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Research Article

A New Generalization of Gegenbauer Polynomials

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Abstract. In this work, the author introduces new generalization of Gegenbauer polynomials of one and two variables by considering new extended gamma function defined by MacDonald function. Certain properties of this new generalized Gegenbauer polynomials like integral formulas, Mellin transform, recurrence relationsand generating function are presented and investigated.

Keywords. MacDonaldfunction; Gegenbauer polynomials; Gamma function; Hermit-Kamp de Friet polynomials; Generating function; Mellin transform

Mathematics Subject Classification (2020). 33B15; 33C20; 33C65; 33D52

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1. Introduction

The classical Gegenbauer polynomials is defined by the series representation as follows [11]:

$$C_E^{\varsigma}(u,v,a)=\frac{1}{\Gamma(\varsigma)}\sum_{\kappa=0}^{\left[\frac{E}{2}\right]}\frac{(2u)^{E-2\kappa}(-v)^{\kappa}\Gamma(\varsigma+E-\kappa)}{(E-2\kappa)!\kappa!a^{\varsigma+E-\kappa}}, \tag{1.1}$$

where $\Gamma(;)$ is the classical gamma function which can expressed as (see [4], [5])

$$\Gamma(E)=\int_0^{\infty}t^{\varsigma-1}\exp(-t)dt, \quad (Re(\varsigma)>0). \tag{1.2}$$

With the following integral representation [11]:

$$C_E^{\varsigma}(u,v,a)=\frac{1}{\Gamma(\varsigma)E!}\int_0^{\infty}t^{\varsigma-1}\exp(-at)H_E(2ut,-vt)dt, \tag{1.3}$$

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here $H_E(2ut,-vt)$ is the two variables Hermit-Kamp de Friet polynomials given by [7]

$$H_E(2ut,-vt) = E! \sum_{\kappa=0}^{\left[\frac{E}{2}\right]} \frac{u^{E-2\kappa} v^{\kappa}}{(E-2\kappa)!\kappa!}. \tag{1.4}$$

Setting $a = 1$, (1.1) and (1.3) reduces to

$$C_E^{\varsigma}(u,v) = \frac{1}{\Gamma(\varsigma)} \sum_{\kappa=0}^{\left[\frac{E}{2}\right]} \frac{(2u)^{E-2\kappa}(-v)^{\kappa}\Gamma(\varsigma+E-\kappa)}{(E-2\kappa)!\kappa!},$$

and

$$C_E^{\varsigma}(u,v) = \frac{1}{\Gamma(\varsigma)E!} \int_0^{\infty} t^{\varsigma-1} \exp(-t) H_E(2ut,-vt)dt.$$

Similarly, $v = a = 1$, (1.1) and (1.3) reduces to Gegenbauer polynomials in [15]

$$C_E^{\varsigma}(u) = \frac{1}{\Gamma(\varsigma)} \sum_{\kappa=0}^{\left[\frac{E}{2}\right]} \frac{(2u)^{E-2\kappa}(-1)^{\kappa}\Gamma(\varsigma+E-\kappa)}{(E-2\kappa)!\kappa!}$$

and

$$C_E^{\varsigma}(u) = \frac{1}{\Gamma(\varsigma)E!} \int_0^{\infty} t^{\varsigma-1} \exp(-t) H_E(2ut,-t)dt.$$

Recently, Atash and Al-Gonah [8] introduced extended Gegenbauer polynomials as follows:

$$C_E^{\varsigma}(u,v,o) = \frac{1}{\Gamma(\varsigma)} \sum_{\kappa=0}^{\left[\frac{E}{2}\right]} \frac{(2u)^{E-2\kappa}(-v)^{\kappa}\Gamma_o(\varsigma+E-\kappa)}{(E-2\kappa)!\kappa!}, \tag{1.5}$$

with the following integral representation:

$$C_E^{\varsigma}(u,v,o) = \frac{1}{\Gamma(\varsigma)E!} \int_0^{\infty} t^{\varsigma-1} \exp(-t-ot^{-1}) H_E(2ut,-vt)dt, \tag{1.6}$$

where $\Gamma_o(;)$ is the extended Euler's beta function given by (refer to [1], [3], [6], [10])

$$\Gamma_o(\varsigma) = \int_0^{\infty} t^{\varsigma-1} \exp(-t-ot^{-1})dt, \tag{1.7}$$

$$(Re(o) > 0,\ Re(\varsigma) > 0).$$

On setting $v = 1$, (1.5) and (1.6) reduces to the following equations [8]

$$C_E^{\varsigma}(u,o) = \frac{1}{\Gamma(\varsigma)} \sum_{\kappa=0}^{\left[\frac{E}{2}\right]} \frac{(2u)^{E-2\kappa}(-1)^{\kappa}\Gamma_o(\varsigma+E-\kappa)}{(E-2\kappa)!\kappa!},$$

with the following integral representation [8]:

$$C_E^{\varsigma}(u,a) = \frac{1}{\Gamma(\varsigma)E!} \int_0^{\infty} t^{\varsigma-1} \exp(-t-ot^{-1}) H_E(2ut,-t)dt.$$

2. The Generalized Gegenbauer Polynomials of Two and Variables

In this work, new generalized Gegenbauer polynomials is presented as follows:

$$C_{E,\varrho}^{\varsigma}(u,v,o) = \frac{1}{\Gamma(\varsigma)} \sum_{\kappa=0}^{\left[\frac{E}{2}\right]} \frac{(2u)^{E-2\kappa}(-v)^{\kappa}\Gamma_{\varrho}(\varsigma+E-\kappa,o)}{(E-2\kappa)!\kappa!}, \tag{2.1}$$

where $\Gamma_{\varrho}(\varsigma,o)$ is the extended gamma function defined by (see for examples [2], [16])

$$\Gamma_{\varrho}(\varsigma,o)=\sqrt{\frac{2o}{\pi}}\int_0^{\infty} t^{\varsigma-\frac{3}{2}}\exp(-t)K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)dt. \tag{2.2}$$

$(Re(\varrho)>0,\ Re(\varsigma)>0)$.

If $\varrho=0$ in (2.2) and considering the MacDonald (modified Bessel) function [14]

$$K_{\frac{1}{2}}(z)=\sqrt{\frac{\pi}{2o}}\exp(-z)$$

it reduces to the extended gamma function given in (1.7) and if $\varrho=o=0$ it reduces to classical gamma function in (1.2).

If $v=1$, (2.1) reduce to the following result

$$C_{E,\varrho}^{\varsigma}(u,o)=\frac{1}{\Gamma(\varsigma)}\sum_{\kappa=0}^{\left[\frac{E}{2}\right]}\frac{(2u)^{E-2\kappa}(-1)^{\kappa}\Gamma_{\varrho}(\varsigma+E-\kappa,o)}{(E-2\kappa)!\kappa!}.$$

3. Integral Representations

Theorem 3.1. *The following integral formula holds*

$$C_{E,\varrho}^{\varsigma}(u,v,o)=\frac{1}{\Gamma(\varsigma)E!}\sqrt{\frac{2o}{\pi}}\int_0^{\infty} t^{\varsigma-\frac{3}{2}}\exp(-t)K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)H_E(2ut,-vt)dt. \tag{3.1}$$

Proof. Using

$$C_{E,\varrho}^{\varsigma}(u,v,o)=\frac{1}{\Gamma(\varsigma)}\sum_{\kappa=0}^{\left[\frac{E}{2}\right]}\frac{(2u)^{E-2\kappa}(-v)^{\kappa}\Gamma_{\varrho}(\varsigma+E-\kappa,o)}{(E-2\kappa)!\kappa!}.$$

Applying (2.2), gives

$$C_{E,\varrho}^{\varsigma}(u,v,o)=\frac{1}{\Gamma(\varsigma)}\sum_{\kappa=0}^{\left[\frac{E}{2}\right]}\frac{(2u)^{E-2\kappa}(-v)^{\kappa}}{(E-2\kappa)!\kappa!}\left\{\sqrt{\frac{2o}{\pi}}\int_0^{\infty} t^{\varsigma+E+\kappa-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)dt\right\}.$$

Interchanging the order of summation and integration, yields

$$C_{E,\varrho}^{\varsigma}(u,v,o)=\frac{1}{\Gamma(\varsigma)}\sqrt{\frac{2o}{\pi}}\int_0^{\infty} t^{\varsigma-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)\left\{\sum_{\kappa=0}^{\left[\frac{E}{2}\right]}\frac{(2u)^{E-2\kappa}(-v)^{\kappa}}{(E-2\kappa)!\kappa!}t^{E+\kappa}\right\}dt.$$

Rewritten this equation, gives

$$C_{E,\varrho}^{\varsigma}(u,v,o)=\frac{1}{\Gamma(\varsigma)}\sqrt{\frac{2o}{\pi}}\int_0^{\infty} t^{\varsigma-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)\left\{t^E\sum_{\kappa=0}^{\left[\frac{E}{2}\right]}\frac{(2u)^{E-2\kappa}(-vt)^{\kappa}}{(E-2\kappa)!\kappa!}\right\}dt.$$

Considering (1.4), yields

$$C_{E,\varrho}^{\varsigma}(u,v,o)=\frac{1}{\Gamma(\varsigma)E!}\sqrt{\frac{2o}{\pi}}\int_0^{\infty} t^{\varsigma-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)t^E H_E\left(2u,-vt^{-1}\right)dt.$$

On using the relation in [12]

$$t^E H_E(u,v)=H_E(ut,vt),$$

leads to

$$C_{E,\varrho}^{\varsigma}(u,v,o)=\frac{1}{\Gamma(\varsigma)E!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)H_E(2ut,-vt)dt.$$

Setting $v=1$; $\varrho=0$ and $v=1$, $\varrho=0$ in (3.1), the following corollaries can be received, respectively:

Corollary 3.2. *The following integral formula hold*

$$C_{E,\varrho}^{\varsigma}(u,o)=\frac{1}{\Gamma(\varsigma)E!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)H_E(2ut,-t)dt.$$

Corollary 3.3. *The following integral representation is valid*

$$C_{E}^{\varsigma}(u,v,o)=\frac{1}{\Gamma(\varsigma)E!}\int_0^\infty t^{\varsigma-1}\exp\left(-t-ot^{-1}\right)H_E(2ut,-vt)dt.$$

Corollary 3.4. *The following integral formula hold true*

$$C_{E}^{\varsigma}(u,v)=\frac{1}{\Gamma(\varsigma)E!}\int_0^\infty t^{\varsigma-1}\exp(-t)\,H_E(2ut,-vt)dt.$$

4. Mellin Transform

Theorem 4.1. *The following Mellin formula hold true*

$$M\left\{C_{E,\varrho}^{\varsigma}(u,v,o);s\right\}=\frac{2^{s-\frac{3}{2}}\Gamma\left(\frac{s-\varrho}{2}\right)\Gamma\left(\frac{s+\varrho+1}{2}\right)}{\Gamma(\varsigma)\sqrt{\pi}E!}C_{E}^{\varsigma+s}(u,v). \tag{4.1}$$

Proof. Using Mellin transform in [5]

$$M\left\{C_{E,\varrho}^{\varsigma}(u,v,o)\right\}=\int_0^\infty \varrho^{s-1}\left\{C_{E,\varrho}^{\varsigma}(u,v,o)\right\}d\varrho.$$

Using the result in (3.1), leads to

$$M\left\{C_{E,\varrho}^{\varsigma}(u,v,o)\right\}=\int_0^\infty o^{s-1}\left\{\frac{1}{\Gamma(\varsigma)E!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)H_E(2ut,-vt)dt\right\}do.$$

Changing the order of integrations, we obtain

$$M\left\{C_{E,\varrho}^{\varsigma}(u,v,o)\right\}=\frac{1}{\Gamma(\varsigma)E!}\int_0^\infty t^{\varsigma-\frac{3}{2}}\exp(-t)\,H_E(2ut,-vt)\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty o^{s-\frac{1}{2}}K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)do\right\}dt.$$

Putting $o=\omega t$, gives

$$M\left\{C_{E,\varrho}^{\varsigma}(u,v,o)\right\}=\frac{1}{\Gamma(\varsigma)E!}\int_0^\infty t^{\varsigma+s-1}\exp(-t)\,H_E(2ut,-vt)\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty \omega^{s-\frac{1}{2}}K_{\varrho+\frac{1}{2}}(\omega)d\omega\right\}dt.$$

Using the relation given in [14]

$$\int_0^\infty \omega^{s-\frac{1}{2}}K_{\varrho+\frac{1}{2}}(\omega)d\omega=2^{s-\frac{3}{2}}\Gamma\left(\frac{s-\varrho}{2}\right)\Gamma\left(\frac{s+\varrho+1}{2}\right),$$

leads to

$$M\left\{C_{E,\varrho}^{\varsigma}(u,v,o)\right\}=\frac{1}{\Gamma(\varsigma)E!}\int_0^\infty t^{\varsigma+s-1}\exp(-t)\,H_E(2ut,-vt)\left\{\sqrt{\frac{2}{\pi}}\left[2^{s-\frac{3}{2}}\Gamma\left(\frac{s-\varrho}{2}\right)\Gamma\left(\frac{s+\varrho+1}{2}\right)\right]\right\}dt.$$

On simplifying

$$M\left\{C_{E,\varrho}^{\varsigma}(u,v,o)\right\}=\frac{2^{s-\frac{3}{2}}\Gamma\left(\frac{s-\varrho}{2}\right)\Gamma\left(\frac{s+\varrho+1}{2}\right)}{\Gamma(\varsigma)\sqrt{\pi}E!}\int_0^\infty t^{\varsigma+s-1}\exp(-t)\,H_E(2ut,-vt)\,dt.$$

Rewritten this equation, gives

$$M\left\{C_{E,\varrho}^{\varsigma}(u,v,o)\right\}=\frac{2^{s-\frac{3}{2}}\Gamma\left(\frac{s-\varrho}{2}\right)\Gamma\left(\frac{s+\varrho+1}{2}\right)}{\Gamma(\varsigma)\sqrt{\pi}E!}\times\left[\Gamma(\varsigma+s)\left\{\frac{1}{\Gamma(\varsigma+s)E!}\int_0^\infty t^{\varsigma+s-1}\exp(-t)\,H_E(2ut,-vt)\,dt.\right\}\right].$$

On using (3.1), yields

$$M\left\{C_{E,\varrho}^{\varsigma}(u,v,o);s\right\}=\frac{2^{s-\frac{3}{2}}\Gamma\left(\frac{s-\varrho}{2}\right)\Gamma\left(\frac{s+\varrho+1}{2}\right)\Gamma(\varsigma+s)}{\Gamma(\varsigma)\sqrt{\pi}}C_E^{\varsigma+s}(u,v).$$ □

Substituting $v=1$ in (4.1), gives

Corollary 4.2. *The following Mellin transform formula also hold true*

$$M\left\{C_{E,\varrho}^{\varsigma}(u,o);s\right\}=\frac{2^{s-\frac{3}{2}}\Gamma\left(\frac{s-\varrho}{2}\right)\Gamma\left(\frac{s+\varrho+1}{2}\right)}{\Gamma(\varsigma)\sqrt{\pi}E!}C_E^{\varsigma+s}(u).$$

5. Recurrence Relations

Theorem 5.1. *The following recurrence relation hold true*

$$(E+1)C_{E+1,\varrho}^{\varsigma}(u,v,o)=2\varsigma uC_{E+1,\varrho}^{\varsigma+1}(u,v,o)-2\varsigma vC_{E-1,\varrho}^{\varsigma+1}(u,v,o). \tag{5.1}$$

Proof. Using the recurrence relation in [9]

$$H_{\varsigma+1}(u,v)=uH_\varsigma(u,v)+2\varsigma vH_{\varsigma-1}(u,v),$$

and setting $u\to 2ut$ and $v\to -vt$, yields

$$H_{\varsigma+1}(2ut,-vt)=2utH_\varsigma(2ut,-vt)-2\varsigma vtH_{\varsigma-1}(2ut,-vt).$$

Multiplying both sides by $\frac{1}{\Gamma(\varsigma)E!}\sqrt{\frac{2o}{\pi}}t^{\varsigma+E+\kappa-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)$ and integrate with respect to t taking limit from 0 to ∞, gives

$$\frac{1}{\Gamma(\varsigma)E!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma-\frac{3}{2}}\exp(-t)K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)H_E(2ut,-vt)\,dt$$
$$=2x\left\{\frac{1}{\Gamma(\varsigma)E!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)tH_E(2ut,-vt)\,dt\right\}$$
$$-2y\varsigma\left\{\frac{1}{\Gamma(\varsigma)E!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)tH_{E-1}(2ut,-vt)\,dt\right\}.$$

Rewritten this equation, gives

$$\frac{(E+1)!}{E!}\left\{\frac{1}{\Gamma(\varsigma)(E+1)!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)H_{E+1}(2ut,-vt)\,dt\right\}$$
$$=2x\frac{\Gamma(\varsigma+1)}{\Gamma(\varsigma)}\left\{\frac{1}{\Gamma(\varsigma+1)E!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma+1-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)H_E(2ut,-vt)\,dt\right\}$$
$$-2y\varsigma\frac{(E-1)!}{E!}\left\{\frac{1}{\Gamma(\varsigma+1)(E-1)!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma+1-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)H_{E-1}(2ut,-vt)\,dt\right\}$$

which readily yields

$$(E+1)C^{\varsigma}_{E+1,\varrho}(u,v,o)=2\varsigma uC^{\varsigma+1}_{E+1,\varrho}(u,v,o)-2\varsigma vC^{\varsigma+1}_{E-1,\varrho}(u,v,o).$$ □

Putting $v=1$ in (5.1), the following corollary can be obtained:

Corollary 5.2. *The following recurrence relation also holds*

$$(E+1)C^{\varsigma}_{E+1,\varrho}(u,o)=2\varsigma uC^{\varsigma+1}_{E+1,\varrho}(u,o)-2\varsigma vC^{\varsigma+1}_{E-1,\varrho}(u,o).$$

Theorem 5.3. *The following summation formulas hold*

$$C^{\varsigma}_{2E,\varrho}(u,v,o)=\frac{2^E(E!)^2}{\Gamma(\varsigma)(2E)!}\sum_{\kappa=0}^{E}\sum_{r=0}^{\kappa}\frac{v^\kappa(-1)^r\Gamma(\varsigma+\kappa)(2r)!}{(E-\kappa)!(\kappa-r)!(r!)^2 2^r}C^{\varsigma+\kappa}_{2r,\varrho}(u,v,o). \tag{5.2}$$

Proof. Using the result in [9]

$$H_{2E}(u,v)=2^E(E!)^2\sum_{\kappa=0}^{E}\frac{\{H_\kappa(u,v)\}^2}{(E-\kappa)!(\kappa!)^2 2^\kappa},$$

and substituting $u\to 2ut$ and $v\to -vt$, gives

$$H_{2E}(2ut,-vt)=2^E(E!)^2\sum_{\kappa=0}^{E}\frac{\{H_E(2ut,-vt)\}^2}{(E-\kappa)!(\kappa!)^2 2^\kappa}.$$

Multiplying both sides by $\frac{1}{\Gamma(\varsigma)E!}\sqrt{\frac{2o}{\pi}}t^{\varsigma+E+\kappa-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)$ and integrate with respect to t taking limit from 0 to ∞, yields

$$\frac{1}{\Gamma(\varsigma)E!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)H_{2E}(2ut,-vt)\,dt$$
$$=\frac{2^E(E!)^2}{\Gamma(\varsigma)E!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)\sum_{\kappa=0}^{E}\frac{\{H_\kappa(2ut,-vt)\}^2}{(E-\kappa)!(\kappa!)^2 2^\kappa}dt.$$

Using the relation [9]

$$\{H_E(u,v)\}^2=(-2v)^\kappa(\kappa!)^2\sum_{r=0}^{\kappa}\frac{(-1)^r H_{2r}(u,v)}{(\kappa-r)!(r!)^2 2^r},$$

leads to

$$\frac{1}{\Gamma(\varsigma)E!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)H_{2E}(2ut,-vt)\,dt$$

$$= \frac{2^E(E!)^2}{\Gamma(\varsigma)E!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)\sum_{\kappa=0}^{E}\frac{(2vt)^\kappa(\kappa!)^2\sum_{r=0}^{\kappa}\frac{(-1)^r H_{2r}(2ut,-vt)}{(\kappa-r)!(r!)^2 2^r},}{(E-\kappa)!(\kappa!)^2 2^\kappa}dt.$$

Interchanging the order of summation, will readily lead to

$$\frac{1}{\Gamma(\varsigma)E!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)H_{2E}(2ut,-vt)\,dt$$
$$= 2^E(E!)^2\sum_{\kappa=0}^{E}\sum_{r=0}^{\kappa}\frac{v^\kappa(-1)^r H_{2r}}{(E-\kappa)!(\kappa-r)!(r!)^2 2^r}$$
$$\times\left\{\frac{1}{\Gamma(\varsigma)E!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma+\kappa-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)H_{2E}(2ut,-vt)\,dt\right\}.$$

Rewritten this equation one can obtain

$$\frac{(2E)!}{E!}\left\{\frac{1}{\Gamma(\varsigma)(2E)!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)H_{2E}(2ut,-vt)\,dt\right\}$$
$$= 2^E(E!)^2\sum_{\kappa=0}^{E}\sum_{r=0}^{\kappa}\frac{v^\kappa(-1)^r H_{2r}}{(E-\kappa)!(\kappa-r)!(r!)^2 2^r}\,\frac{\Gamma(\varsigma+\kappa)(2r)!}{\Gamma(\varsigma)E!}$$
$$\times\left\{\frac{1}{\Gamma(\varsigma+\kappa)(2r)!}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma+\kappa-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)H_{2r}(2ut,-vt)\,dt\right\}$$

which can be written in simplified form as

$$C^{\varsigma}_{2E,\varrho}(u,v,o) = \frac{2^E(E!)^2}{\Gamma(\varsigma)(2E)!}\sum_{\kappa=0}^{E}\sum_{r=0}^{\kappa}\frac{v^\kappa(-1)^r\Gamma(\varsigma+\kappa)(2r)!}{(E-\kappa)!(\kappa-r)!(r!)^2 2^r}C^{\varsigma+\kappa}_{2r,\varrho}(u,v,o).$$ □

On putting $v = 1$ in (5.2), the following corollary can be obtained

Corollary 5.4. *The following result hold true*

$$C^{\varsigma}_{2E,\varrho}(u,o) = \frac{2^E(E!)^2}{\Gamma(\varsigma)(2E)!}\sum_{\kappa=0}^{E}\sum_{r=0}^{\kappa}\frac{(-1)^r\Gamma(\varsigma+\kappa)(2r)!}{(E-\kappa)!(\kappa-r)!(r!)^2 2^r}C^{\varsigma+\kappa}_{2r,\varrho}(u,o).$$

6. Generating Functions

Theorem 6.1. *The following generating function is valid*

$$\sum_{E=0}^{\infty}\frac{((a))_E}{((b))_E}C^{\varsigma}_{E,\varrho}(u,v,o) = \sum_{E=0}^{\infty}\frac{((a))_E(\varsigma,o,\varrho)_E(2ut)^E}{((b))_E}A+1F_B\left[\begin{matrix}-E,\ (a+E);\ \frac{vt}{2u}\\ (b+E);\end{matrix}\right]. \tag{6.1}$$

Proof. By direct computation

$$\sum_{E=0}^{\infty}\frac{((a))_E}{((b))_E}C^{\varsigma}_{E,\varrho}(u,v,o) = \sum_{E=0}^{\infty}\frac{(a_1)_E\cdots(a_A)_E}{(b_1)_E\cdots(b_B)_E}\left\{\frac{1}{\Gamma(\varsigma)}\sum_{\kappa=0}^{\left[\frac{E}{2}\right]}\frac{(2u)^{E-2\kappa}(-1)^\kappa\Gamma_\varrho(\varsigma+E-\kappa,o)}{(E-2\kappa)!\kappa!}\right\}.$$

Rewritten this as

$$\sum_{E=0}^{\infty}\frac{((a))_E}{((b))_E}C^{\varsigma}_{E,\varrho}(u,v,o)t^E = \sum_{E=0}^{\infty}\sum_{\kappa=0}^{\left[\frac{E}{2}\right]}\frac{(a_1)_E\cdots(a_A)_E}{(b_1)_E\cdots(b_B)_E}\frac{(2u)^{E-2\kappa}(-v)^\kappa}{(E-2\kappa)!\kappa!}$$

$$\times\left\{\frac{1}{\Gamma(\varsigma)}\sqrt{\frac{2o}{\pi}}\int_0^\infty t^{\varsigma+E+\kappa-\frac{3}{2}}\exp(-t)\,K_{\varrho+\frac{1}{2}}\left(\frac{o}{t}\right)dt\right\}t^E.$$

which can be written as

$$\begin{aligned}
&\sum_{E=0}^{\infty}\frac{((a))_E}{((b))_E}C^{\varsigma}_{E,\varrho}(u,v,o)t^E\\
&=\sum_{E=0}^{\infty}\sum_{\kappa=0}^{\left[\frac{E}{2}\right]}\frac{(a_1)_E\cdots(a_A)_E}{(b_1)_E\cdots(b_B)_E}\frac{(2u)^{E-2\kappa}(-v)^{\kappa}(\varsigma,o;\varrho)_{E-\kappa}}{(E-2\kappa)!\kappa!}t^E\\
&=\sum_{E=0}^{\infty}\sum_{\kappa=0}^{E}\frac{(a_1)_{E+\kappa}\cdots(a_A)_{E+\kappa}}{(b_1)_{E+\kappa}\cdots(b_B)_{E+\kappa}}\frac{(2u)^{E-\kappa}(-v)^{\kappa}(\varsigma,o;\varrho)_E}{(E-2\kappa)!\kappa!}t^{E+\kappa}\\
&=\sum_{E=0}^{\infty}\sum_{\kappa=0}^{E}\frac{(a_1)_{E+\kappa}\cdots(a_A)_{E+\kappa}}{(b_1)_{E+\kappa}\cdots(b_B)_{E+\kappa}}\frac{(2ut)^{E-\kappa}(-vt)^{\kappa}(\varsigma,o;\varrho)_E}{(E-2\kappa)!\kappa!}\\
&=\sum_{E=0}^{\infty}\frac{(a_1)_E\cdots(a_A)_E(2ut)^E(\varsigma,o;\varrho)_E}{(b_1)_E\cdots(b_B)_E E!}\sum_{\kappa=0}^{E}\frac{(-E)_\kappa(a_1+E)_\kappa\cdots(a_A+E)_\kappa}{(b_1+E)_\kappa\cdots(b_B+E)_\kappa\kappa!}\left(\frac{vt}{2u}\right)^{\kappa}.
\end{aligned}$$

Applying the definition of hypergeometric function [16], yields

$$\sum_{E=0}^{\infty}\frac{((a))_E}{((b))_E}C^{\varsigma}_{E,\varrho}(u,v,o)t^E=\sum_{E=0}^{\infty}\frac{((a))_E(\varsigma,o,\varrho)_E(2ut)^E}{((b))_E}A+1F_B\left[\begin{matrix}-E,\ (a+E);\ \frac{vt}{2u}\\ (b+E);\end{matrix}\right].$$

Setting $v=1$ in (6.1), the following corollary can be obtained. □

Corollary 6.2. *The following generating function also holds*

$$\sum_{E=0}^{\infty}\frac{((a))_E}{((b))_E}C^{\varsigma}_{E,\varrho}(u,o)t^E=\sum_{E=0}^{\infty}\frac{((a))_E(\varsigma,o,\varrho)_E(2ut)^E}{((b))_E}A+1F_B\left[\begin{matrix}-E,\ (a+E);\ \frac{t}{2u}\\ (b+E);\end{matrix}\right].$$

On substituting $A=0$, $B=1$ and $b_1=\tau$ in (6.1)*, the following corollary is obtained.*

Corollary 6.3. *The following generating function also holds*

$$\sum_{E=0}^{\infty}\frac{C^{\varsigma}_{E,\varrho}(u,v,o)t^E}{(b_1)_E}=\sum_{E=0}^{\infty}\frac{(\varsigma,o,\varrho)_E(2ut)^E}{(b_1)_{2E}}R_E\left(\tau,\frac{vt}{2u}\right).$$

Where $R_E(d,u)$ is defined in [15] as

$$R_E(d,u)=\frac{(b_1)_{2E}}{(b_1)_E E!}1F_1\left[\begin{matrix}-E\ ;\\ (d+E);\end{matrix}\ u\right]$$

7. Conclusion

The new generalized Gegenbauer polynomials of one and two variables are obtained by using new extended gamma function defined by the MacDonald function (modified Bessel function). Certain properties of this new generalized Gegenbauer polynomials such as integral formulas, Mellin transform, recurrence relations and generating function are investigated. If $\varrho=0$ in the new generalized Gegenbauer polynomials $C^{\varsigma}_{E,\varrho}(u,v,o)$, the new extended Gegenbauer polynomial $C^{\varsigma}_{E}(u,v;o)$ is obtained; and if $\varrho=o=0$ the new generalized Gegenbauer polynomials $C^{\varsigma}_{E,\varrho}(u,v,o)$, reduces to classical Gegenbauer polynomials $C^{\varsigma}_{E}(u,v)$.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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