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Research Article

Stochastic Integrals and Random Sums of Power Contractions in Systemics

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Abstract. Stochastic integrals, random sums, random contractions, and selfdecomposable random variables constitute fundamental concepts of probability theory with significant applications in several areas of systemics. The main results of the paper are a characterization of a selfdecomposable distribution and a formulation of a Poisson random sum of power contractions. These results are established by incorporating a type of stochastic integral for a continuous in probability, homogeneous stochastic process with independent increments, and the same type of stochastic integral for a compound Poisson stochastic process with positive jumps. Interpretations of the results in treatment of risks threatening various systems are also provided.

Keywords. Stochastic integral; Random sum; Random contraction; Selfdecomposability; Risk management; Systemics

Mathematics Subject Classification (2020). 37A50; 60E05; 60E10; 60H05; 91B70; 97M30; 97M40

1. Introduction

Let $\{X(t), t \geq 0\}$ be a continuous in probability, homogeneous stochastic process with independent increments and denote by $\varphi(u,\tau)$ the characteristic function of the increment $X(t+\tau)-X(t)$. It is easily shown that $\varphi(u,\tau)$ is infinite divisible [5] and that

$$\varphi(u,\tau) = \varphi^{\tau}(u,1).$$

For the sake of brevity we write

$$\varphi_L(u) = \varphi(u, 1),$$

where

$$L = X(t+1) - X(t).$$

Moreover, it is known that the stochastic integral

$$C = \int_0^1 t^{\frac{1}{a}} dX(t), \quad a > 0 \tag{1.1}$$

exists in the sense of convergence in probability and the function

$$\varphi_C(u) = \exp\left\{a \int_0^1 \log \varphi_L(uw) w^{a-1} dw\right\} \tag{1.2}$$

is the characteristic function of C [14].

We consider the compound Poisson stochastic process $\{Y(t), t \geq 0\}$ with positive jumps. It is also known that $\{Y(t), t \geq 0\}$ is continuous in probability, homogeneous stochastic process with independent increments, and the characteristic function of the increment

$$V = Y(t+1) - Y(t)$$

has the form

$$\varphi_V(u) = \exp\{\lambda[\varphi_S(u) - 1]\},$$

where $\varphi_S(u)$ is the characteristic function of a positive random variable S and $\lambda > 0$ [16].

In consequence, the stochastic integral

$$R = \int_0^1 t^{\frac{1}{a}} dY(t) \tag{1.3}$$

exists in the sense of convergence in probability and the function

$$\varphi_R(u) = \exp\left\{\lambda\left[a \int_0^1 \varphi_S(uw) w^{a-1} dw - 1\right]\right\} \tag{1.4}$$

is readily recognized as the characteristic function of the stochastic integral R in (1.3).

The present paper is devoted to the implementation of three purposes. The first purpose is the establishment of a characterization of a selfdecomposable distribution by incorporating the increment L, the stochastic integral C in (1.1), and the stochastic integral R in (1.3). The second purpose is the representation of the characteristic function $\varphi_R(u)$ in (1.4) as the characteristic function of a Poisson random sum of power contractions. In addition, the third purpose is the interpretation of such a Poisson random sum in modeling important operations in the discipline of systemics.

2. Embedding Selfdecomposability

The present section mainly concentrates on interconnecting the very important property of selfdecomposability with the increment L and the stochastic integrals C, R. In classical probability theory selfdecomposability is usually defined via some decomposability property or as a certain class of limit distributions. However, Jurek and Vervaat (1983) [10] have established the method of stochastic integral representation that permits to describe selfdecomposable and some other distributions as distributions of some stochastic integrals with respect to Levy stochastic processes. One of the advantages of sucha stochastic integral representation is that it permits easily to incorporate space and time changes. It is readily recognized that such changes constitute structural concepts of systemic [25].

Theorem. *We suppose that the stochastic integrals C and R are independent, the characteristic function $\varphi_L(u)$ is differentiable, and $S$is a positive random variable with finite mean then $\varphi_L(u)$ has the form*

$$\varphi_L(u)=\exp\left\{a\lambda\int_0^u \frac{\varphi_S(w)-1}{w}dw\right\}$$

and L is a positive selfdecomposable random variable with finite mean if, and only if

$$L\overset{d}{=}C+R, \tag{2.1}$$

where $\overset{d}{=}$ denotes equality in distribution.

Proof. Only the sufficiency condition will be proved since the necessity condition can be proved by reversing the argument. If we write (2.1) in terms of the characteristic function $\varphi_L(u)$, the characteristic function $\varphi_C(u)$ in (1.2), and the characteristic function $\varphi_R(u)$ in (1.4) we get the integral equation

$$\varphi_L(u)=\exp\left\{a\int_0^1 \log\varphi_L(uw)w^{a-1}dw\right\}\exp\left\{\lambda\left[a\int_0^1 \varphi_S(uw)w^{a-1}dw-1\right]\right\}$$

which can be written in the form

$$\varphi_L(u)=\exp\left\{\frac{a}{u^a}\int_0^u \log\varphi_L(w)w^{a-1}dw\right\}\exp\left\{\lambda\left[\frac{a}{u^a}\int_0^u \varphi_S(w)w^{a-1}dw-1\right]\right\}. \tag{2.2}$$

From the integral equation (2.2), we get the integral equation

$$u^a\log\varphi_L(u)=a\int_0^u \log\varphi_L(w)w^{a-1}dw+\lambda a\int_0^u \varphi_S(w)w^{a-1}dw-\lambda u^a. \tag{2.3}$$

The infinite divisibility of $\varphi_L(u)$ implies that $\varphi_L(u)$ has no real roots. Moreover, $\varphi_L(u)$ is differentiable. In consequence, the differentiation of the integral equation (2.3) leads to the differential equation

$$au^{a-1}\log\varphi_L(u)+\frac{u^a}{\varphi_L(u)}\frac{d\varphi_L(u)}{du}=au^{a-1}\log\varphi_L(u)+\lambda a\varphi_S(u)u^{a-1}-\lambda au^{a-1} \tag{2.4}$$

satisfying the boundary conditions

$$\varphi_L(0)=1,$$

$$\varphi_S(0)=1.$$

For $u\neq 0$, the differential equation (2.4) can be written in the form

$$\frac{1}{\varphi_L(u)}\frac{d\varphi_L(u)}{du}=a\lambda\frac{\varphi_S(u)-1}{u}. \tag{2.5}$$

From (2.5) we get that

$$\int_0^u \frac{d\varphi_L(u)}{du}=a\lambda\int_0^u \frac{\varphi_S(w)-1}{w}dw. \tag{2.6}$$

Integrating (2.6) with due regard to the above boundary conditions we get that

$$\varphi_L(u) = \exp\left\{a\lambda \int_0^u \frac{\varphi_S(w)-1}{w} dw\right\}. \tag{2.7}$$

The assumptionthat S is a positive random variable with finite mean and (2.7) imply that L is a positive selfdecomposable random variable with finite mean [24].

The selfdecomposable characteristic function $\varphi_L(u)$ constitutes a strong analytical tool for stochastic modeling in practical disciplines such as physics [11, 13, 20, 23], nuclear technology, biology, sociology [21], queuing theory [4], finance [6, 17], economics [8], and systemics [25]. □

3. Poissonrandom Sum of Power Contractions

We suppose that D is a positive random variable and W is a random variable with values in the interval $(0,1)$. If the random variables D and W are independent then the random variable

$$Q = DW$$

is said a contraction of the random variable D via the random variable W [9]. Random contractions have useful applications in practical disciplines such as income distribution analysis [12], risk control [1], continuous discounting [7], reliability theory [3], inventory control [22], operations research [19], engineering [18], systemics [25], and informatics [2]. If the random variable W follows the power distribution then the random variable Q is said a power contraction [15]. The present section establishes the equality in distribution between the stochastic integral R and a Poisson random sum of power contractions. Moreover, the present section provides interpretation of such a Poisson random sum in systemics.

Let N be a discrete random variable following the Poisson distribution with parameter λ, $\{S_n, n = 1,2,\dots\}$ be a sequence of positive independent random variables distributed as the variable S with characteristic function $\varphi_S(u)$, $\{W_n, n = 1,2,\dots\}$ be a sequence of independent random variables with distribution function

$$F_W(w) = w^a, \quad 0 < w < 1,\ a > 0,$$

and we consider the sequence $\{H_n = S_n W_n, n = 1,2,\dots\}$ of positive random variables. If N $\{S_n, n = 1,2,\dots\}$ and $\{W_n, n = 1,2,\dots\}$ are independent then it is readily shown that $\{H_n = S_n W_n, n = 1,2,\dots\}$ is a sequence of independent power contractions distributed as the random variable

$$H = SW$$

and $\{H_n = S_n W_n, n = 1,2,\dots\}$ is independent of N. Hence

$$J = H_1 + H_2 + \dots + H_N$$

is a Poisson random sum of power contractions with characteristic function

$$\varphi_J(u) = \exp\left\{\lambda\left[a\int_0^1 \varphi_S(uw) w^{a-1} dw - 1\right]\right\}. \tag{3.1}$$

From (1.4) and (3.1) it follows that the stochastic integral R and the random sum J are equal in distribution. In the following, we provide a practical interpretation of the random sum J in the leading discipline of proactive risk management.

We suppose that the random variable N denotes the frequency of a risk threatening a system in a time interval of unit length, and the random variable S_n denotes the severity of the nth occurrence of the risk in this time interval. Hence the random variable $H_n = S_n W_n$ can be interpreted as the severity of the nth occurrence of the risk after the application of a severity reduction operation to that risk. Moreover, the random sum J denotes the severity of a risk in a time interval of unit length after the application of a severity reduction operation to that risk. In consequence, the characteristic function $\varphi_J(u)$ of a Poisson random sum of power contractions constitutes an analytical tool for implementing severity reduction operations of proactive risk treatment under the principles and purposes of systemics.

4. Concluding Remarks

It can be said that the presence in the characterization of a selfdecomposable distribution and the interpretation in practical operations of a Poisson random sum of power contractions substantially contribute to the selection and use of such a Poisson random sum in the formulation of stochastic models, suitable for the description and solution of problems arising in various areas of systemics. In addition, it seems to be of some particular importance the undertaking of further research activities in the area of formulating, investigating, and applying random contractions for operations of complex system.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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