031, 16.5164, and 15.4031. We note that no two graphs have equal energy. Thus, no two non-isomorphic sc comparability graphs with 9 vertices are L-equi-energetic. Similarly, Laplacian energies of the three nonisomorphic sc chordal graphs with 9 vertices, shown in Figure 5 are 23.8299, 23.4031 and 23.4891. Clearly no two graphs have same energy. □

Theorem 11. *No two non isomorphic sc comparability graphs and sc chordal graphs with 12 vertices are L-equi-energetic.*

Proof. Laplacian energies of the fourteen non-isomorphic sc comparability graphs with 12 vertices, shown in Figure 9 are 40.8058, 39.6529, 33.6529, 38.4852, 35.0746, 30.4852, 27.0746, 27.0746, 24.4852, 22.5919, 19.8284, 20.6264, 19.8284 and 18.4852.

We note that no two graphs have equal energy. Thus, no two non-isomorphic sc comparability graphs with 12 vertices are L-equi-energetic. Similarly, Laplacian energies of the 16 non-isomorphic sc chordal graphs with 12 vertices, shown in Figure 6 are 40.8058, 39.6529, 39.794, 39.2992, 39.3588, 39.1925, 39.3045, 40.7009, 40.7271, 39.2684, 39.2218, 38.8301, 38.4852, 38.6538, 38.7468 and 38.8021. Clearly, no two graphs have same energy. Hence the result. □

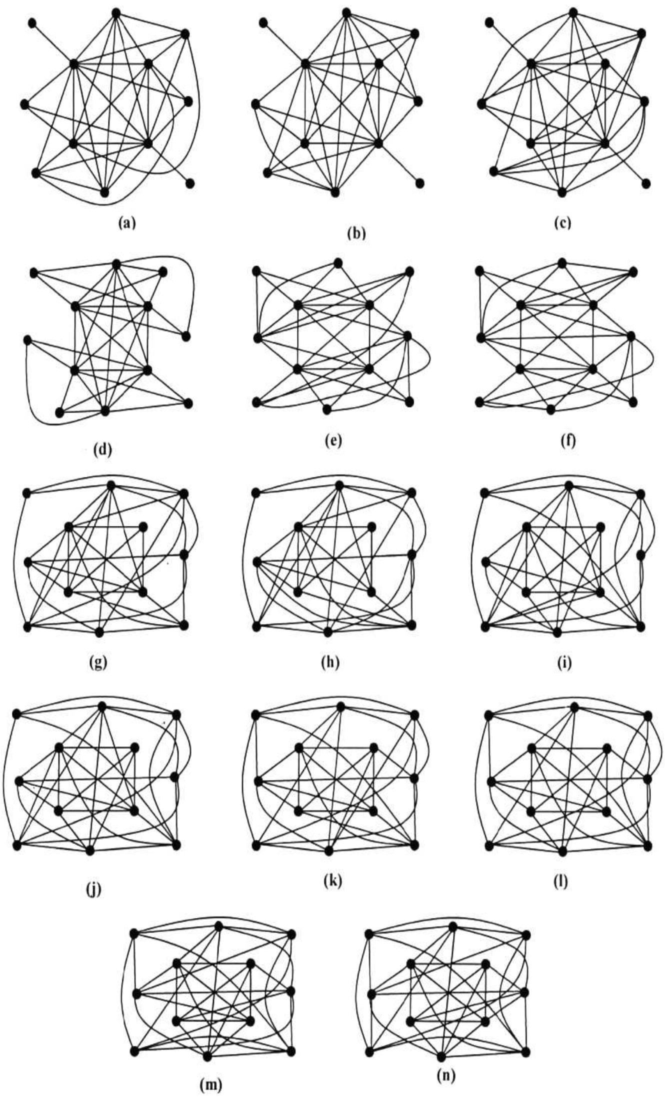

Figure 9

Theorem 12. *No two non isomorphic sc comparability graphs and sc chordal graphs with* 13 *vertices are L-equi-energetic.*

Proof. Laplacian energies of the 31 non-isomorphic sc comparability graphs with 13 vertices (graphs are not shown) are 47.827, 47.4001, 41.8633, 39.4001, 40.1111, 40.2605, 35.0591, 40.1418, 39.4272, 33.2973, 32.0159, 40.0432, 34.2947, 39.7527, 46.8251, 38.9249, 38.8251, 34.0966, 29.1397, 33.1397, 32.4315, 32.6227, 30.8251, 30.9249, 27.4757, 27.2, 27.1562, 27.442, 24.6227, 25.6307 and 22.8251. We note that no two graphs have equal energy. Thus, no two non-isomorphic sc comparability graphs with 13 vertices are L-equi-energetic. Similarly, Laplacian energies of the 16 non-isomorphic sc chordal graphs with 13 vertices, shown in Figure 7 are 47.827, 47.4001, 47.4861, 47.2234, 47.2631, 47.1621, 47.2377, 47.7527, 47.7718, 47.2123, 47.184, 47.0664, 46.8251, 46.9451, 47.0102, and 47.0476. Clearly no two graphs have same energy. This proves the result. □

In the view of above results we conclude with the following conjecture.

Conjecture 2. *There do not exist non-isomorphic non L-cospectral L-equi-energetic sc comparability and sc chordal graphs on $n = 4k$ or $n = 4k + 1$ vertices.*

References

[1] R. Grone and R. Merris, The Laplacian spectrum of a graph II*, *SIAM J. Discrete Math.* **7**(2) (1994), 221–229.

[2] I. Gutman and B. Zhou, Laplacian energy of graph, *Linear Algebra and its Applications* **414** (2006), 29–37.

[3] Merajuddin, S.A.K. Kirmani, P. Ali and S. Pirzada, Cospectral and hyper-energetic self complementary comparability graphs, *J. KSIAM* **11**(3) (2007), 65–75.

[4] R. Merris, A note on Laplacian graph eigenvalues, *Linear Algebra and its Applications* **285** (1998), 33–35.

[5] R. Merris, A survey of graph Laplacian, *Linear and Multilinear Algebra* **39** (1995), 19–31.

[6] B. Mohar, The Laplacian spectral of graphs, in *Graph Theory Combinatorics and Applications* (Y. Alavi, G. Chartrand, O.R. Oellermann and A.J. Schwenk (editors)), **Vol. 2** (1991), 53–64.

[7] M.R. Sridharan and K. Balaji, On cospectral self-complementary chordal graphs, *National Academy Science Letters* **22** (1999), 19–23.

S.A.K. Kirmani, *Department of Mathematics, Galgotia College of Engineering and Technology, Greater Noida, India.*
E-mail: azazkirmani@rediffmail.com

Merajuddin, *Department of Applied Mathematics, Faculty of Engineering & Technology, Aligarh Muslim University, Aligarh 202002, India.*
E-mail: meraj1957@rediffmail.com

Madhukar Sharma, *Department of Applied Sciences, IIMT College of Engineering, Greater Noida, India.*
E-mail: madhukar13777@gmail.com

Received February 22, 2011
Accepted September 29, 2011