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Research Article

Common Fixed Point Results for Contractive Mapping in Complex Valued A_b-metric Space

Surendra Kumar Tiwari* and Mridusmita Gauratra

Department of Mathematics, Dr. C. V. Raman University, Kota, Bilaspur, Chattishgarh, India
*Corresponding author: sk10tiwari.@gmail.com

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Abstract. In this article, we prove common fixed point results for two self mappings in complex valued A_b-metric space. Our results extend and generalize the common fixed point result of Singh and Singh [15].

Keywords. A_b-metric space; Complex valued metric space; Complex valued b-metric space; Complex valued A_b-metric space; Common fixed point

MSC. 47H10; 54H25; 37C25; 55M20

1. Introduction

The concept of complex valued metric space was introduced by Azam *et al.* [6], which is the generalization of the classical metric space and proved some fixed point results for a pair of mappings for contractive condition satisfying a rational expression.

Subsequently, many authors have obtained fixed point and common fixed points of set mappings in complex valued metric spaces (see for instance [1, 3, 4, 7–9, 14, 17–19, 21]).

In 2013, Rao *et al.* [13] introduced the concept of complex valued b-metric space, which was general than the well known complex valued metric space. After that, many authors have generalize and extend the results in complex valued b-metric spaces (see for instance [10–13]).

In 2016, Ughade *et al.* [20], introduce the notation of A_b-metric space and proved some fixed point theorems under contraction and expansion type condition.

Recently, in 2019, Singh and Singh [15] introduced the concept of complex valued A_b-metric space and proved fixed point theorem, and also in 2020, he is proved common fixed point for two self mappings in rational expression and complex valued A_b-metric spaces, which is generalization of the results giving by Mukheimer [11].

In this paper, we describe and extend common fixed point theorem in complex valued A_b-metric space. Our results generalized the results of Singh and Singh [15].

2. Basic Concept and Mathematical Preliminary

In this section, we recall some properties of A-metric space, A_b-metric space, complex valued metric space, complex valued b-metric space and complex valued A_b-metric space.

Definition 2.1 ([2]). Let X be a nonempty set. A function $A : X^n \to [0,\infty)$ is called an A-metric on X if for any x_i, $a \in X$, $i = 1,2,3,\cdots,n$, the following conditions hold:

(A1) $A(x_1,x_2,x_3,\cdots,x_{(n-1)},x_n) \ge 0$,

(A2) $A(x_1,x_2,x_3,\cdots,x_{(n-1)},x_n) = 0$ if and only if $x_1 = x_2 = x_3 = \cdots = x_{n-1} = x_n$,

(A3)
$$\begin{aligned} A(x_1,x_2,x_3,\cdots,x_{(n-1)},x_n) \le\ & A(x_1,x_1,x_1,\cdots,(x_1)_{(n-1)},a) \\ & + A(x_2,x_2,x_2,\cdots,(x_2)_{(n-1)},a) \\ & + A(x_3,x_3,x_3,\cdots,(x_3)_{(n-1)},a) \\ & \vdots \\ & + A(x_{(n-1)},x_{(n-1)},x_{(n-1)},\cdots,(x_{(n-1)})_{(n-1)},a) \\ & + A(x_n,x_n,x_n,\cdots,(x_n)_{(n-1)},a). \end{aligned}$$

The pair (X,A) is called an A-metric space.

Definition 2.2 ([20]). Let X be a nonempty set and $b \ge 1$ be a given number. A function $A : X^n \to [0,\infty)$ is called an A_b-metric on X if for any x_i, $a \in X$, $i = 1,2,3,\cdots,n$, the following conditions hold:

(A_b1) $A(x_1,x_2,x_3,\cdots,x_{n-1},x_n) \ge 0$,

(A_b2) $A(x_1,x_2,x_3,\cdots,x_{n-1},x_n) = 0$ if and only if $x_1 = x_2 = x_3 = \cdots = x_{n-1} = x_n$,

(A_b3)
$$\begin{aligned} A(x_1,x_2,x_3,\cdots,x_{n-1},x_n) \le b\big[& A(x_1,x_1,x_1,\cdots,(x_1)_{(n-1)},a) \\ & + A(x_2,x_2,x_2,\cdots,(x_2)_{(n-1)},a) \\ & + A(x_3,x_3,x_3,\cdots,(x_3)_{(n-1)},a) \\ & \vdots \\ & + A(x_{n-1},x_{n-1},x_{n-1},\cdots,(x_{(n-1)})_{(n-1)},a) \\ & + A(x_n,x_n,x_n,\cdots,(x_n)_{(n-1)},a)\big]. \end{aligned}$$

The pair (X,A) is called an A_b-metric space.

Remark 2.3. A_b-metric space is more general than A-metric space. Moreover, A-metric space is a special case of A_b-metric space with $b = 1$.

Example 2.4. Let $X = [1,+\infty)$. Define $A_b : X^n \to [0,\infty)$ by

$$A_b(x_1,x_2,x_3,\cdots,x_{n-1},x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|^2$$

for all $x_i \in X$, $i = 1,2,3,\cdots,n$.
Then (X,A_b) is an A_b-metric space with $b = 2 > 1$.

The concept of complex valued metric space was initiated by Azam *et al.* [6].

Let C be the set of complex numbers and $z_1,z_2 \in C$. Define a partial order $\precsim$ on C as follows:

$$z_1 \precsim z_2 \text{ if and only if } \operatorname{Re}(z_1) \le \operatorname{Re}(z_2) \text{ and } \operatorname{Im}(z_1) \le \operatorname{Im}(z_2).$$

It follows that $z_1 \precsim z_2$ if one of the following conditions are satisfied:

(C_1) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,

(C_2) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,

(C_3) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,

(C_4) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$.

Particularly, we write $z_1 \precsim z_2$ if $z_1 \neq z_2$ and one of (C_2), (C_3) and (C_4) is satisfied and we write $z_1 \precsim z_2$ if only (C_4) is satisfied. The following statements hold:

(1) If $a,b \in R$ with $a \le b$, then $az \precsim bz$ for all $0 \precsim z \in C$.

(2) If $z_1 \precsim z_2$, then $az_1 \precsim az_2$ for all $0 \le a \in R$.

(3) If $0 \precsim z_1 \precsim z_2$, then $|z_1| \le |z_2|$.

(4) If $0 \prec z_1 \prec z_2$, then $|z_1| < |z_2|$.

(5) If $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 2.5 ([6])**.** Let X be a nonempty set. A function $d : X \times X \to C$ is called a complex valued metric on X if for all $x,y,z \in X$, the following conditions are satisfied:

(i) $0 \precsim d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \precsim d(x,z) + d(z,y)$.

The pair (X,d) is called a complex valued metric space.

Definition 2.6 ([11])**.** Let X be a nonempty set and let $s \ge 1$. A function $d : X \times X \to C$ is called a complex valued b-metric on X if for all $x,y,z \in X$, the following conditions are satisfied:

(i) $0 \precsim d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \precsim s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a complex valued b-metric space.

Definition 2.7 ([15])**.** Let X be a nonempty set and $b \ge 1$ be a given real number. Suppose that a mapping $A : X^n \to C$ satisfies for all x_i, $a \in X$, $i = 1,2,3,\cdots,n$:

(CA_b1) $0 \precsim A(x_1,x_2,x_3,\cdots,x_n)$,

(CA_b2) $A(x_1,x_2,x_3,\cdots,x_n)=0 \Leftrightarrow x_1=x_2=x_3=\cdots=x_n$,

(CA_b3) $$A(x_1,x_2,x_3,\cdots,x_n)\precsim b\big[A(x_1,x_1,x_1,\cdots,(x_1)_{(n-1)},a) + A(x_2,x_2,x_2,\cdots,(x_2)_{(n-1)},a) + A(x_3,x_3,x_3,\cdots,(x_3)_{(n-1)},a)+\cdots + A(x_{n-1},x_{n-1},x_{n-1},\cdots,(x_{(n-1)})_{n-1},a) + A(x_n,x_n,x_n,\cdots,(x_n)_{(n-1)},a)\big].$$

Then A is called a complex valued A_b-metric on X and the pair (X,A) is called a complex valued A_b-metric space.

Example 2.8 ([15])**.** Let $X=R$ and $A:X^n\to C$ be such that

$$A(x_1,x_2,x_3,\cdots,x_n)=(\alpha+i\beta)A_*(x_1,x_2,x_3,\cdots,x_n),$$

where $\alpha,\beta\geq 0$ are constants and A_* is an A_b-metric on X. Then A is a complex valued A_b-metric on X. As a particular case, we have the following example of complex valued A_b-metric on X. The mapping $A:X^n\to C$ defined by $A(x_1,x_2,x_3,\cdots,x_n)=(1+i)\sum_{i=1}^{n}\sum_{i<j}|x_i-x_j|^2$ is a complex valued A_b-metric on $X=\mathbb{R}$ with $b=2$.

Definition 2.9 ([15])**.** A complex valued A_b-metric space (X,A) is said to be symmetric if

$$A(x_1,x_1,x_1,\cdots,(x_1)_{(n-1)},x_2)=A(x_2,x_2,x_2,\cdots,(x_2)_{(n-1)},x_1).$$

for all $x_1,x_2\in X$.

Definition 2.10 ([15])**.** Let (X,A) be a complex valued A_b-metric space.

(i) A sequence $\{x_p\}$ in X is said to be complex valued A_b-convergent to x if for every $a\in C$ with $0\prec a$, there exists $k\in\mathbb{N}$ such that $A(x_p,x_p,\cdots,x_p,x)\prec a$ or $A(x,x,\cdots,x,x_p)\prec a$ for all $p\geq k$ and is denoted by $\lim_{p\to\infty} x_p=x$ or $x_p\to x$ as $p\to\infty$.

(ii) A sequence $\{x_p\}$ in X is called complex valued A_b-Cauchy if for every $a\in C$ with $0\prec a$, there exists $k\in\mathbb{N}$ such that $A(x_p,x_p,\cdots,x_p,x_q)\prec a$ for each $p,q\geq k$.

(iii) If every complex valued A_b-Cauchy sequence is complex valued A_b-convergent in X, then (X,A) is said to be complex valued A_b-complete.

Lemma 2.11 ([15])**.** *Let (X,A) be a complex valued A_b-metric space and let $\{x_p\}$ be a sequence in X. Then $\{x_p\}$ is complex valued A_b-convergent to x if and only if $|A(x_p,x_p,\cdots,x_p,x)|\to 0$ as $p\to\infty$ or $|A(x,x,\cdots,x,x_p)|\to 0$ as $p\to\infty$.*

Lemma 2.12. *Let (X,A) be a complex valued A_b-metric space and let $\{x_p\}$ be a sequence in X. Then $\{x_p\}$ is complex valued A_b-Cauchy sequence if and only if $|A(x_p,x_p,\cdots,x_p,x_q)|\to 0$ as $p,q\to\infty$.*

Lemma 2.13. *Let (X,A) be a complex valued A_b-metric space. Then*

$$A(x,x,\cdots,x,y)\precsim bA(y,y,\cdots,y,x).$$

for all $x,y\in X$.

3. Main Results

Theorem 3.1. *Let (X,A) be a complete complex valued A_b-metric space and $f,g:X\to X$ be any two mapping satisfying*

$$A(fx,fx,\cdots,fx,gy)\precsim \alpha A(x,x,\cdots,y). \tag{3.1}$$

for all $x,y\in X$ where $\alpha\in[0,\frac{1}{b^2}]$. Then f and g have a unique common fixed point in X.

Proof. Let $x_0\in X$ be an arbitrary point and let $\{x_k\}$ in X be defined as

$$x_{2k+1}=fx_{2k}=f^{2k+1}x_0,$$
$$x_{2k+2}=gx_{2k+1}=g^{2k+2}x_0,$$

for $k=0,1,2,3,\cdots$.

Then, we show that the sequence $\{x_k\}$ is complex valued A_b-Cauchy.

From (3.1), we have

$$\begin{aligned}A(x_{2k+1},x_{2k+1},\cdots,x_{2k+1},x_{2k+2})&\precsim A(fx_{2k},fx_{2k},\cdots,fx_{2k},gy_{2k+1})\\&\precsim \alpha A(x_{2k},x_{2k},\cdots,x_{2k},y_{2k+1})\\&\ \vdots\\&\precsim \alpha^k A(x_0,x_0,\cdots,x_0,y_1).\end{aligned} \tag{3.2}$$

Using (CA_b3) and (3.2), for $k,l\in\mathbb{N}$ with $k<l$, we have

$$\begin{aligned}A(x_k,x_k,\cdots,x_k,x_l)&\precsim (n-1)bA(x_k,x_k,\cdots,x_k,x_{k+1})+b^2A(x_{k+1},x_{k+1},\cdots,x_{k+1},x_l)\\&\precsim (n-1)b[A(x_k,x_k,\cdots,x_k,x_{k+1})+b^2A(x_{k+1},x_{k+1},\cdots,x_{k+1},x_{k+2})+\cdots\\&\quad+b^{2(l-k-1)}A(x_{l-1},x_{l-1},\cdots,x_{l-1},x_l)]\\&\precsim (n-1)b(\alpha^k+b^2\alpha^{k+1}+b^4\alpha^{k+2}+\cdots+b^{2(l-k-1)}\alpha^{l-1})A(x_0,x_0,\cdots,x_0,x_1)\\&\precsim (n-1)b\alpha^k(1+b^2\alpha+b^4\alpha^2+\cdots+b^{2(l-k-1)}\alpha^{l-p-1})A(x_0,x_0,\cdots,x_0,x_1)\\&\precsim \frac{(n-1)b\alpha^k}{1-b^2\alpha}A(x_0,x_0,\cdots,x_0,x_1).\end{aligned} \tag{3.3}$$

Thus, we obtain

$$|A(x_k,x_k,\cdots,x_k,x_l)|\le \frac{(n-1)(b\alpha)^k}{1-b^2\alpha}|A(x_0,x_0,\cdots,x_0,x_1)|.$$

Since $\alpha\in[0,\frac{1}{b^2})$ where $b>1$, taking limit as $k,l\to\infty$, we have

$$|A(x_0,x_0,\cdots,x_0,x_1)|\le \frac{(n-1)(b\alpha)^k}{1-b^2\alpha}|A(x_0,x_0,\cdots,x_0,x_1)|\to 0.$$

Therefore, $|A(x_0,x_0,\cdots,x_0,x_1)|\to 0$ as $k,l\to\infty$.

So, by Lemma 2.11, $\{x_k\}$ is a complex valued A_b-Cauchy sequence. Since (X,A) is complete, there exist $u\in X$ such that the sequence $\{x_k\}$ is complex valued A_b-convergent to u.

Now, we show that u is fixed point of f. We have

$$\begin{aligned}A(fu,fu,\cdots,fu,u)&\precsim (n-1)bA(fu,fu,\cdots,fu,x_{2k+2})+bA(u,u,\cdots,u,x_{2k+2})\\&=(n-1)bA(fu,fu,\cdots,fu,gx_{2k+1})+bA(u,u,\cdots,u,x_{2k+2})\end{aligned}$$

$$\precsim (n-1)bA(u,u,\cdots,u,x_{2k+1}) + bA(u,u,\cdots,u,x_{2k+2}).$$

$\Rightarrow\quad |A(fu,fu,\cdots,fu,u)| \le (n-1)b|A(u,u,\cdots,u,x_{2k+1}) + b|A(u,u,\cdots,u,x_{2k+2})| \to 0$ as $k \to \infty$.

$\Rightarrow\quad |A(fu,fu,\cdots,fa,u)| = 0.$

$\Rightarrow\quad fu = u.$

$\Rightarrow\quad u$ is a fixed point of f.

Similarly, we can show that $gu = u$ is the fixed point g.
Therefore, u is common fixed point of f and g i.e. $fu = u = gu$.
Finally, to show that the uniqueness of the common fixed point of f and g. Now, let v is another common fixed point of f and g. Then, we have

$$\begin{aligned} A(u,u,\cdots,u,v) &= A(fu,fu,\cdots,fu,gv) \\ &\precsim \alpha A(u,u,\cdots,u,v). \end{aligned}$$

Hence

$$|A(u,u,\cdots,u,v)| \le \alpha|A(u,u,\cdots,u,v)|.$$

Since $\alpha \in \left(0,\frac{1}{b^2}\right)$ ans $b > 1$, we must have

$$|A(u,u,\cdots,u,v)| = 0$$

$\Rightarrow\quad u = v.$

So u is the unique common fixed point of f and g. □

Corollary 3.2. *Let (X,A) be a complete complex valued A_b-metric space and $f,g : X \to X$ be any two mapping for some positive constant k*

$$A(f^{2k+1}x, f^{2k+1}x,\cdots,f^{2k+1}x, g^{2k+2}y) \precsim \alpha A(x,x,\cdots,x,y),$$

for all $x,y \in X$ where $\alpha \in \left(0,\frac{1}{b^2}\right)$, then f and g have a unique common fixed point in X.

From Theorem 3.1 that $f^{2k+1}x$ has a unique fixed point u in X. But $f^{2k+1}(fu) = f(f^{2k+1}u) = fu$. So, fu is also fixed point of f^{2k+1}. Hence $fu = u$ is a fixed point of f. Since the fixed point of f is also fixed point of f^{2k+1}, the fixed point of f is unique. Similarly it can be established that $gu = u$. Then $fu = u = gu$. Thus u is common fixed point of f and g.

Theorem 3.3. *Let (X,A) be a complete complex valued A_b-metric space and let $f,g : X \to X$ be any two mapping satisfying the following condition*

$$A(fx,fx,\cdots,fx,gy) \precsim \alpha[A(x,x,\cdots,x,fx) + A(y,y,\cdots,y,gy)], \tag{3.4}$$

for all $x,y \in X$ and $\alpha \in \left[0,\frac{1}{2(n-1)b^2}\right)$. Then f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point and let us define a sequence $\{x_k\}$ in X as

$$x_{2k+1} = fx_{2k} = f^{2k+1}x_0,$$
$$x_{2k+2} = gx_{2k+1} = g^{2k+2}x_0,$$

for $k = 0,1,2,3,\cdots$.
Then, we show that the sequence $\{x_k\}$ is complex valued A_b-Cauchy sequence.

From (3.4), we have

$$
\begin{aligned}
&A(x_{2k+1},x_{2k+1},\cdots,x_{2k+1},x_{2k+2})\\
&\quad = A(fx_{2k},fx_{2k},\cdots,fx_{2k},gy_{2k+1})\\
&\quad \precsim \alpha[A(x_{2k},x_{2k},\cdots,x_{2k},fx_{2k})+A(x_{2k+1},x_{2k+1},\cdots,x_{2k+1},gx_{2k+1})]\\
&\quad = \alpha[A(x_{2k},x_{2k},\cdots,x_{2k},x_{2k+1})+A(x_{2k+1},x_{2k+1},\cdots,x_{2k+1},x_{2k+2})]\\
&\quad \precsim \frac{\alpha}{1-\alpha}|A(x_{2k},x_{2k},\cdots,x_{2k},x_{2k+1})|
\end{aligned}
$$

or

$$\Rightarrow \quad |A(x_{2k+1},x_{2k+1},\cdots,x_{2k+1},x_{2k+2})| \le \frac{\alpha}{1-\alpha}|A(x_{2k},x_{2k},\cdots,x_{2k},x_{2k+1})|. \tag{3.5}$$

Similarly, using the symmetry of X, we get

$$|A(x_{2k+2},x_{2k+2},\cdots,x_{2k+2},x_{2k+3})| \le \frac{\alpha}{1-\alpha}|A(x_{2k+1},x_{2k+1},\cdots,x_{2k+1},x_{2k+2})|. \tag{3.6}$$

From (3.5) and (3.6), we have

$$|A(x_{2k},x_{2k},\cdots,x_{2k},x_{2k+1})| \le h|A(x_{2k-1},x_{2k-1},\cdots,x_{2k-1},x_{2k})|, \tag{3.7}$$

for all $k\in\mathbb{N}$, where $h=\frac{\alpha}{1-\alpha}<1$.
By repeatedly applying (3.7), we get

$$|A(x_{2k},x_{2k},\cdots,x_{2k},x_{2k+1})| \le h^{2k}|A(x_0,x_0,\cdots,x_0,x_1)|. \tag{3.8}$$

Using (CA_b3) and (3.8), we have for $k,l\in\mathbb{N}$ with $k<l$ we get

$$
\begin{aligned}
&|A(x_{2k},x_{2k},\cdots,x_{2k},x_{2l})|\\
&\quad \le (n-1)b[|A(x_{2k},x_{2k},\cdots,x_{2k},x_{2k+1})|+b|A(x_{2k+1},x_{2k+1},\cdots,x_{2k+1},x_{2l})|]\\
&\quad \le (n-1)b|A(x_{2k},x_{2k},\cdots,x_{2k},x_{2k+1})|+(n-1)b^2|A(x_{2k+1},x_{2k+1},\cdots,x_{2k+1},x_{2k+2})|\\
&\qquad + b^3|A(x_{2k+2},x_{2k+2},\cdots,x_{2k+2},x_{2l})|\\
&\quad \le (n-1)b|A(x_{2k},x_{2k},\cdots,x_{2k},x_{2k+1})|+(n-1)b^2|A(x_{2k+1},x_{2k+1},\cdots,x_{2k+1},x_{2k+2})|\\
&\qquad +(n-1)b^3|A(x_{2k+2},x_{2k+2},\cdots,x_{2k+2},x_{2k+3})|+\cdots\\
&\qquad +(n-1)b^{2k-2l-1}|A(x_{2l-2},x_{2l-2},\cdots,x_{2l-2},x_{2l-1})|\\
&\qquad + b^{2l-2k-1}|A(x_{2l-1},x_{2l-1},\cdots,x_{2l-1},x_{2l})|\\
&\quad \le [(n-1)b\alpha^{2k}+(n-1)b^2\alpha^{2k+1}+\cdots+(n-1)b^{2l-2k-1}\alpha^{2l-2}\\
&\qquad +(n-1)b^{2l-2k}\alpha^{2l-1}]|A(x_0,x_0,\cdots,x_0,x_1)|\\
&\quad = (n-1)[(b\alpha)^{2k}+(b\alpha)^{2k+1}+\cdots+(b\alpha)^{2l-2}+(b\alpha)^{2l-1}]|A(x_0,x_0,\cdots,x_0,x_1)|\\
&\quad = (n-1)[(b\alpha)^{2k}+(b\alpha)^{2k+1}+\cdots]|A(x_0,x_0,\cdots,x_0,x_1)|\\
&\quad \le \frac{(n-1)(b\alpha)^{2k}}{1-b\alpha}|A(x_0,x_0,\cdots,x_0,x_1)| \to 0 \ \text{ as } \ k,l\to\infty \ \text{(by Lemma 2.11)}.
\end{aligned} \tag{3.9}
$$

Hence the sequence x_{2k} is complex valued A_b-Cauchy in X. Since (X,A) is a complete, there exists $x^*\in X$ such that $\lim_{k\to\infty} x_{2k}=x^*$.
We show that x^* is a fixed point of f.

$$A(fx^*,fx^*,\cdots,fx^*,x^*) \precsim (n-1)bA(fx^*,fx^*,\cdots,fx^*,fx_{2k+1})+b^2A(fx^{2k},fx^{2k},\cdots,fx^{2k},x^*).$$

$$\begin{aligned}
&A(fx^*,fx^*,\cdots,fx^*,x^*)\\
&\quad\precsim (n-1)bA(fx^*,fx^*,\cdots,fx^*,fx_{2k+1})+b^2A(fx_{2k+1},fx_{2k+1},\cdots,fx_{2k+1},x^*)\\
&\quad\precsim (n-1)b\alpha[A(x^*,x^*,\cdots,x^*,fx^*)+A(fx_{2k},fx_{2k},\cdots,fx_{2k},fx_{2k+1})]\\
&\qquad +b^2A(fx_{2k+1},fx_{2k+1},\cdots,fx_{2k+1},x^*)\\
&\quad\precsim (n-1)b\alpha A(x^*,x^*,\cdots,x^*,fx^*)+(n-1)b\alpha A(fx_{2k},fx_{2k},\cdots,fx_{2k},fx_{2k+1})\\
&\qquad +b^2A(fx_{2k+1},fx_{2k+1},\cdots,fx_{2k+1},x^*)\\
&\quad\precsim (n-1)b^2\alpha A(fx^*,fx^*,\cdots,fx^*,x)+(n-1)b\alpha A(fx_{2k},fx_{2k},\cdots,fx_{2k},fx_{2k+1})\\
&\qquad +b^2A(fx_{2k+1},fx_{2k+1},\cdots,fx_{2k+1},x^*).
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad |A(fx^*,fx^*,\cdots,fx^*,x^*)| &\le (n-1)b^2\alpha|A(fx^*,fx^*,\cdots,fx^*,x^*)|\\
&\quad +(n-1)b\alpha|A(fx_{2k},fx_{2k},\cdots,fx_{2k},fx_{2k+1})|\\
&\quad +b^2|A(fx_{2k+1},fx_{2k+1},\cdots,fx_{2k+1},x^*)|\\
&\le \frac{1}{1-(n-1)b^2\alpha}[(n-1)b\alpha|A(fx_{2k},fx_{2k},\cdots,fx_{2k},fx_{2k+1})|\\
&\quad +b^2|A(fx_{2k+1},fx_{2k+1},\cdots,fx_{2k+1},x^*)|]\to 0,\ \text{as } k\to\infty.
\end{aligned}$$

$\Rightarrow\quad |A(fx^*,fx^*,\cdots,fx^*,x^*)|=0.$

$\Rightarrow\quad fx^*=x^*.$

Therefore, x^* is a fixed point of f.

Similarly, we can show that x^* is also fixed point of g i.e. $gx^*=x^*$.

Thus $fx^*=x^*=gx^*$.

Hence x^* is common fixed point of f and g.

Now, we show that the uniqueness of the common fixed point of f and g.
Let us assume that $y^*\in X$ is another common fixed point of f and g. Then we have

$$\begin{aligned}
A(x^*,x^*,\cdots,x^*,y^*) &\precsim A(fx^*,fx^*,\cdots,fx^*,gy^*)\\
&\precsim \alpha[A(x^*,x^*,\cdots,x^*,fx^*)+A(y^*,y^*,\cdots,y^*,gy^*)]\\
&\precsim \alpha[A(x^*,x^*,\cdots,x^*,x^*)+A(y^*,y^*,\cdots,y^*,y^*)]\\
&\precsim 0.
\end{aligned}$$

Hence

$$|A(x^*,x^*,\cdots,x^*,y^*)|\le 0.$$

$\Rightarrow\qquad x^*=y^*.$

Thus x^* is the unique common fixed point of f and g. This completes the proof of the theorem. □

Theorem 3.4. *Let (X,d) be a complete complex valued A_b-metric space and let $f,g:X\to X$ be any two mappings satisfying the following condition*

$$A(fx,fx,\cdots,fx,gy)\precsim \alpha[A(x,x,\cdots,x,gy)+A(y,y,\cdots,y,fx)], \tag{3.10}$$

for all $x,y\in X$ and $\alpha\in\left[0,\frac{1}{b^2\{(n-1)b+1\}}\right)$, then f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point and let us define a sequence $\{x_{2n}\}$ in X as

$$x_{2n+1} = fx_{2n} = f^{2n+1}x_0,$$
$$x_{2n+2} = gx_{2n+1} = g^{2n+2}x_0,$$

for $n = 0,1,2,3,\cdots$.

Put $x = x_{2n-1}$, $y = x_{2n}$ in (3.10) we have

$$\begin{aligned}
&A(x_{2n},x_{2n},\cdots,x_{2n},x_{2n+1})\\
&\quad = A(fx_{2n-1},fx_{2n-1},\cdots,fx_{2n-1},gx_{2n})\\
&\quad \precsim \alpha[A(x_{2n-1},x_{2n-1},\cdots,x_{2n-1},gx_{2n}) + A(fx_{2n},fx_{2n},\cdots,fx_{2n},fx_{2n-1})]\\
&\quad = \alpha[A(x_{2n-1},x_{2n-1},\cdots,x_{2n-1},x_{2n+1}) + A(x_{2n},x_{2n},\cdots,x_{2n},x_{2n})]\\
&\quad = \alpha A(x_{2n-1},x_{2n-1},\cdots,x_{2n-1},x_{2n+1})\\
&\quad \precsim (n-1)\alpha b A(x_{2n-1},x_{2n-1},\cdots,x_{2n-1},x_{2n}) + \alpha b^2 A(x_{2n+1},x_{2n+1},\cdots,x_{2n+1},x_{2n}).
\end{aligned}$$

Therefore

$$\begin{aligned}
|A(x_{2n},x_{2n},\cdots,x_{2n},x_{2n+1})| &\le (n-1)\alpha b|A(x_{2n-1},x_{2n-1},\cdots,x_{2n-1},x_{2n})|\\
&\quad + \alpha b^2|A(x_{2n+1},x_{2n+1},\cdots,x_{2n+1},x_{2n})|\\
&\le \frac{(2n-1)\alpha b}{1-\alpha b^2}|A(x_{2n-1},x_{2n-1},\cdots,x_{2n-1},x_{2n})|. \qquad (3.11)
\end{aligned}$$

If we put $x_{2n},x_{2n},\cdots,x_{2n+1} = A_{2n}$ and $x_{2n-1},x_{2n-1},\cdots,x_{2n} = A_{2n-1}$.

Then, from (3.11), we have

$$\begin{aligned}
&|A_{2n}| \le \frac{(2n-1)\alpha b}{1-\alpha b^2}|A_{2n-1}|\\
\Rightarrow\quad &|A_{2n}| \le k|A_{2n-1}|, \qquad (3.12)
\end{aligned}$$

where $\frac{(2n-1)\alpha b}{1-\alpha b^2} < 1$.

Repeating this process, we get

$$\begin{aligned}
|A(x_{2n},x_{2n},\cdots,x_{2n+1})| &\le k|A(x_{2n-1},x_{2n-1},\cdots,x_{2n})|\\
&\le k^2|A(x_{2n-2},x_{2n-2},\cdots,x_{2n-1})|\\
&\ \ \vdots\\
&\le k^{2n}|A(x_0,x_0,\cdots,x_1)|, \qquad (3.13)
\end{aligned}$$

for all $n \ge 1$.

Now

$$\begin{aligned}
\alpha < \frac{1}{b^2\{(2n-1)b+1\}} &\Rightarrow \alpha b^2 < \frac{1}{(2n-1)b+1}\\
&\Rightarrow 1-\alpha b^2 > 1 - \frac{1}{(2n-1)b+1}\\
&\Rightarrow \frac{(2n-1)b}{(2n-1)b+1} > 0.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\alpha < \frac{1}{b^3\{(2n-1)+b^2\}} &\Rightarrow \alpha b^3(2n-1)+\alpha b^2<1\\
&\Rightarrow \alpha b^3(2n-1)<1-\alpha b^2\\
&\Rightarrow \frac{\alpha b^3(2n-1)}{1-\alpha b^2}<1\\
&\Rightarrow \frac{\alpha(2n-1)b}{1-\alpha b^2}<\frac{1}{b^2}<1\\
&\Rightarrow k<1.
\end{aligned}$$

Using (CA_b3) and (3.13)f, we have for all $n,m\in N$, with $n<m$

$$\begin{aligned}
&A(f^{2n}x_0,f^{2n}x_0,\cdots,f^{2n}x_0)\\
&\quad\le b[(n-1)|A(f^{2n}x_0,\cdots,f^{2n}x_0,f^{2n+1}x_0)|+|A(f^{2n}x_0,\cdots,f^{2n}x_0,f^{2n+1}x_0)|]\\
&\quad\le b(n-1)|A(f^{2n}x_0,\cdots,f^{2n}x_0,f^{2n+1}x_0)|+b^2|A(f^{2n+1}x_0,\cdots,f^{2n+1}x_0,f^{2m}x_0)|\\
&\quad\le b(n-1)|A(f^{2n}x_0,\cdots,f^{2n}x_0,f^{2n+1}x_0)|+b^3(n-1)|A(f^{2n+1}x_0,\cdots,f^{2n+1}x_0,f^{2n+2}x_0)|\\
&\qquad+b^4|A(f^{2n+2}x_0,\cdots,f^{2n+2}x_0,f^{2m}x_0)|\\
&\quad\le b(n-1)|A(f^{2n}x_0,\cdots,f^{2n}x_0,f^{2n+1}x_0)|+b^2|A(f^{2n+1}x_0,\cdots,f^{2n+1}x_0,f^{2n+2}x_0)|+\cdots\\
&\qquad+b^{2n-2m-1}|A(f^{2n-1}x_0,\cdots,f^{2m-1}x_0,f^{2m}x_0)|\\
&\quad\le (n-1)b[k^{2n}+b^2k^{2n+1}+\cdots+b^{2(2m-2n-1}k^{2n-1}]|A(x_0,x_0,\cdots,x_0,x_1)|\\
&\quad=(n-1)bk^{2n}[1+b^2k+(b^2k)^2+\cdots+(b^2k)^{2m-2n-1}]|A(x_0,x_0,\cdots,x_0,x_1)|\\
&\quad\le\frac{(n-1)bk^{2n}}{1-b^2k}|A(x_0,x_0,\cdots,x_0,x_1)|\to0,\ \text{ as }\ n,m\to\infty.
\end{aligned}$$

Hence $\{x_{2n}\}$ is complex valued A_b-Cauchy sequence in X. Since X is complex, there exists $v\in X$ such that $\lim_{n\to\infty}x_{2n}=v$. We show that v is fixed point of f.

We have

$$\begin{aligned}
&A(fv,fv,\ldots,fv,v)\\
&\quad\precsim(n-1)bA(fv,fv,\cdots,fv,f^{2n+1}x_0)+bA(v,v,\cdots,v,f^{2n+1}x_0)\\
&\quad\precsim(n-1)b[kA(v,v,\cdots,v,f^{2n+1}x_0)+A(f^{2n}x_0,f^{2n}x_0,\cdots,f^{2n}x_0,fv)]+bA(v,v,\cdots,v,f^{2n+1}x_0)\\
&\quad=[(n-1)b\alpha+b]A(v,v,\cdots,v,f^{2n+1}x_0)+(n-1)b\alpha A(f^{2n}x_0,f^{2n}x_0,\cdots,f^{2n}x_0,fv)\\
&\quad\precsim[(n-1)b\alpha+b]A(v,v,\cdots,v,f^{2n+1}x_0)+(n-1)b\alpha A(f^{2n}x_0,f^{2n}x_0,\cdots,f^{2n}x_0,fv)\\
&\qquad+bA(fv,\cdots,fv,v)\\
&\quad\precsim[(n-1)b\alpha+b]A(v,v,\cdots,v,f^{2n+1}x_0)+(n-1)^2b^2\alpha A(f^{2n}x_0,f^{2n}x_0,\cdots,f^{2n}x_0,fv)\\
&\qquad+(n-1)b^2\alpha A(fv,\cdots,fv,v).
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad |A(fv,\cdots,fv,v)|&\le\frac{1}{1-(n-1)b^2\alpha}[(n-1)b\alpha+b]|A(v,\cdots,v,f^{2n+1}x_0)|\\
&\quad+(n-1)^2b^2\alpha|A(f^{2n}x_0,f^{2n}x_0,\cdots,f^{2n}x_0,fv)|]\to0\ \text{as}\ n\to\infty.
\end{aligned}$$

$$\Rightarrow\quad |A(fv,\cdots,fv,v)|=0.$$

$$\Rightarrow\quad fv=v.$$

Therefore, v is a fixed point of f. Similarly, we can show that, v is a fixed point of g i.e. $gv = v$. Thus $fv = v = gv$.
Hence $v \in X$ is common fixed point of f and g.
Now, we show that the common fixed point of f and g are unique.
Let $w \in X$ be another common fixed point of f and g. Then we have

$$\begin{aligned} A(v,v,\cdots,v,w) &= A(fv,fv,\cdots,fv,w) \\ &\precsim \alpha[A(v,v,\cdots,v,fw) + A(w,w,\cdots,w,fv)] \\ &= \alpha[A(v,v,\cdots,v,w) + A(w,w,\cdots,w,v)] \\ &\precsim \alpha[A(v,v,\cdots,v,w) + bA(v,v,\cdots,v,w)] \\ &\precsim \alpha(1+b)A(v,v,\cdots,v,w) \end{aligned}$$

$$\Rightarrow \qquad |A(v,v,\cdots,v,w)| \le \alpha(1+b)|A(v,v,\cdots,v,w)|.$$

But

$$\begin{aligned} \alpha &< \frac{1}{b^2\{(2n-1)b+1\}} \\ &< \frac{1}{b^2(b+1)} \end{aligned}$$

$$\Rightarrow \qquad \alpha(b+1) < \frac{1}{b^2} < 1.$$

Therefore, we must have

$$|A(v,v,\cdots,v,w)| = 0 \Rightarrow v = w.$$

Hence v is the unique common fixed point of f and g. This completes the proof of the theorem. □

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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