Journal of Informatics and Mathematical Sciences
Vol. 14, No. 1, pp. 2–21, 2022
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/jims.v14i1.1798

Research Article

Some Results on the Spectral Radius of Banach Algebra

Luma J. Barghooth

Department of Mathematics, College of Basic Education, Mustansiriyah University, Baghdad, Iraq
lumaj.barghooth@uomustansiriyah.edu.iq

Received: February 1, 2022 **Accepted:** March 9, 2022

Abstract. Al-Banach algebra was introduced in 1940 by Russian mathematician I.M. Gelfand and arose from the observation that some Banach spaces exhibit interesting properties when they can be provided with operations and examples. Among the linear functions adopted in Al-Banach commutative algebra and its maxima, as well as with the spectra of its elements in one way or another that leads us towards the central component of the spectra mapping theory and the spectral radius formula. In this paper, we get some results on the spectral radius of Al-Banach algebra.

Keywords. Banach algebra, Normed algebra, Spectral radius

Mathematics Subject Classification (2020). 18G40

1. Introduction

Because of uncertain data in the real world, various problems in mathematics, engineering, environmental sciences, economics, and medical sciences cannot be solved by the usual mathematical methods. The difficulty of the usual mathematical method is the lack of tools for determining coefficients-determining tools to describe problems arising in areas of ambiguity and uncertainty to deal with such problems climate algebra was introduced by Russian mathematics and interesting properties arose when it could be provided with operations. At the present time, Al-Banach algebra is a broad discipline with a variety of disciplines and applications. This paper is based on a number of theories that represent the relationship between the multiple linear functions in the Al-Banach algebra and its extremes, as well as with the spectra of its elements in one way or another towards the central component of the well-known

Gelfand-Mazur theory [5,6]. The spectral radius formula [6,11,13] and the Stone-Weierstrass theory [3,12] and mapping theory [2,12] where this topic was addressed through a research published in 2017, where the paper used the source [7,8,14] for the concept of Al-Banach soft algebra and they studied some of its primary properties and from the sources used in this research [4,9,12] and [1,10], where the researcher presented the ideas of the soft spectrum, radius of a soft algebra of Banach [8]. In this paper, we get some results on the spectral radius of Al-Banach algebra. Them ain difference between the previous works and this paper is the use of reversible elements, spectral radius and reversible element, spectral algebra and some basic properties of these ideas in Banach algebra. Firstly, in sections two and three, we defined a number of concepts and basic theorems, and then, in section four, we verify some results and theories.

2. Preliminaries

In this section, we state some basic definitions and theorems [4,9,12], those we will use thorough out the paper.

Definition 2.1. A normed algebra A is an algebra which is a normed space $(A, \|\cdot\|)$ and in which

$$\|xy\| \leq \|x\|\,\|y\|, \quad x, y \in A.$$

We will mention and prove some results related to normed algebras.

Lemma 2.1. *Let A be a normed algebra with unit e. Then, $\|e\| \geq 1$.*

Proof. Assume $y \in A$ a normed algebra with $y \neq 0$. Then

$$ye = ey = y.$$

Thus $\|ye\| = \|y\|$. We get $\|ye\| \leq \|y\|\,\|e\|$. Therefore, $\|y\| \leq \|y\|\,\|e\|$ and thus $\|e\| \geq 1$. Similarly, if $ey = y$, then $\|e\| \geq 1$. □

Remark 2.1. We have to assume that additional $\|e\| = 1$.

Lemma 2.2. *Let A be a normed algebra. Let $y \in A$, $n \in N$. Then*

$$\|y^n\| \leq \|y\|^n.$$

Proof. Using the mathematical induction:
Let $n = 1$. Then

$$\|y\| \leq \|y\|, \quad y \in A.$$

Now, let's assume the statement is true for $n = k$,

$$\|y^k\| \leq \|y\|^k, \quad y \in A.$$

Now, we shall prove that it is true for $n = k+1$.
We have

$$\begin{aligned}\|y^{k+1}\| &= \|y^k\| \qquad \text{(where } y \in A\text{)}\\ &\leq \|y^k\|\,\|y\|\end{aligned}$$

$$\leq \|y\|^k \|y\|$$
$$= \|y\|^{k+1}.$$

Thus

$$\|y^{k+1}\| \leq \|y\|^{k+1}.$$

Hence $\|y^n\| \leq \|y\|^n$. □

Lemma 2.3. *Let A be a normed algebra. Let $y \in A$ and $n, m \in \mathbb{N}$. Then.*

$$\|y^{n+m}\| \leq \|y\|^{n+m}.$$

Proof. Let $y \in A$. Then

$$\begin{aligned}\|y^{n+m}\| &= \|y^n y^m\| \qquad (\text{where } y \in A)\\ &\leq \|y^n\| \, \|y^m\|\\ &\leq \|y\|^n \|y\|^m\\ &= \|y\|^{n+m}.\end{aligned}$$

□

Theorem 2.4. *Let A be a normed algebra. If $x_n \to x$, $y_n \to y$ $(n \to \infty)$ in A, then $x_n y_n \to xy$.*

Proof. Let $x_n \to x$ and $y_n \to y$ in A. Then

$$\begin{aligned}\|x_n y_n - xy\| &= \|x_n y_n - x_n y + x_n y - xy\|\\ &= \|x_n(y_n - y) + y(x_n - x)\|\\ &\leq \|x_n(y_n - y)\| + \|y(x_n - x)\|\\ &\leq \|x_n\| \, \|y_n - y\| + \|y\| \, \|x_n - x\|\\ &\to 0 \quad (n \to \infty).\end{aligned}$$

Hence $x_n y_n \to xy$. □

Theorem 2.5. *Let (x_n) and (y_n) be bounded sequences in a normed in a normed algebra A. Then, $(x_n y_n)$ is a bounded sequence in A.*

Proof. Let (x_n) be a bounded sequence in A. Then, there exists a positive integer M_1 such that $\|x_n\| \leq M_1$, for all n.
Let (y_n) be a bounded sequence in A. Then, there exists a positive integer M_2 such that $\|y_n\| \leq M_2$, for all n.
We have

$$\|x_n y_n\| \leq \|x_n\| \, \|y_n\| \leq M_1 M_2.$$

Choose $M = M_1 M_2 > 0$.
It follows that

$$\|x_n y_n\| \leq M, \quad \text{for all } n.$$

Hence $(x_n y_n)$ is a bounded sequence. □

Theorem 2.6. *Let A be a normed algebra, if (x_n) and (y_n) are Cauchy sequences in A, then $(x_n y_n)$ is a Cauchy sequence in A.*

Proof. Since x_n is a Cauchy sequence in A, so (x_n) is a bounded sequence. Then, there exists a positive integer M such that

$$\|x_n\| \le M, \quad n \in \mathbb{N}, \text{ for each } \varepsilon > 0.$$

There exists a positive integer N such that

$$\|x_n - x_m\| < \frac{\varepsilon}{2M}, \quad n,m > N.$$

Also, since (y_n) is a Cauchy sequence, so (y_n) is a bounded sequence. Then, there exists a positive integer M such that

$$\|y_n\| \le M, \quad n \in \mathbb{N}.$$

Similarly, for each $\varepsilon > 0$, there exists a positive integer N such that

$$\|y_n - y_m\| < \frac{\varepsilon}{2M}, \quad n,m > N.$$

We have

$$\begin{aligned}
\|x_n y_n - x_m y_m\| &= \|x_n y_n - x_m y_n + x_m y_n - x_m y_m\| \\
&= \|y_n(x_n - x_m) + x_m(y_n - y_m)\| \\
&\le \|y_n(x_n - x_m)\| + \|x_m(y_n - y_m)\| \\
&\le \|y_n\| \|x_n - x_m\| + \|x_m\| \|y_n - y_m\| \\
&\le M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence $(x_n y_n)$ is a Cauchy sequence in A. □

Definition 2.2. Let $(A, \|\cdot\|)$ be a normed algebra. If A is complete with relative to this norm (i.e., A is a Banach space), then A is called a Banach algebra.

We give some examples on Banach algebras.

Example 2.1. (i) The space $\mathbb{R}$ is a Banach space with the norm $\|x\| = |x|$, $x \in \mathbb{R}$. Then, for $x, y \in \mathbb{R}$:

$$\|xy\| = |xy| = |x|\,|y| = \|x\|\,\|y\|.$$

Hence $\mathbb{R}$ is a normed algebra. Then, $\mathbb{R}$ with addition to the usual and standard multiplication, the dot multiplication is a commutative product Banach algebra.
Also, $\mathbb{C}$ with the usual structure and the norm $\|x\| = |x|$ $(x \in \mathbb{C})$, is a commutative Banach algebra.

(ii) The norm on $M_{n\times n}$ is given by

$$\|A\| = \max\left\{\sum_{j=1}^{n} |a_{ij}| : 1 \le i \le n\right\}, \quad A \in M_{n\times n}.$$

Then, it is a Banach space.
Let $A = (a_{ij})$, $B = (b_{ij})$. Let $A, B \in M_{n\times n}$. Then

$$\|AB\| \le \|A\|\,\|B\|.$$

Hence $M_{n\times n}$ is a Banach algebra. Hence well-known matrix multiplication is not commutative.

(iii) The norm on $C[a,b]$ is given by

$$\|f\| = \sup_{a\le x\le b} (|f(x)|), \quad f \in C[a,b].$$

Then, $C[a,b]$ is a Banach space.
Let $f,g \in C[a,b]$. Then

$$\|fg\| = \sup_{a\le x\le b} (|f(x)g(x)|).$$

By Theorem there exists x_0 in $[a,b]$ such that

$$\|fg\| = |f(x_0)|\ |g(x_0)| \le \|f\|\ \|g\|.$$

Hence $C[a,b]$ is a commutative Banach algebra.

(iv) The norm on $C^n[a,b]$ is given by

$$\|f\| = \sum_{K=0}^{n} \frac{1}{k!} \|f^{(k)}\|_\infty, \quad f \in C^n[a,b].$$

Then, $C^n[a,b]$ is a Banach space.
Let $f,g \in C^n[a,b]$. Then

$$\begin{aligned}
\|fg\| &= \sum_{K=0}^{n} \frac{1}{k!} \|(fg)^{(k)}\|_\infty \\
&= \sum_{K=0}^{n} \frac{1}{k!} \left\| \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j)} \right\|_\infty \\
&= \sum_{K=0}^{n} \left\| \sum_{j=0}^{k} \frac{1}{j!(k-j)} f^{(j)} g^{(k-j)} \right\|_\infty \\
&\le \sum_{K=0}^{n} \sum_{j=0}^{n} \frac{1}{j!} \left\| f^{(j)} \left\| \sum_{j=0}^{k} \frac{1}{j!(k-j)} \right\| g^{(k-j)} \right\|_\infty \\
&\le \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{1}{j!} \|f^{(j)}\|_\infty \frac{1}{i!} \|g^{(I)}\|_\infty \\
&= \|f\|\ \|g\|.
\end{aligned}$$

Then, $\|fg\| \le \|f\|\ \|g\|$.
Hence $C^n[a,b]$ is a commutative Banach algebra.

(v) The norm on $B(X,X)$ is Banach space.

$$\|T\| = \sup\{\|T(x)\| : \|x\| \le 1\}, \quad T \in B(X,X).$$

Then, $B(X,X)$ is a Banach space.
Let $T_1, T_2 \in B(X,X)$. Then

$$\begin{aligned}
\|(T_1T_2)(x)\| &= \|T_1(T_2(x))\| \\
&\le \|T_1\|\ \|T_2(x)\| \\
&\le \|T_1\|\ \|T_2\|\ \|x\|.
\end{aligned}$$

Then

$$\|T_1T_2\| \le \|T_1\|\,\|T_2\|.$$

Hence $B(X,X)$ is a Banach algebra.

(vi) Let $A(D)$ be the disc algebra with the norm

$$\|f\| = \sup_{z\in D}(|f(z)|), \quad f \in A(D).$$

Then, $A(D)$ is a Banach space.
Let $f,g \in A(D)$. Then

$$\|fg\| \le \|f\|\,\|g\|.$$

Hence $A(D)$ is a commutative Banach algebra.

(vii) The norm on $L^1(\mathbb{R})$ is

$$\|f\| = \int_{-\infty}^{\infty} |f(x)|dx, \quad x \in \mathbb{R},\ f \in L^1(R).$$

Then, $L^1(R)$ is a Banach space and the product is given

$$(f*g)(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx.$$

Let $f,g \in L^1(\mathbb{R})$. Then

$$\begin{aligned}\|f*g\| &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x)g(t-x)|dt\,dx \\ &= \int_{-\infty}^{\infty} |f(x)| \left(\int_{-\infty}^{\infty} |g(t-x)|dt\right) dx \\ &= \int_{-\infty}^{\infty} |f(x)|\ \|g\|dx = \|f\|\,\|g\|.\end{aligned}$$

Hence $L^1(\mathbb{R})$ is a commutative Banach algebra.

(viii) The norm on t'^1 is given by

$$\|a\| = \sum_{n=-\infty}^{\infty} |a_n|, \quad a \in t'^1.$$

Then, t'^1 is a Banach space and the product is given by

$$(a*b)_n = \sum_{n=-\infty}^{\infty} a_{n-K} b_k\,, \quad n \in \mathbb{Z}.$$

Let $a,b \in t'^1$. Then

$$\begin{aligned}\sum_{n\in Z} |(a*b)_n| &= \sum_{n\in Z} \left|\sum_{k\in Z} |a_{n-K} b_k|\right| \\ &\le \sum_{n\in Z}\sum_{k\in Z} |a_{n-k}|\ |b_k| \\ &= \left(\sum_{k\in Z} |b_k|\right)\left(\sum_{k\in Z} |a_{n-k}|\right) = \|a\|\,\|b\|.\end{aligned}$$

Hence t'^1 is a norm algebra. Thus t'^1 is a Banach algebra.

$$(a*b)_n = \sum_{k\in Z} a_{n-K} b_k = \sum_{k\in Z} b_k a_{n-K}.$$

Set $u = n - k$,

$$(a * b)_n = \sum_{k \in Z} b_{n-u} a_u.$$

Hence t'^1 is commutative.

(iv) Let A be a norm space over K. Let A^* be the set of all ordered pairs (x, λ), where $x \in A$ and $\lambda \in \mathbb{C}$.
The norm on A^* is given by

$$\|(x, \lambda)\| = \|x\| + |\lambda|.$$

Then, A^* is a Banach space.
Let A be a normed algebra. Let (x_1, λ_1), $(x_2, \lambda_2) \in A^*$. Then

$$\begin{aligned}
\|(x_1, \lambda_1), (x_2 \lambda_2)\| &= \|(x_1 x_2 + \lambda_1 x_2 + \lambda_2 x_1, \lambda_1 \lambda_2)\| \\
&= \|x_1 x_2 + \lambda_1 x_2 + \lambda_2 x_1\| + |\lambda_1 \lambda_2| \\
&\le \|x_1 x_2\| + \|\lambda_1 x_2\| + \|\lambda_2 x_1\| + |\lambda_1 \lambda_2| \\
&\le \|x_1\|\,\|x_2\| + |\lambda_1|\,\|x_2\| + |\lambda_2|\,\|x_1\| + |\lambda_1|\,|\lambda_2| \\
&= (\|x_1\| + |\lambda_1|)(\|x_2\| + |\lambda_2|) \\
&= \|(x_1, \lambda_1)\|\,\|(x_2, \lambda_2)\|.
\end{aligned}$$

Thus A^* is a normed algebra.
Hence A^* is a Banach algebra with unit $\tilde{e} = (0, 1)$. If A is commutative, then A^* is commutative.

Definition 2.3. Let X be a compact T_2. Let A be a subset of $C(X)$. Then, A is called separates the points of X, if for each $x, y \in X$ with $x \neq y$, there exists $f \in A$ such that $f(x) \neq f(y)$.

Definition 2.4. Let A be a subset of $C(X)$. Then, A is called self-adjoint points of X. If $f \in A$, then $\overline{f} = A$.

Theorem 2.7 (Stone-Weierstrass, [2])**.** *Let X be a compact T_2. Let A be a subset of $C(X)$ and separating the points of X. If A is A is self-adjoint, then*

$$\overline{A} = C(x).$$

Remark 2.2. There are some Banach algebras which are not closed, e.g., let $A = C^1[0, 1]$.
Then, A is Banach algebra (Example 2.1(iv)).
By Stone-Weierstrass theorem, we obtain $\overline{C^1[0, 1]} = C[0, 1]$.
It follows that $C^1[0, 1]$ is not closed.

Theorem 2.8 ([5])**.** *Let A be a complex Banach algebra with unit. Then, every closed subalgebra of A is itself a Banach algebra.*

Theorem 2.9. *Let A be a complex Banach algebra and suppose x in A is such that $\|x\| < 1$. Then, there exists $y \in A$ such that $y = x + y$.*

Proof. Since $\|x\| < 1$ and $\|x^n\| \le \|x\|^n$, the series $-x - x^2 - x^3 - \dots$ is absolutely convergent. Since A is a Banach space, so the converges.

Let the sum of the series be y. Then

$$xy = -x^2 - x^3 - x^4 - \cdots = x + y.$$ □

Theorem 2.10 ([10])**.** *Let A be a complex Banach algebra with unit. Then, every maximal ideal of A is closed.*

Theorem 2.11 ([10])**.** *Let A be a complex Banach algebra with unit. Let I be an ideal of A. Then, the closure of I is an ideal.*

3. Reversible Elements of Banach's Algebra

Theorem 3.1 ([5])**.** *Let A be a complex Banach algebra with unit. If $x \in A$ satisfies $\|x\| < 1$, then $e - x$ is invertible, and*

$$(e-x)^{-1} = e + \sum_{n=1}^{\infty} x^n.$$

Theorem 3.2 ([5])**.** *Let A be a complex Banach algebra with unit. If $x \in A$ and $\|x\| < 1$, then $e - x$ is invertible, $(e-x)^{-1} = e + \sum_{n=1}^{\infty}(-1)^n x^n$, and*

$$\|(e+x)^{-1} - e + x\| \le \frac{\|x\|^2}{1 - \|x\|}.$$

Theorem 3.3 ([5])**.** *Let A be a complex Banach algebra with unit. $\|x - e\| < 1$, then x is invertible and $x^{-1} = e + \sum_{n=1}^{\infty}(e-x)^n$.*

Theorem 3.4. *Let A be a complex Banach algebra with unit. If A^{-1} is an open subset of A.*

Proof. Let $x_0 \in A^{-1}$. Let $B(x_0, \varepsilon)$ be an open ball with center x_0 and radius ε.
Set $\varepsilon = \frac{-1}{\|x_0^{-1}\|} > 0$.
We will show that $B(x_0,\varepsilon) \subseteq A^{-1}$. Let $x \in B(x_0, \varepsilon)$. Then

$$\|x - x_0\| < \frac{-1}{\|x_0^{-1}\|}.$$

Let $y = x_0^{-1}x$ and $z = e - y$. Then

$$\begin{aligned}\|z\| &= \|-z\| \\ &= \|y - e\| \\ &= \|{x_0}^{-1}x - {x_0}^{-1}x_0\| \\ &= \|{x_0}^{-1}(x - x_0)\| \\ &\le \|{x_0}^{-1}\|\,\|x - x_0\| \\ &< \|{x_0}^{-1}\| \frac{1}{\|{x_0}^{-1}\|} = 1.\end{aligned}$$

Thus $\|z\| < 1$. So $e - z$ is invertible in A (Theorem 3.1), and hence $y = e - z \in A^{-1}$.
Now, we have $x_0, y \in A^{-1}$. So $x_0 y \in A^{-1}$ (Theorem 2.5). Therefore

$$x_0 y = x_0 {x_0}^{-1} x = ex = x \in A^{-1}.$$

Hence A^{-1} is open. □

Corollary 3.5. *Let A be a complex Banach's algebra with unit. Then, the set of all non-reversible elements is closed.*

Proof. Since A^{-1} is open (Theorem 3.4), and the set of all non-reversible elements is complement of A^{-1}, so it closed. □

Theorem 3.6 ([10])**.** *Let A be a complex Banach algebra with unit e. Let $x \in A^{-1}$ and $y \in A$ such that*

$$\|x-y\| < \frac{1}{\|x^{-1}\|}.$$

Then, $y \in A^{-1}$ and $\|x^{-1}-y^{-1}\| \le \frac{\|x^{-1}\|^2\,\|x-y\|}{1-\|x^{-1}\|\,\|x-y\|}$.

Proof. Let $x \in A^{-1}$ and $y \in A$. Then

$$\begin{aligned}\|e-x^{-1}y\| &= \|xx^{-1}-x^{-1}y\|\\ &= \|x^{-1}(x-y)\|\\ &\le \|x^{-1}\|\,\|x-y\| \le 1.\end{aligned}$$

So $x^{-1}y$ invertible (Theorem 3.3) and has an inverse in A say Z. Then

$$x^{-1}yz = e. \tag{3.1}$$

Multiplying (3.1) on the left by, we have $xx^{-1}yz = xe$ and so $yz = x$. We obtain $yzx^{-1} = xx^{-1} = e$. Hence $yzx^{-1} = e$.

Again multiplying (3.1) on the right by x^{-1}, we have

$$(x^{-1}yz)x^{-1} = ex^{-1} \quad \text{thus} \quad x^{-1}(x^{-1}yz) = x^{-1}.$$

It follows that

$$zx^{-1}y = \frac{1}{y}y = e.$$

Thus, it is the inverse of Theorem 3.3, this gives us

$$\begin{aligned}z &= \sum_{n=0}^{\infty}(e-x^{-1}y)^n\\ &= \sum_{n=0}^{\infty}(x^{-1}x-x^{-1}y)^n\\ &= \sum_{n=0}^{\infty}(x^{-1}(x-y))^n.\end{aligned}$$

We have

$$\begin{aligned}\|x^{-1}-y^{-1}\| = \|x^{-1}-zx^{-1}\| = \|x^{-1}(e-z)\| &\le \|e-z\|\,\|x^{-1}\|\\ &\le \|x^{-1}\|\sum_{n=1}^{\infty}\|x^{-1}\|^n\,\|x-y\|^n\\ &\le \|x^{-1}\|\sum_{n=1}^{\infty}(\|x^{-1}\|\,\|x-y\|)^n\\ &= \frac{\|x^{-1}\|^2\|x-y\|}{1-\|x^{-1}\|\,\|x-y\|}.\end{aligned}$$

□

Theorem 3.7. *Let A be a complex Banach algebra with unit. Let $x \in A^{-1}$ and $a \in A$ such that $\|a\| \le \frac{1}{2}\|x^{-1}\|^{-1}$. Then, $x + a \in A^{-1}$.*

Proof. Let $x \in A^{-1}$, $a \in A$ and $\|a\| \le \frac{1}{2}\|x^{-1}\|^{-1}$.
Then, $\|x^{-1}a\| \le \frac{1}{2}$.
Hence, $ex^{-1}a \in A^{-1}$, and so writing

$$x + a = x(ex^{-1}a).$$

Now, we have $x \in A^{-1}$ and $e + x^{-1}a \in A^{-1}$. Thus $x(ex^{-1}a) \in A^{-1}$. Hence $x + a \in A^{-1}$. □

Theorem 3.8. *Let A be a complex Banach algebra with unit. Let $y \in A^{-1}$ such that $\|y^{-1}\| = \frac{1}{\alpha}$, $h \in A$ and $\|h\| = \beta < \alpha$. Then, $y + h \in A^{-1}$ and*

$$\|(y+h)^{-1} - y^{-1} + y^{-1}hy^{-1}\| \le \frac{\beta^2}{\alpha^2(\alpha-\beta)}.$$

Proof. Let $y \in A^{-1}$, $h \in A$. Then, $\|y^{-1}h\| \le \frac{\beta}{\alpha} < 1$.
Hence $e + y^{-1}h \in A^{-1}$ (Theorem 3.2).
Since $y + h = x(e + y^{-1}h)$, so we have $y + h \in A^{-1}$.
Then

$$\begin{aligned}(y+h)^{-1} &= (y(e+y^{-1}h))^{-1}\\ &= (e+y^{-1}h)^{-1}y^{-1}.\end{aligned}$$

Now, we have

$$(y+h)^{-1} - y^{-1} + y^{-1}hy^{-1} = [(e+y^{-1}h)^{-1} - e + y^{-1}h]x^{-1}.$$

Therefore

$$\begin{aligned}\|(y+h)^{-1} - y^{-1} + y^{-1}hy^{-1}\| &= \|[(e+y^{-1}h)^{-1} - e + y^{-1}h]y^{-1}\|\\ &\le \|(e+y^{-1}h)^{-1} - e + y^{-1}h\|\,\|y^{-1}\|.\end{aligned}$$

It follows from (Theorem 3.2) with $y^{-1}h$ in place of y:

$$\begin{aligned}\|(y+h)^{-1} - y^{-1} + y^{-1}hy^{-1}\| &\le \frac{\|y^{-1}h\|^2}{1-\|y^{-1}h\|}\|y^{-1}\|\\ &\le \frac{\frac{\beta^2}{\alpha^2}\frac{1}{\alpha}}{1-\frac{\beta}{\alpha}}\\ &\le \frac{\beta^2}{\alpha^2(\alpha-\beta)}.\end{aligned}$$

□

Theorem 3.9. *Let A be a complex Banach algebra with unit. Let $y \in A$ and $\lambda \in \mathbb{C}$ such that $\|y\| < |\lambda|$. Then, $y - \lambda e \in A^{-1}$.*

Proof. Let $\|y\| < |\lambda|$. Then, $\frac{\|y\|}{|\lambda|} < 1$.
Thus, we obtain $\frac{\|y\|}{|\lambda|} < 1$.
Then, it is reversible. Since $-\lambda(e - \lambda^{-1}y) = y - \lambda e$, so $y - \lambda e$ is reversible.
Hence $y - \lambda e \in A^{-1}$. □

Theorem 3.10. *Let A be a commutative Banach algebra with unit. Let $a \in A$. Then, the inversion mapping $a \to a^{-1}$ is continuous in A.*

Proof. Suppose $x_n \in A^{-1}$ and $y_n \to a$ in A. will show that $y_n^{-1} \to a^{-1}$ as $n \to \infty$. Let $a \in A$ such that

$$\|y_n - a\| \le \frac{1}{2\|y_n - a\|}.$$

Then

$$\begin{aligned} \|y_n^{-1} - a^{-1}\| &= \|y_n^{-1}(a - y_n)a^{-1}\| \\ &\le \|y_n^{-1}\| \, \| a - y_n\| \, \|a^{-1}\| \\ &\le \frac{1}{2}\|y_n^{-1}\|. \end{aligned} \tag{3.2}$$

Since $\|y_n^{-1}\| - \|a^{-1}\| \le \|y_n^{-1} - a^{-1}\|$.
Thus, $\|y_n^{-1}\| - \|a^{-1}\| \le \frac{1}{2}\|y_n^{-1}\|$.
It follows that $\|y_n^{-1}\| \le 2\|a^{-1}\|$.
By (3.2), we can get

$$\|y_n^{-1} - a^{-1}\| \le 2\|a^{-1}\|^2 \, \|a - y_n\| \to 0, \quad n \to \infty.$$

Thus $y_n^{-1} \to a^{-1}$. □

Theorem 3.11. *Let A be a commutative complex Banach algebra with unit. Let $a \in A$. Then, the inversion mapping $a \to a^{-1}$ is a homomorphisms of A^{-1} to itself.*

Proof. Clearly, the mapping $a \to a^{-1}$ is onto. Let $a_1, a_2 \in A$ with $a_1^{-1} = a_2^{-1}$. Then, $(a_1^{-1})^{-1} = (a_2^{-1})^{-1}$ and so $a_1 = a_2$.
Then, $a \to a^{-1}$ is one-one.
We have $a \to a^{-1}$ is continuous too (Theorem 2.10), and the inverse map from A onto A is continuous too.
Hence $a \to a^{-1}$ is a homomorphisms. □

Theorem 3.12. *Let A be a commutative complex Banach algebra with unit. Let (a_n) be a sequence in A^{-1} such that $a_n \to a$ in A as $n \to \infty$. If there exists a positive integer such that $\|a_n^{-1}\| \le M$ for all $n \in \mathbb{N}$, then $a \in A^{-1}$ and $a_n^{-1} \to a^{-1}$ as $n \to \infty$.*

Proof. Let $M > 0$ and let $a_n \to a$ as $n \to \infty$. Then, (a_n) is a Cauchy sequence. Then, for each $\varepsilon > 0$ there exists a positive N such that

$$\|a_n - a_m\| < \frac{\varepsilon}{M^2}, \quad \text{for all } n, m \in \mathbb{N}.$$

Therefore,

$$\begin{aligned} \|a_n^{-1} - a_m{}^{-1}\| &= \|a_n^{-1}(a_n - a_m)a_m{}^{-1}\| \\ &\le \|a_n^{-1}\| \, \|a_n - a_m\| \, \|a_m{}^{-1}\| \\ &\le M^2 \frac{\varepsilon}{M^2} = \varepsilon. \end{aligned}$$

Hence (a_n^{-1}) is Cauchy sequence in A. Since A is a Banach algebra, so a_n^{-1} converges to an element in A, say x.

Then

$$x = \lim_{n\to\infty}(a_n^{-1}).$$

So $xa = \lim_{n\to\infty}(a_n^{-1})(a_n) = e$.
Hence a is invertible in A and $x = a^{-1}$.
Thus $a \in A^{-1}$ and $a_n^{-1} \to a^{-1}$ as $n \to \infty$. □

Theorem 3.13. *Let A be a complex Banach algebra with unit. Let (a_n) be a boundary point of A. Let $y_n \in A^{-1}$ such that $y_n \to y$ in A as $n \to \infty$. Then, $\|y_n^{-1}\| \to \infty$ $(n \to \infty)$.*

Proof. If the conclusion is false, then there exists $M < \infty$ such that

$$\|y_n^{-1}\| < M, \quad \text{for all } n.$$

Let y be a boundary point of A and let $y_n \to y$ $(n \to \infty)$. Then, for each $\varepsilon > 0$, there exists $N > 0$ such that

$$\begin{aligned}\|y_n - y\| &< \varepsilon \qquad (\text{where } n > N)\\ &< \frac{1}{M},\end{aligned}$$

$$\begin{aligned}\|e - y_n^{-1}y\| &= \|y_n^{-1}(y_n - y)\|\\ &\le \|y_n^{-1}\|\,\|y_n - y\|\\ &< M\frac{1}{M} = 1.\end{aligned}$$

Thus $\|e - y_n^{-1}y\| < 1$. So $y_n^{-1}y \in A^{-1}$. Then

$$y = y_n(y_n^{-1}y) \in A^{-1}.$$

We have $y \in A^{-1}$ and $y \in \partial(A)$.
It follows that $A^{-1} \cap \partial(A) \neq \emptyset$.
This is contradicts to A^{-1} is open (Theorem 3.4).
Hence $\|y_n^{-1} \to \infty\|$ $(n \to \infty)$. □

Theorem 3.14. *Let A be a complex Banach algebra with unit $e = 1$. Let $(a_n) \subseteq A^{-1}$ be and $a_n \to a$ $(n \to \infty)$ in A. Then, there exists a sequence $(b_n) \subseteq A$ with $\|b_n\| = 1$ and $b_n a \to 0$ $(n \to \infty)$.*

Proof. Set $b_n = \frac{a_n^{-1}}{\|a_n^{-1}\|}$.
Then, $\|b_n\| = 1$ and so (b_n) is a bounded sequence.
Also, $b_n a_n = \frac{1}{\|a_n^{-1}\|} \to 0$.
We have $b_n(a - a_n) \to 0$.
Adding, we obtain $b_n a \to 0$ $(n \to \infty)$. □

Definition 3.1. Let A be a complex Banach algebra with unit. We define the exponential function $\exp : A \to A$ by

$$\exp(y) = \sum_{n=0}^{\infty} \frac{1}{n!} y^n \quad (y \in A), \text{ and } \exp(0) = 1.$$

Theorem 3.15. *Let A be a commutative Banach algebra with unit $e = 1$. Let $x, y \in A$. Then*

(i) $\exp(xy) = \exp(x)\exp(y)$,

(ii) $\exp(x) \in A^{-1}$ *and* $(\exp(x))^{-1} = \exp(-x)$.

Proof. Let $x, y \in A$. Then

(i)
$$\begin{aligned}\exp(x+y) &= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^{-j} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{j!(n-j)!} x^{n-j} y^{-j} \\ &= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{j!n!} x^n y^j \\ &= \exp(x)\exp(y).\end{aligned}$$

(ii) Take $y = -x$ in (i). Then

$$\begin{aligned}\exp(0) &= \exp(x)\exp(-x) \\ 1 &= \exp(x)\exp(-x)\end{aligned}$$

Thus $(\exp(x))^{-1} = \exp(-x)$. □

Theorem 3.16 ([2])**.** *Let A be a complex Banach algebra with unit $e = 1$. Let $x \in A$ such that $\|1 - x\| < 1$. Then, there exists $y \in A$ such that $\exp(y) = x$.*

Definition 3.2. Let A be a complex Banach algebra with unit. We define

$$\exp(A) = \{\exp(x) : x \in A\}.$$

It is clear that $\exp(A) \subset A^{-1}$.

Theorem 3.17. *Let A be a commutative Banach algebra with unit $e = 1$. Then, $\exp(A)$ is open in A^{-1}.*

Proof. Let $x \in \exp(A)$. Then, a^{-1},

$$x = \exp(h), \quad h \in A.$$

Let $y \in A$ with $\|x - y\| < \frac{1}{\|x^{-1}\|}$.
Then,

$$\begin{aligned}\|1 - x^{-1}y\| &= \|x^{-1}\| \, \|x - y\| \\ &\le \|x^{-1}\| \frac{1}{\|x^{-1}\|} = 1.\end{aligned}$$

There exists $z \in A$ such that $x^{-1}y = \exp(z)$. We have

$$\begin{aligned}y &= \exp(h)\exp(z) \\ &= \exp(h+z) \in \exp(A).\end{aligned}$$

Hence $\exp(A)$ is open in A^{-1}. □

4. Spectral Radius and Spectroscopic of Banach's Algebra

Definition 4.1. Let A be a complex Banach algebra with unit e. Then, spectrum of an element $y \in A$, denoted by $\sigma_A(y)$, is defined by

$$\sigma_A(y) = \{\lambda \in \mathbb{C} : y - \lambda e \notin A^{-1}\}.$$

The complement of $\sigma_A(y)$ in $\mathbb{C}$ is called the resolvent set of y. It is denoted by $\rho_A(y)$. That is

$$\rho_A(y) = \frac{\mathbb{C}}{\sigma_A(y)}.$$

Remark 4.1. Let A be a Banach algebra with unit. It is clear that y is reversible in A if and only if $0 \notin \sigma_A(y)$.

Example 4.1. Let $A = M_{2\times 2}$ with complex entries.

Then, $A = M_{2\times 2}$ is a complex Banach algebra with unit $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Let $x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in M_{2\times 2}$.

By an elementary of matrix algebra it is known that $x - \lambda I$ has no inverse if and only if $\det(x - \lambda I) = 0$.

Then

$$\begin{aligned}
\sigma_A(x) &= \left\{\lambda \in \mathbb{C} : \det\left(\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0\right\} \\
&= \left\{\lambda \in \mathbb{C} : \det\left(\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} - \lambda \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0\right\} \\
&= \left\{\lambda \in \mathbb{C} : \det\left(\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} - \lambda \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0\right\} \\
&= \left\{\lambda \in \mathbb{C} : \det\begin{bmatrix} -\lambda & i \\ -i & -\lambda \end{bmatrix} = 0\right\} \\
&= \left\{\lambda \in \mathbb{C} : \lambda^2 + i^2 = 0\right\} \\
&= \{-1, +1\}.
\end{aligned}$$

Lemma 4.1. *Let be a complex Banach algebra with unit e. Then*

$$\sigma_A(0) = \{0\}.$$

Proof.

$$\begin{aligned}
\sigma_A(0) &= \{\lambda \in \mathbb{C} : 0 - \lambda e \notin A^{-1}\} \\
&= \{\lambda \in \mathbb{C} : -\lambda e \notin A^{-1}\} \\
&= \{\lambda \in \mathbb{C} : -\lambda \notin A^{-1}\} \\
&= \{0\}.
\end{aligned}$$

□

Theorem 4.2. *Let A be a complex Banach algebra with unit e. Let $y \in A$. Then, $\sigma_A(y)$ is non-empty.*

Proof. Suppose for a contradiction that $y \in A$ has an empty spectrum.
Define $u(\lambda) = (y - \lambda e)^{-1} (\lambda \in \mathbb{C})$.
Then, u is well-defined and a continuous mapping of $\mathbb{C}$ into A.
Let $\lambda_0 \in \mathbb{C}$. Then

$$\begin{aligned} u(\lambda) - u(\lambda_0) &= (y - \lambda e)^{-1} - (y - \lambda_0 e)^{-1} \\ &= u(\lambda)u(\lambda_0)((y - \lambda e) - (y - \lambda_0 e)) \\ &= (\lambda - \lambda_0) e u(\lambda) u(\lambda_0). \end{aligned}$$

It follows that

$$\frac{u(\lambda) - u(\lambda_0)}{\lambda - \lambda_0} = u(\lambda)u(\lambda_0).$$

So

$$\lim_{\lambda - \lambda_0} \frac{u(\lambda) - u(\lambda_0)}{\lambda - \lambda_0} = (u(\lambda_0))^2. \tag{4.1}$$

Let f be a continuous linear functional on A. We define a function h by

$$h(\lambda) = f(u(\lambda)), \quad \lambda \in \mathbb{C}.$$

Since f and u are continuous, so is h.
Applying f to (4.1), we thus obtain

$$\lim_{\lambda \to \lambda_0} \frac{h(\lambda) - h(\lambda_0)}{\lambda - \lambda_0} = f(u(\lambda_0))^2.$$

Then, h is an entire function from $\mathbb{C}$ into $\mathbb{C}$.
Since

$$u(\lambda) = -\lambda^{-1}(e - \lambda^{-1}x)^{-1}$$

and

$$(e - \lambda^{-1}x)^{-1} \to e^{-1} = e \text{ as } |\lambda| \to \infty,$$

we obtain

$$\begin{aligned} |h(\lambda)| &= |f(u(\lambda))| \\ &\le \|f\| \, \|u(\lambda)\| \\ &= \frac{1}{|\lambda|} \|f\| \left\| \left(e - \frac{1}{\lambda} x \right)^{-1} \right\| \\ &\to 0 \text{ as } |\lambda| \to \infty. \end{aligned} \tag{4.2}$$

This shows that h would be bounded on $\mathbb{C}$.
By Liouville's theorem, h is constant which is zero by (4.2). Then

$$h(\lambda) = f(u(\lambda)) = 0.$$

It follows that. So

$$\begin{aligned} \|e\| &= \|(y - \lambda e)(e - \lambda^{-1} y)^{-1}\| \\ &= \|(y - \lambda e) u(\lambda)\| = \|0\| = 0 \end{aligned}$$

and contradicts to $\|e\| = 1$.
Hence $\rho_A(y) \neq \emptyset$. □

Remark 4.2. If A be a real Banach algebra with unit, then it is possible that there exists $\rho_A(y) \in A$ such that $\sigma(y) = \emptyset$.

Example 4.2. Let $A = M_{2\times 2}$ be a real Banach algebra with unit $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Let $y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in M_{2\times 2}$. Then

$$\begin{aligned}\sigma_A(y) &= \left\{\lambda \in \mathbb{R} : \det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0\right\} \\ &= \left\{\lambda \in \mathbb{R} : \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = 0\right\} \\ &= \{\lambda \in \mathbb{R} : \lambda^2 + 1 = 0\} = \emptyset.\end{aligned}$$

Lemma 4.3 ([2]). *Let A be a complex Banach algebra with unit. Let $y \in A$. The resolvent set $\rho_A(y)$ of y is open in $\mathbb{C}$.*

Theorem 4.4. *Let A be a Banach algebra with unit. Let $y \in A$. Then, $\rho_A(y)$ is a compact subset of $\mathbb{C}$.*

Proof. By the Heine-Borel Theorem, it is enough to show that $\sigma_A(y)$ is bounded and closed. Let $\lambda \in \sigma_A(y)$. Then, $y - \lambda e \notin A^{-1}$.
By Theorem 2.9,

$$\|y\| \leq |\lambda|.$$

So

$$\sigma_A(y) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|y\|\}.$$

Thus $\sigma_A(y)$ is bounded.
Since $\rho_A(y)$ is open in $\mathbb{C}$ (Lemma 4.3), so $\sigma_A(y)$ is closed. □

Theorem 4.5. *Let A be a complex Banach algebra with unit $= 1$. Let $y \in A$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. If $\lambda \in \sigma_A(y)$, then $x^n \in \sigma(y^n)$.*

Proof. Let $y \in A$ and let $\lambda \in \mathbb{C}$. Assume $\lambda^n \notin \sigma(y^n)$.
We have

$$(y^n - \lambda^n e) = (y - \lambda e)(y^{n-1} + \lambda y^{n-2} + \cdots + \lambda^{n-1} e). \tag{4.3}$$

If multiply both sides of (4.3) by $(y^n - \lambda^n e)^{-1}$, then $(y - \lambda e)$ is invertible in A. So $\lambda \notin \sigma(y)$. This completes the proof. □

Theorem 4.6 ([12]). *Let A be a complex Banach algebra with unit. Let B be a closed subalgebra of A containing e. If $y \in B$, then $\sigma_A(y) \subseteq \sigma_B(y)$, and*

$$\partial(\sigma_A(y)) \subseteq \partial(\sigma_B(y)).$$

Theorem 4.7 ([7]). *Let A be a closed subalgebra of a complex Banach algebra B. Let $y \in A$. If $\sigma_A(y)$ has empty interior, then $\sigma_A(y) = \sigma_B(y)$.*

Theorem 4.8 ([2])**.** *Let A be a commutative complex Banach algebra with unit. Let $x \in A$. Then*

$$\sigma_A(\exp(y)) = \exp(\sigma_B(y)).$$

Remark 4.3. In fact, there are some non-zero element of complex Banach algebras which are not invertible. For examples:

(i) Let $A = M_{2\times 2}$ with complex entries. Then, $M_{2\times 2}$ is a complex Banach algebra with unit $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Let $x = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \in M_{2\times 2}$. Then, y is a non-zero element of $M_{2\times 2}$ but x is not invertible.

(ii) Let $A = C[0,1]$.
Then, $C[0,1]$ is a complex Banach algebra with unit $e = 1$.
Define f by

$$f(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2}, \\ x - \frac{1}{2}, & \frac{1}{2} \le x \le 1. \end{cases}$$

Then, f is a non-zero element of $C[0,1]$ but f is not invertible.

Proposition 4.9 ([1])**.** *Let A be a complex Banach algebra with unit e in which each non-zero element in A is invertible. Let $y \in A$. Then, there exists a unique $\lambda \in \mathbb{C}$ such that $y = \lambda e$.*

Proof. Let $y \in A$. Then, $\sigma_A(y) \neq \emptyset$ (Theorem 3.2). Hence there exists $\lambda \in \sigma_A(y)$ such that $(y - \lambda e)$ is not invertible. So $y - \lambda e = 0$. Thus $y = \lambda e$. For uniqueness, let $y = \lambda e$, $x = \mu e$ ($\mu \in \mathbb{C}, \lambda \neq \mu$). Let $\alpha = \lambda - \mu \neq 0$.
Then, $\alpha e = 0$, and so $e = 0$ which is a contradiction. □

Corollary 4.10. *Let A be a complex Banach algebra with unit e in which each non-zero element in A is invertible. Then, Ais commutative.*

Proof. Let $x, y \in A$. Then, there exists unique $\lambda, \mu \in \mathbb{C}$ ($\lambda \neq \mu$), such that

$$x = \lambda e, \quad y = \mu e \qquad \text{(Proposition 4.9)}.$$

Then

$$xy = (\lambda e)(\mu e) = (\lambda\mu)e = (\mu\lambda)e = (yx).$$

Hence A is commutative. □

Theorem 4.11 (Gelfand-Mazur, [6])**.** *Let A be a complex Banach algebra with unit e in which each non-zero element in A is invertible. Then, A is isomorphic to $\mathbb{C}$.*

Theorem 4.12 (Spectral Mapping Theorem, [6])**.** *Let A be a complex Banach algebra with unit, and $y \in A$. Let P be a polynomial function with complex coefficients in A. Then*

$$P(\sigma_A(y)) \subseteq P(\sigma_B(y)).$$

Lemma 4.13. *Let A be a commutative Banach algebra with unit. Let $y \in A$ and P be a polynomial function such that $P(y) = 0$. Then, $P(\sigma_A(y)) = \{0\}$.*

Proof. Let $y \in A$. By spectral mapping theorem,

$$\begin{aligned} P(\sigma_A(y)) &= \sigma_A(P(y)) \\ &= \sigma_A(0) \\ &= \{0\}. \quad \text{(Lemma 4.1)} \end{aligned}$$

□

Definition 4.2. Let A be a complex Banach algebra with unit e. Let $xy \in A$. The spectral radius of y, denoted by $r_A(y)$, is defined by

$$r_A(y) = \sup\{|\lambda| : \lambda \in \sigma_A(y)\}.$$

Remarks. (i) $0 \le r_A(y) < \infty$, for all y.

(ii) If $r_A(y) = 0$, then $0 \in \sigma_A(y)$.

Example 4.3. Let $A = M_{2\times 2}$ with complex entries.
Let $y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \in M_{2\times 2}$. Then, $\sigma_A(y) = \{-1, +1\}$.
So $r_A(y) = \sup\{|-1|, |1|\} = 1$.

Lemma 4.14. *Let A be a complex Banach algebra with unit e. Let $y \in A$. Then*

$$r_A(y) \le \|y\|.$$

Proof. If $|\lambda| \ge \|y\|$, then $\|\lambda^{-1} y\| < 1$.
So $e - \lambda^{-1} y$ is invertible (Theorem 3.1). Since

$$-\lambda(e - \lambda^{-1} y) = y - \lambda e.$$

So $y - \lambda e$ is invertible in A. Thus $\lambda \notin \sigma_A(y)$. So $\lambda \in \sigma_A(y)$ implies $|\lambda| < \|y\|$.
Taking supremum over $\lambda \in \sigma_A(y)$, we obtain

$$\sup_{\lambda \in \sigma_A(x)} (|\lambda|) \le \|y\|.$$

Hence $r_A(y) \le \|y\|$. □

Lemma 4.15. *Let A be a complex Banach algebra with unit e, and $y \in A$, $n \in \mathbb{N}$. Then*

$$r_A(y^n) = r_A(y)^n.$$

Proof. Let $y \in A$. Then, $r_A(y) = \sup\{|\lambda| : \lambda \in \sigma_A(y)\}$.
Therefore, $r_A(y^n) = \sup\{|\lambda| : \lambda \in \sigma_A(y^n)\}$.
The spectral mapping theorem gives us:

$$\begin{aligned} \sigma_A(P(y)) &= P(\sigma_A(y)) \\ &= \{p(\lambda) : \lambda \in \sigma_A(y)\}. \end{aligned}$$

Let $p(y) = y^n$. Then

$$\sigma_A(y^n) = \{\lambda^n : \lambda \in \sigma_A(y)\}.$$

It follows that

$$\begin{aligned} r_A(y^n) &= \sup\{|\lambda|^n : \lambda \in \sigma_A(y)\} \\ &= r_A(y^n). \end{aligned}$$

□

Theorem 4.16 (Spectral Radius Formula, [6]). *Let A be a complex Banach algebra with unit e, and $x \in A$. Then*

$$\begin{aligned} r_A(y) &= \lim_{n\to\infty} \|y^n\|^{\frac{1}{n}} \qquad (\textit{where } n = 1,2,3,\dots) \\ &= \inf_{n\geq 1}(\|y^n\|^{\frac{1}{n}}). \end{aligned}$$

5. Conclusion

We presented the ideas of (reversible elements, spectral radius and reversible spectral). We studied some of the basic properties of these ideas in the Banach algebra.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] Y. Abilov and M. Acikgoz, On some properties of Banach spaces and Banach algebras, *International Mathematical Forum* **2**(21-24) (2007), 1193 – 1206, DOI: 10.12988/imf.2007.07105.

[2] F.F. Bonsall and J. Duncan, *Complete Normed Algebras*, 1st edition, Ergebnisse der Mathematik und ihrer Grenzgebiete 2. Folge (MATHE2, Volume **80**), Springer-Verlag, Berlin — Heidelberg, x + 304 pages (1973), URL: 10.1007/978-3-642-65669-9.

[3] E.K. Gaur, *Pure States of the Commutative Banach Algebras*, Springer Science Media, LLC, New York (2009).

[4] A.W. Knapp, *Advanced Real Analysis*, 2nd edition, Springer Science Media Inc., USA (2017).

[5] E. Kreyszig, *Introductory Functional Analysis With Applications*, John Wiley & Sons, USA (1978).

[6] I.J. Maddox, *Elements of Functional Analysis*, Cambridge University Press, x + 208 pages (1970).

[7] T.W. Palmer, Banach algebras and the general theory of $*$-algebras, in: *Algebras and Banach Algebras*, Vol. **1**, Cambridge University Press (1994), DOI: 10.1017/CBO9781107325777.

[8] S.H.J. Petroudi, S.A. Sadati and A. Yaghobi, New results on soft banach algebra, *International Journal of Science and Engineering Investigations* **6**(9) (2017), 17 – 25, URL: https://arastirmax.com/en/system/files/dergiler/65306/makaleler/6/9/arastirmax-new-results-soft-banach-algebra.pdf.

[9] A. Rizai and M. Adib, φ-Multipliers on Banach algebras without order, *International Journal of Mathematical Analysis* **3**(3) (2009), 121 – 132.

[10] W. Rudin, *Real and Complex Analysis*, 3rd edition, McGraw-Hill Book Company, New York (1987), xiv + 416 pages.

[11] K. Saxe, *Beginning Functional Analysis*, Undergraduate Texts in Mathematics series, Springer, New York, USA, xi + 197 pages (2002), DOI: 10.1007/978-1-4757-3687-8.

[12] A.H. Siddiqi, *Applied Functional Analysis: Numerical Methods, Wavelet Methods, and Image Processing*, 1st edition, CRC Press, 660 pages (2003).

[13] M. Thill, Introduction to normed $*$-algebras and their representations, 7th edition, *arXiv:1011.1558*, (2008), DOI: 10.48550/arXiv.1011.1558.

[14] M. Tomforde, Continuity of ring $*$-homomorphisms between C^*-algebras, *New York Journal of Mathematics* **15** (2009), 161 – 167, URL: http://nyjm.albany.edu/j/2009/15-8.pdf.