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Research Article

Solution of Nonlinear Random Partial Differential Equations by Using Finite Element Method

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Abstract. In this paper, a new technique is proposed to solve some classes of nonlinear random partial differential equations using finite element method. Through this technique we were able to deal with the random variable in the presence of a nonlinear function. The idea of this technique is based on assuming that the nodal coefficients are functions of the random variable. Then by discretization of the random variable and using fitting over the discretized values of the random variable, and utilizing the shape functions of the finite element method, we get the approximate solution as a function in both space and random variable. Some numerical examples, in different domains, are presented to show the effectiveness of this technique.

Keywords. Random differential equations; Random finite element method; Nonlinear partial differential equations

MSC. 35R60; 65N30; 60H15; 34F15

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1. Introduction

Physical phenomena of interest in science and technology are often simulated by means of models which correspond to *partial differential equations* (PDEs). These equations are in general nonlinear and, as such, their solution is usually a difficult task.

The coefficients of a PDE can be deterministic values or *random variables* (RVs). Randomness in the coefficients of PDEs describes the real behavior of quantities of interest than their counterpart with deterministic coefficients. It may arise because of errors in the observed or measured data, variability conditions of the experiment or uncertainties. The model in that case is a *random partial differential equation* (RPDE) [4,10,15,18,30].

There are several techniques which have been considered to obtain approximate solutions of random nonlinear differential equations. These techniques include Adomian decomposition method [12, 20], homotopy perturbation method [3, 11], variational iteration method [13], differential transformation method [14], finite difference method [5], Euler Maruyama method [32], Milstein method [7] and stochastic finite element method [6,21,25–27].

In this paper, a new technique is proposed to solve nonlinear RPDEs using Galerkin finite element method FEM [1,2,8,9,16,17,22,23,29,31,33]. This technique is based on discretization of the RV and solving the nonlinear RPDE at discretized values of the RV. Then by using the process of curve fitting over the values of the RV, we obtain the nodal coefficients. Then by utilization the shape functions of FEM, we obtain the approximate solution. The remainder of this paper is structured as follows. Section 2 presents the technique of solution. Section 3 presents some numerical examples. Finally, Section 4 presents the general conclusion of this work.

2. The Proposed Finite Element Technique

Consider the following nonlinear RPDE

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+g\left(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=f(\beta,x,y), \quad \beta\in[\beta_i,\beta_f], \tag{2.1}$$

and its boundary conditions are prescribed as functions of β, x and y where β is a second order RV and g is a nonlinear function.

The weighted residual statement of equation (2.1) is

$$\int_\Omega w(x,y)\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+g\left(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)-f(\beta,x,y)\right]d\Omega=0, \tag{2.2}$$

where Ω is the problem domain and w is the weight function. By applying the divergence theorem to the terms which have the second derivative in equation (2.2), we obtain the following weak form

$$\begin{aligned}-\int_\Omega \frac{\partial w}{\partial x}\frac{\partial u}{\partial x}d\Omega-\int_\Omega \frac{\partial w}{\partial y}\frac{\partial u}{\partial y}d\Omega+\int_\Omega w(x,y)g\left(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)d\Omega\\ =\int_\Omega w(x,y)f(\beta,x,y)d\Omega-\int_\Gamma w(x,y)\frac{\partial u}{\partial x}\eta_x d\Gamma-\int_\Gamma w(x,y)\frac{\partial u}{\partial y}\eta_y d\Gamma,\end{aligned} \tag{2.3}$$

where Γ is the domain boundary and $\eta_{x,y}$ are the cartesian components of the unit outward normal to the boundary. By dividing Ω into N^e elements, equation (2.3) can be written as

$$\begin{aligned}-\sum_{e=1}^{N^e}\int_{\Omega^e} \frac{\partial w}{\partial x}\frac{\partial u}{\partial x}d\Omega-\sum_{e=1}^{N^e}\int_{\Omega^e} \frac{\partial w}{\partial y}\frac{\partial u}{\partial y}d\Omega+\sum_{e=1}^{N^e}\int_{\Omega^e} w(x,y)g\left(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)d\Omega\\ =\sum_{e=1}^{N^e}\int_{\Omega^e} w(x,y)f(\beta,x,y)d\Omega-\sum_{e=1}^{N^e}\int_{\Gamma^e} w(x,y)\frac{\partial u}{\partial x}\eta_x d\Gamma-\sum_{e=1}^{N^e}\int_{\Gamma^e} w(x,y)\frac{\partial u}{\partial y}\eta_y d\Gamma.\end{aligned} \tag{2.4}$$

In our technique, we propose the approximate solution over an element in the form

$$u(x,y,\beta)=\sum_{j=1}^{N_h} u_j(\beta)s_j(x,y), \tag{2.5}$$

where $u_j(\beta)$ are the nodal unknown values formulated as functions of β, N_h is the number of nodes in the finite element mesh and $s_j(x,y)$ are prescribed functions of position called shape functions. Clearly, in general

$$s_j(x_i,y_i)=\begin{cases}1 & i=j,\\ 0 & i\neq j.\end{cases} \tag{2.6}$$

According to Galerkin method, we take $w(x)=s_j$. Equation (2.4) can be written in the form

$$\begin{aligned}
&-\sum_{e=1}^{N^e}\int_{\Omega^e}\frac{\partial s_i}{\partial x}\frac{\partial}{\partial x}\left(\sum_{j=1}^{N_h}u_j(\beta)s_j(x,y)\right)d\Omega-\sum_{e=1}^{N^e}\int_{\Omega^e}\frac{\partial s_i}{\partial y}\frac{\partial}{\partial y}\left(\sum_{j=1}^{N_h}u_j(\beta)s_j(x,y)\right)d\Omega\\
&+\sum_{e=1}^{N^e}\int_{\Omega^e}s_i g\left(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)d\Omega\\
&\qquad=\sum_{e=1}^{N^e}\int_{\Omega^e}s_i f(\beta,x,y)d\Omega-\sum_{e=1}^{N^e}\int_{\Gamma^e}s_i\frac{\partial u}{\partial x}\eta_x d\Gamma-\sum_{e=1}^{N^e}\int_{\Gamma^e}s_i\frac{\partial u}{\partial y}\eta_y d\Gamma.
\end{aligned} \tag{2.7}$$

The nonlinear term in Equation (2.7) is approximated by

$$g\left(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=g\left(\overline{u},\frac{\partial \overline{u}}{\partial x},\frac{\partial \overline{u}}{\partial y}\right), \tag{2.8}$$

where $\overline{u}$ is the initial guess of the approximate solution which is approximated by

$$\overline{u}=\sum_{j=1}^{N_h}\overline{u_j}(\beta)s_j(x,y), \tag{2.9}$$

in which $\overline{u_j}(\beta)$ is the initial guess of the nodal unknown values. Equation (2.7) becomes

$$\begin{aligned}
&-\sum_{e=1}^{N^e}\int_{\Omega^e}\frac{\partial s_i}{\partial x}\frac{\partial}{\partial x}\left(\sum_{j=1}^{N_h}u_j(\beta)s_j(x,y)\right)d\Omega-\sum_{e=1}^{N^e}\int_{\Omega^e}\frac{\partial s_i}{\partial y}\frac{\partial}{\partial y}\left(\sum_{j=1}^{N_h}u_j(\beta)s_j(x,y)\right)d\Omega\\
&\qquad=-\sum_{e=1}^{N^e}\int_{\Omega^e}s_i g\left(\overline{u},\frac{\partial \overline{u}}{\partial x},\frac{\partial \overline{u}}{\partial y}\right)d\Omega+\sum_{e=1}^{N^e}\int_{\Omega^e}s_i f(\beta,x,y)d\Omega-\sum_{e=1}^{N^e}\int_{\Gamma^e}s_i\frac{\partial u}{\partial x}\eta_x d\Gamma\\
&\qquad\quad-\sum_{e=1}^{N^e}\int_{\Gamma^e}s_i\frac{\partial u}{\partial y}\eta_y d\Gamma.
\end{aligned} \tag{2.10}$$

By discretization of the RV, we solve the nonlinear algebraic system resulting from equation (2.10) at each value of the specified values of the random variable. This system was solved using iterative method [19, 23] such as fixed point iteration. Then by applying a process of constructing a curve such as curve fitting over the specified values of RV, we get $u_j(\beta)$. Finally, by utilizing the shape functions of FEM, we obtain the approximate solution at every element taking the form of equation (2.5). In the next section, some numerical examples are presented. In each example, the error over any element (e) between the approximate solution (u) and the exact solution (U) is calculated as a function of random variable using L^2 error norm with the form

$$e = \sqrt{\int_{\Omega^e} (u-U)^2 d\Omega} \tag{2.11}$$

in which the integration is numerically calculated using Gauss quadrature points. Then by summation of element's errors, we obtain the error over the whole domain (e_w) as a function of random variable.

The steps is summarized as follows:

Step 1. Choose a tolerance of error over the whole domain (e_t).

Step 2. Construct suitable mesh for spatial variables x and y.

Step 3. Discretize the random variable β

$$\beta \in [\beta_i, \beta_f], \quad M \in Z^+, \quad h_\beta = (\beta_f - \beta_i)/M, \quad \beta_i = i h_\beta \quad i = 0, 1, \dots, (n-1).$$

Step 4. Solve the discretized problems by FEM.

Step 5. utilize the curve fitting to construct the nodal values as functions of β.

Step 6. utilize the FEM shape functions to construct the approximate solution.

Step 7. Compute e_w.

Step 8. Define the maximum error of e_w by $e_{\max} = \max[e_w]$.

Step 9. If $e_{\max}$ less or equal e_t then stop.

Step 10. Increasing the discretized values of β.

Step 11. Go to 4.

Finally, expectation (E) of the approximate solution over any element and variance (V) of the approximate solution over any element are computed by [24, 28]

$$E[u(x,y,\beta)] = \sum_{j=1}^{N_h} s_j(x,y) E[u_j(\beta)], \tag{2.12}$$

$$V[u(x,y,\beta)] = E[(u(x,y,\beta))]^2 - [E(u(x,y,\beta))]^2, \tag{2.13}$$

and compared with expectation and variance of the exact solution to illustrate the efficiency and accuracy of the proposed technique.

3. Illustrative Examples

In this section, we solve some numerical examples to illustrate the efficiency of the previous presented technique for solving nonlinear RPDEs in different domains with different types of nonlinearities using FEM.

Example 1. Consider the following nonlinear 2-D problem which on a domain shown in Figure 1

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = f(x,y,\beta), \qquad \beta \in [0.1, 0.5], \tag{3.1}$$

where the source term and boundary conditions are prescribed so that the exact solution is given by $u = \beta x y$.

By discretization the two dimensional domain as shown in Figure 1 and using Matlab code, we obtain the results listed in the following tables. Table 1 illustrates FEM solution for the internal nodes at discretized values of RV. Table 2 illustrates the internal nodal values as functions of the RV. Table 3 represents the approximate solution for every element. The error over whole domain e_w is illustrated in Figure 2 as a function of RV. For this example we chose a tolerance of error over the whole domain $e_t = 0.004$. Figure 3 illustrates expectations of the approximate and exact solutions at $y = 0.2$. Figure 4 illustrates variances of the approximate and exact solutions at $y = 0.2$.

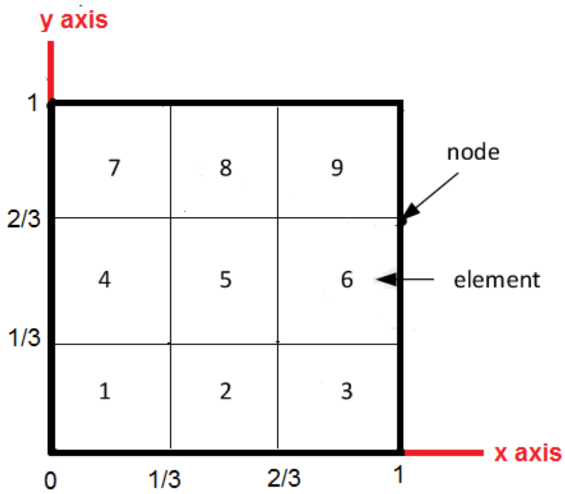

Figure 1. Domain discretization for Example 1

Table 1. FEM solution for the internal nodes at selected values of the random variable β for Example 1

	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.3$	$\beta = 0.4$	$\beta = 0.5$
$u(1/3, 1/3)$	0.0111	0.0221	0.0331	0.044	0.0549
$u(2/3, 1/3)$	0.0222	0.0443	0.0663	0.0882	0.11
$u(1/3, 2/3)$	0.0222	0.0443	0.0663	0.0882	0.11
$u(2/3, 2/3)$	0.0444	0.0887	0.1329	0.177	0.2219

Table 2. Internal nodal values as functions of random variable β for Example 1

$u(1/3, 1/3)$	$0.083333\beta^4 - 0.1\beta^3 + 0.039167\beta^2 + 0.104\beta + 0.0004$
$u(2/3, 1/3)$	$-0.005\beta^2 + 0.2225\beta$
$u(1/3, 2/3)$	$-0.005\beta^2 + 0.2225\beta$
$u(2/3, 2/3)$	$0.375\beta^4 - 0.375\beta^3 + 0.12625\beta^2 + 0.42575\beta + 0.0009$

Table 3. Approximate random solution over every element for Example 1

element(1)
$0.7497\beta^4xy-0.9\beta^3xy+0.3528\beta^2xy+0.936\beta xy+0.0036xy$
element(2)
$-0.7497\beta^4xy+0.9\beta^3xy-0.3978\beta^2xy+1.0665\beta xy-0.0036xy$ $+0.4998\beta^4y-0.6\beta^3y+0.2502\beta^2y-0.0435\beta y+0.0024y$
element(3)
$0.045\beta^2xy+0.9975\beta xy-0.045\beta^2xy+0.0025\beta y$
element(4)
$-0.7497\beta^4xy+0.9\beta^3xy-0.3978\beta^2xy+1.0665\beta xy-0.0036xy$ $+0.4998\beta^4x-0.6\beta^3x+0.2502\beta^2x-0.0435\beta x+0.0024x$
element(5)
$4.1247\beta^4xy-4.275\beta^3xy+1.5795\beta^2xy+0.7632\beta xy+0.0117xy$ $-1.6248\beta^4y+1.725\beta^3y-0.6591\beta^2y+0.1011\beta y-0.0051y$ $-1.6248\beta^4x+1.725\beta^3x-0.6591\beta^2x+0.1011\beta x-0.0051x$ $+0.7082\beta^4-0.775\beta^3+0.3031\beta^2-0.0482\beta+0.0025$
element(6)
$-3.375\beta^4xy+3.375\beta^3xy-1.18125\beta^2xy+1.17075\beta xy-0.0081xy$ $+3.375\beta^4y-3.375\beta^3y+1.18125\beta^2y-0.17075\beta y+0.0081y$ $+1.125\beta^4x-1.125\beta^3x+0.40875\beta^2x-0.05775\beta x+0.0027x$ $-1.125\beta^4+1.125\beta^3-0.40875\beta^2+0.05775\beta-0.0027$
element(7)
$0.405\beta^2xy+0.9975\beta xy-0.405\beta^2x+0.0025\beta x$
element(8)
$-3.375\beta^4xy+3.375\beta^3xy-1.18125\beta^2xy+4.17075\beta xy-0.0081xy$ $+1.125\beta^4y-1.125\beta^3y+0.40875\beta^2y-0.05775\beta y+0.0027y$ $+3.375\beta^4x-3.375\beta^3x+1.18125\beta^2x-7.82925\beta x+0.0081x$ $-1.125\beta^4+1.125\beta^3-0.40875\beta^2+0.05772\beta-0.0027$
element(9)
$3.375\beta^4xy-3.375\beta^3xy+1.13625\beta^2xy+0.83175\beta xy+0.0081xy$ $-3.375\beta^4y+3.375\beta^3y-1.13625\beta^2y+0.16825\beta y-0.0081y$ $-3.375\beta^4x+3.375\beta^3x-1.13625\beta^2x+0.16825\beta x-0.0081x$ $+3.375\beta^4-3.375\beta^3+1.13625\beta^2-0.16823\beta+0.0081$

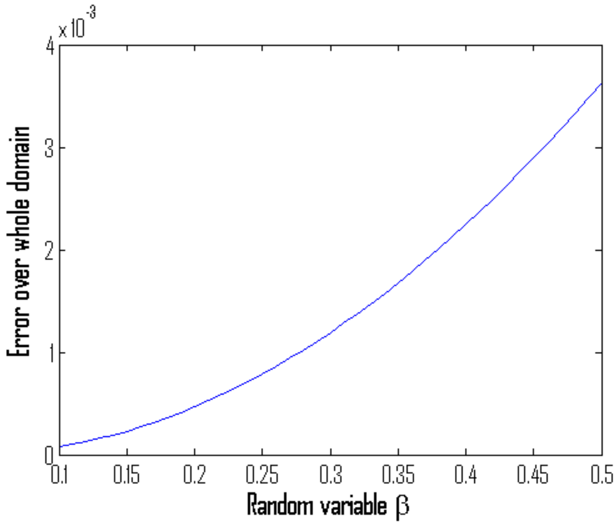

Figure 2. Error over whole domain e_w as a function of β for Example 1

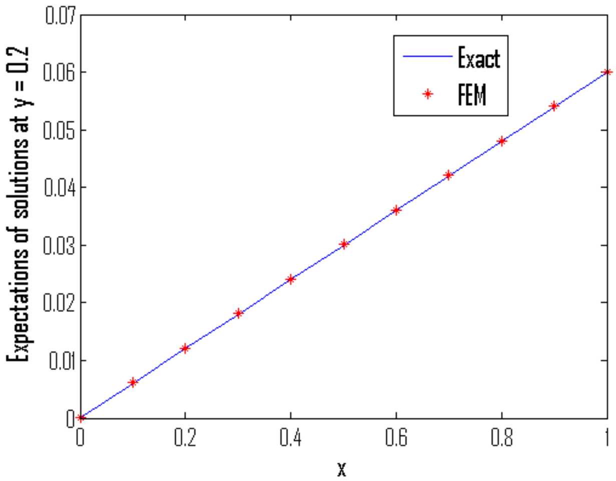

Figure 3. Expectations of the approximate and exact solutions at $y = 0.2$ for Example 1

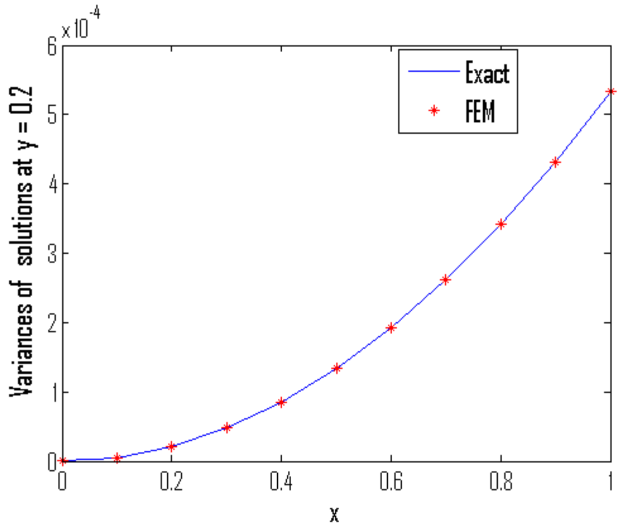

Figure 4. Variances of the approximate and exact solutions at $y = 0.2$ for Example 1

Example 2. Consider the following nonlinear 2-D problem which on a domain shown in Figure 5

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = f(x,y,\beta), \quad \beta \in [0.5, 0.9], \tag{3.2}$$

where the source term and boundary conditions are prescribed so that the exact solution is given by $u = \beta(x+y)$.

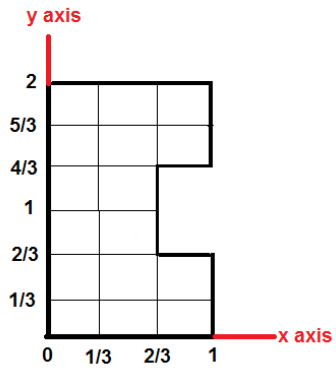

Figure 5. Domain discretization for Example 2

Table 4. Finite element solution for internal nodes at discretized values of random variable β for Example 2

	$\beta = 0.5$	$\beta = 0.6$	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$
$u(1/3, 1/3)$	0.3278	0.3921	0.456	0.5194	0.5825
$u(2/3, 1/3)$	0.4960	0.5943	0.6923	0.79	0.8874
$u(1/3, 2/3)$	0.4933	0.5905	0.6873	0.7836	0.8794
$u(1/3, 1)$	0.6569	0.7866	0.9157	1.0443	1.1722
$u(1/3, 4/3)$	0.8179	0.9922	1.1409	1.3016	1.4617
$u(1/3, 5/3)$	0.9671	1.1589	1.3502	1.5409	1.7311
$u(2/3, 5/3)$	1.1414	1.3700	1.5985	1.8272	2.0559

Table 5. Internal nodal values as functions of random variable β for Example 2

$u(1/3, 1/3)$	$0.125\beta^4 - 0.34167\beta^3 + 0.32375\beta^2 + 0.51392\beta + 0.0248$
$u(2/3, 1/3)$	$-0.015\beta^2 + 0.9995\beta$
$u(1/3, 2/3)$	$0.041667\beta^4 - 0.125\beta^3 + 0.11458\beta^2 + 0.93175\beta + 0.0118$
$u(1/3, 1)$	$-0.125\beta^4 + 0.34167\beta^3 - 0.37375\beta^2 + 1.4811\beta - 0.0251$
$u(1/3, 4/3)$	$-20.917\beta^4 + 60.65\beta^3 - 65.061\beta^2 + 32.153\beta - 5.2676$
$u(1/3, 5/3)$	$0.083333\beta^4 - 0.23333\beta^3 + 0.21417\beta^2 + 1.8388\beta + 0.0181$
$u(2/3, 5/3)$	$-0.20833\beta^4 + 0.59167\beta^3 - 0.61792\beta^2 + 2.5671\beta - 0.0486$

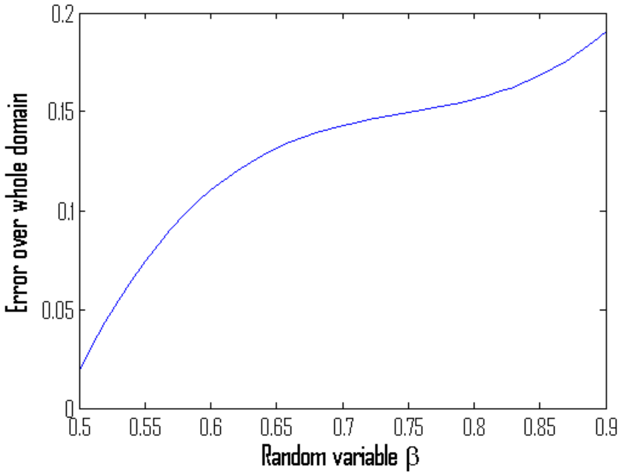

Figure 6. Error over whole domain e_w as a function of random variable β for Example 2

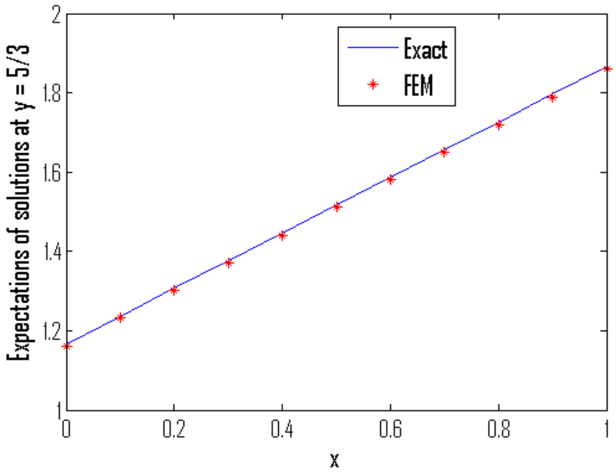

Figure 7. Expectations of the approximate and exact solutions at $y = 5/3$ for Example 2

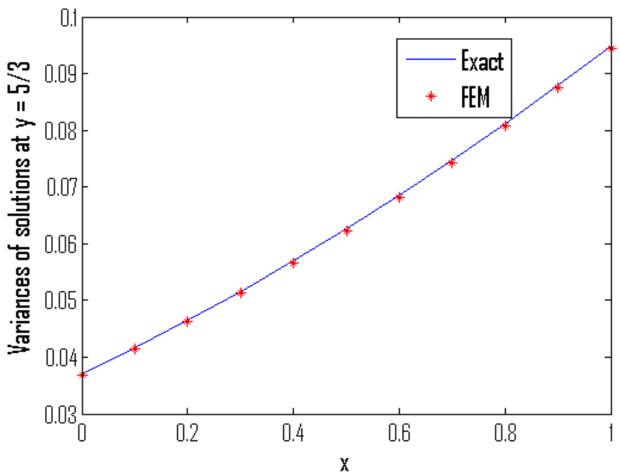

Figure 8. Variances of the approximate and exact solutions at $y = 5/3$ for Example 2

By discretization the two-dimensional domain as shown in Figure 5 and using Matlab code, we obtain the results listed in the following tables. Table 4 illustrates FEM solution for the internal nodes at discretized values of RV. Table 5 illustrates the internal nodal values as functions of the RV. The error over whole domain e_w is illustrated in Figure 6 as a function of RV. For Example 2 we chose a tolerance of error over whole domain $e_t = 0.2$. Figure 7 illustrates expectations of the approximate and exact solutions at $y = 5/3$. Figure 8 illustrates variances of the approximate and exact solutions at $y = 5/3$.

Example 3. Consider the following nonlinear 2-D problem which on a domain shown in Figure 9

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \beta u^2 = f(x,y,\beta), \quad \beta \in [0.1, 0.5], \tag{3.3}$$

where the source term and boundary conditions are prescribed so that the exact solution is given by $u = \beta(x-y)^2$.

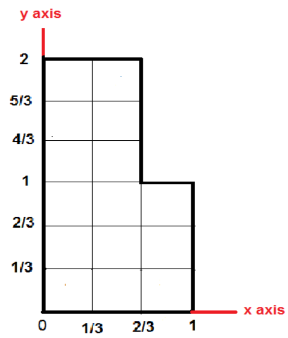

Figure 9. Domain discretization for Example 3

By discretization the two dimensional domain as shown in Figure 9 and using Matlab code, we obtain the results listed in the following tables. Table 6 illustrates FEM solution for the internal nodes at discretized values of RV. Table 7 illustrates the internal nodal values as functions of the RV. The error over whole domain e_w is illustrated in Figure 10 as a function of RV. For Example 3 we chose a tolerance of error over whole domain $e_t = 0.1$. Figure 11 illustrates expectations of the approximate and exact solutions at $y = 0.5$. Figure 12 illustrates variances of the approximate and exact solutions at $y = 0.5$.

Table 6. Finite element solution for internal nodes at discretized values of random variable β for Example 3

	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.3$	$\beta = 0.4$	$\beta = 0.5$
$u(1/3, 1/3)$	0.000694	0.0013	0.0018	0.0022	0.0024
$u(2/3, 1/3)$	0.01	0.021	0.032	0.043	0.053
$u(1/3, 2/3)$	0.011	0.0172	0.028	0.038	0.045
$u(2/3, 2/3)$	0.002	0.005	0.007	0.009	0.01
$u(1/3, 1)$	0.04	0.073	0.11	0.146	0.184
$u(1/3, 4/3)$	0.092	0.19	0.28	0.37	0.47
$u(1/3, 5/3)$	0.172	0.345	0.52	0.7	0.8813

Table 7. Internal nodal values as functions of random variable β for Example 3

$u(1/3,1/3)$	$-0.044167\beta^4+0.045167\beta^3-0.021358\beta^2+0.0099683\beta-0.00013$
$u(2/3,1/3)$	$-0.41667\beta^4+0.41667\beta^3-0.14583\beta^2+0.13083\beta-0.002$
$u(1/3,2/3)$	$14.508\beta^3-15.186\beta^2+6.7743\beta-0.52$
$u(1/3,2/3)$	$1.3333\beta^4-2.2333\beta^3+1.2367\beta^2-0.17267\beta+0.018$
$u(2/3,2/3)$	$-0.83333\beta^4+\beta^3-0.44167\beta^2+0.105\beta-0.005$
$u(1/3,1)$	$3.3333\beta^4-4.1667\beta^3+1.8667\beta^2+0.011667\beta+0.024$
$u(1/3,4/3)$	$0.83333\beta^4+0.5\beta^3-0.90833\beta^2+1.205\beta-0.02$
$u(1/3,5/3)$	$-2.7917\beta^4+3.2917\beta^3-1.1771\beta^2+1.8946\beta-0.0087$

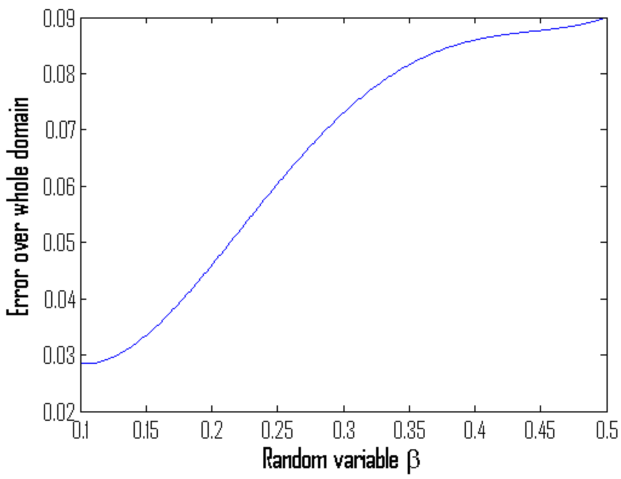

Figure 10. Error over whole domain e_w as a function of β for Example 3

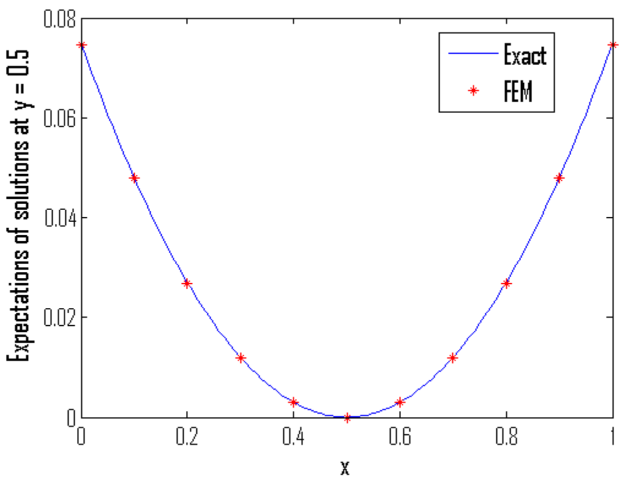

Figure 11. Expectations of the approximate and exact solutions at $y=0.5$ for Example 3

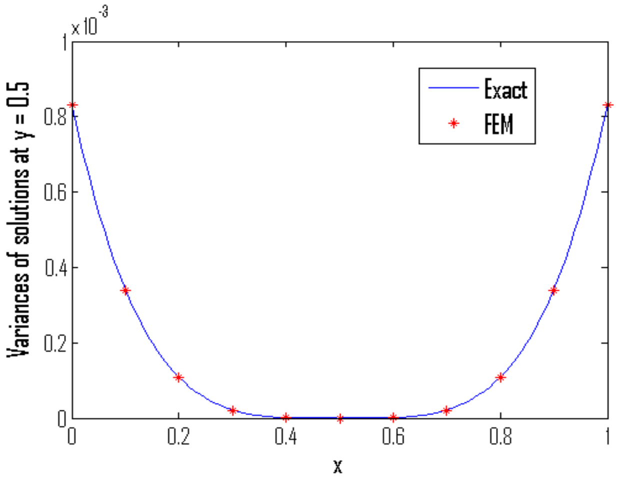

Figure 12. Variances of the approximate and exact solutions at $y = 0.5$ for Example 3

4. Conclusion

In this paper, we discuss a new technique for solving nonlinear random partial differential equation using finite element method. The nodal coefficients are proposed as functions of the random variable. So for some selected values of the random variable, the systems of nonlinear algebraic equations are solved. Then by applying the curve fitting and the basis of the finite element method, we obtain the approximate solution as a function of both space and random variable. The number of discretized values of the random variable depends on the comparison of the maximum error over the whole domain with the desired chosen tolerance of error over the whole domain. The results obtained in the numerical examples illustrate the accuracy of the proposed scheme as the expectations and variances of the approximate solutions agree with those of the exact solutions.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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