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Stability of Two Superposed Rivlin-Ericksen Viscoelastic Dusty Fluids in the Presence of Magnetic Field

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Abstract. The stability of the plane interface separating two Rivlin-Ericksen viscoelastic superposed fluids permeated with suspended particles and uniform horizontal magnetic field is considered following the linearized perturbation theory and normal mode analysis. The stability analysis has been carried out, for mathematical simplicity, for two highly viscoelastic fluids of equal kinematic viscosities and equal kinematic viscoelasticities. For potentially stable configuration, the system is found to be stable for disturbances of all wave numbers. The magnetic field succeeds in stabilizing certain wave-number range, for the potentially unstable configuration. The case of exponentially varying density, viscosity, viscoelasticity, magnetic field and particle number density is also considered. For stable density stratification, the system is found to be stable for disturbances of all wave numbers. The magnetic field succeeds in stabilizing the potentially unstable stratifications for a certain wave-number range which were unstable in the absence of the magnetic field. Discussion of oscillatory modes and non-oscillatory modes are also made.

1. **Introduction**

Several authors have studied the instability of two plane interface separating two Newtonian fluids where one is accelerated towards the other or when one is superposed over the other. Chandrasekhar [1] has discussed the theoretical and experimental results on the onset of thermal instability (B'enard convection) in a fluid layer under varying assumptions of hydrodynamics. Bhatia [2] has considered the Rayleigh-Taylor instability of two viscous superposed conducting fluids in the presence of a uniform horizontal magnetic field. Kumar [3] have studied the problem of Rayleigh-Taylor instability of Rivlin-Ericksen elastico-viscous fluid in the presence of suspended particles through porous medium and found that in the case of two uniform elastico-viscous fluids separated by a horizontal boundary and exponentially varying density, the perturbation decay with time for potentially stable configuration/stable stratification and grow with time for potentially unstable configuration/unstable stratification.

Key words and phrases. Rayleigh-Taylor instability; Rivlin-Ericksen visco-elastic fluid; Suspended particles; Magnetic field.

With the growing importance of non-Newtonian fluids in the modern technology and industries, the investigations of such fluids are desirable. Rivlin-Ericksen [4] is an important class of visco-elastic fluids. Khan and Bhatia [5] have considered the problem of stability of two superposed visco-elastic fluids in a horizontal magnetic field and found that elasticity has a stabilizing effect and viscosity has a destabilizing effect on the growth rate of unstable mode of disturbances. Kumar and Lal [6] have studied the stability of two superposed Rivlin-Ericksen viscous-viscoelastic fluids and found that both kinematic viscosity and kinematic viscoelasticity have stabilizing effect.

Recent spacecraft observations have confirmed that the dust particles play an important role in the dynamics of atmosphere as well as in the diurnal and surface variations in the temperature of the Martin weather. It is, therefore, of interest to study the presence of dust particles in astrophysical situation. The problem of stability of stratified visco-elastic Walter's (Model B') dusty fluid in porous medium has been studied by Rajbahadur and Pundir [7] and found that the system is stable for $\beta < 0$ and unstable for $\beta > 0$ under certain conditions. The Rayleigh-Taylor instability of two superposed couple-stress fluids of uniform densities in a porous medium in the presence of a uniform horizontal magnetic field is studied by Sunil, Sharma and Chandel [8] and found that magnetic field stabilizes a certain wave number range $k > k^*$, which is unstable in the absence of the magnetic field. Kumar, Mohan and Singh [9] have studied the stability of two superposed viscoelastic fluid-particle mixtures and found that system is stable or unstable for the wave number range $k \leq$ or $> \dfrac{1}{\sqrt{2\nu'}}$ depending on the kinematic viscoelasticity ν'. The problem of stability of superposed visco-elastic (Walters B') fluids in the presence of suspended particles through porous medium has been studied by Kumar and Sharma [10] and found that for the unstable configuration, the magnetic field and viscoelasticity have a stabilizing effect. Kumar and Singh [11] have studied the stability of two superposed Rivlin-Ericksen visco-elastic fluids in the presence of suspended particles. It is found that system is stable for stable configuration and unstable for unstable configuration.

The present paper is devoted to the consideration of hydromagnetic stability of two superposed Rivlin-Ericksen visco-elastic fluids in the presence of suspended particles. Since the Rivlin-Ericksen visco-elastic fluid plays a significant role in industrial application, it would be of much interest to examine the stability conditions of Rivlin-Ericksen fluid. Since the stability of two superposed Rivlin-Ericksen visco-elastic dusty fluids in the presence of magnetic field seems to the best of our knowledge uninvestigated so far. Hence, in this paper, we shall discuss the effect of magnetic field on stability of two superposed Rivlin-Ericksen visco-elastic fluids in the presence of suspended particles.

2. Notations

ρ	Density of fluid,
μ	Coefficient of viscosity,
μ'	Coefficient of viscoelasticity,
μ_e	Magnetic permeability,
∂	Curly operator,
∇	Del operator,
β	Constant,
ν	Kinematic viscosity (μ/ρ),
ν'	Kinematic viscoelasticity (μ'/ρ),
p	Fluid pressure,
$g(0,0,\mathbf{g})$	Acceleration due to gravity,
$\mathbf{H}(H,0,0)$	Magnetic field vector having components $(H,0,0)$,
$\delta\rho$	Perturbation in density $\rho(z)$,
δp	Perturbation in pressure, $p(z)$,
$\mathbf{q}(u,v,w)$	Perturbation in fluid velocity $\mathbf{q}(0,0,0)$,
$\mathbf{q}_d(l,r,s)$	Perturbations in particle velocity $\mathbf{q}_d(0,0,0)$
$\mathbf{h}(h_x,h_y,h_z)$	Perturbation in magnetic field $\mathbf{H}(H,0,0)$,
k_x, k_y	Wave numbers in x and y directions respectively,
$k=\sqrt{k_x^2+k_y^2}$	Wave number of the disturbance,
n	Growth rate of disturbance,
V_A^2	Square of the Alfven velocity $\left(V_A^2=\dfrac{\mu_e H^2}{4\pi}\right)$,
$\rho_0, \nu_0, \nu_0', \mu_0, \mu_0', N_0$	Constants,
π	Constant value,
D	Derivative with respect to z $\left(=\dfrac{d}{dz}\right)$

3. Formulation of the problem

Let $T_{ij}, \tau_{ij}, e_{ij}, \delta_{ij}, q_i, x_i, p, \mu$ and μ' denote respectively, the stress tensor, shear stress tensor, rate of strain tensor, Kronecker delta, velocity vector, position vector, isotropic pressure, viscosity and visco-elasticity. The constitutive relations for the Rivlin-Ericksen visco-elastic fluid are

$$\left.\begin{aligned} T_{ij} &= -p\delta_{ij}+\tau_{ij}, \\ \tau_{ij} &= 2\left[\mu+\mu'\frac{\partial}{\partial t}\right]e_{ij} \\ \text{and}\quad e_{ij} &= \frac{1}{2}\left[\frac{\partial q_i}{\partial x_j}+\frac{\partial q_j}{\partial x_i}\right]. \end{aligned}\right\} \tag{3.1}$$

Consider a static state in which an incompressible, visco-elastic Rivlin-Ericksen fluid layer containing suspended particles of variable density is arranged in horizontal strata and the pressure p and density ρ are functions of vertical coordinate z only. The fluid layer is under the action of gravity $\mathbf{g}(0,0,-g)$ and the horizontal magnetic field $\mathbf{H}(H,0,0)$. The particles are assumed to be non-conducting.

Let $\mathbf{q}(u,v,w)$, ρ and p denote respectively the velocity, density and pressure of the hydromagnetic fluid. $\mathbf{q}_d(\bar{x},t)$ and $\mathbf{N}(\bar{x},t)$ denote the velocity and number density of particles, respectively. $K=6\pi\mu\eta$, where η particle radius, is a constant and $\bar{x}=(x,y,z)$. Then the equation of motion and continuity for the Rivlin-Ericksen visco-elastic fluid are

$$\rho\left[\frac{\partial \mathbf{q}}{\partial t}+(\mathbf{q}\cdot\nabla)\mathbf{q}\right]=-\nabla p+\mathbf{g}\rho+\left(\mu+\mu'\frac{\partial}{\partial t}\right)\nabla^2\mathbf{q} \\ +KN(\mathbf{q}_d-\mathbf{q})+\frac{\mu_e}{4\pi}\left[(\nabla\times\mathbf{H})\times\mathbf{H}\right], \tag{3.2}$$

$$\nabla\cdot\mathbf{q}=0, \tag{3.3}$$

$$\frac{\partial\rho}{\partial t}+(\mathbf{q}\cdot\nabla)\rho=0, \tag{3.4}$$

$$\frac{\partial\mathbf{H}}{\partial t}=(\mathbf{H}\cdot\nabla)\mathbf{q}-(\mathbf{q}\cdot\nabla)H \tag{3.5}$$

and

$$\nabla\cdot\mathbf{H}=0, \tag{3.6}$$

where μ_e, the magnetic permeability is assumed to be constant and fluid is assumed to be infinitely conducting.

The presence of particles adds an extra force term, proportional to the velocity difference between particles and appears in equations of motion (3.2). Since the force exerted by the fluid on the particles is equal and opposite to the exerted by the particles on the fluid, there must be an extra force term, equal in magnitude but opposite in sign, in the equation of motion for the particles. The buoyancy force on the particles is neglected. Interparticle reactions are not considered for we assume that the distance between particles is quite large as compared with their diameter. The equations of motion and continuity for the particles, under the above approximation, are

$$mN\left[\frac{\partial\mathbf{q}_d}{\partial t}+(\mathbf{q}_d\cdot\nabla)\mathbf{q}_d\right]=KN(\mathbf{q}-\mathbf{q}_d) \tag{3.7}$$

and

$$\frac{\partial N}{\partial t}+\nabla\cdot(N\mathbf{q}_d)=0, \tag{3.8}$$

where mN is the mass of the particles per unit volume.

4. **Perturbation equations and normal mode analysis**

The time dependent solution of (3.2) to (3.8) known as the basic state, whose stability we wish to examine is that of an incompressible, Rivlin-Ericksen viscoelastic fluid layer of variable density arranged in horizontal strata. The character of equilibrium is examined by supposing that the system is slightly disturbed and then by following its further evolution.

Let $\delta\rho$, δp, $\mathbf{q}(u,v,w)$, $\mathbf{q}_d(l,r,s)$ and $\mathbf{h}(h_x,h_y,h_z)$ denote respectively the perturbations in the hydromagnetic fluid density ρ, pressure p, velocity $q(0,0,0)$, particles velocity $q_d(0,0,0)$ and the magnetic field $H(H,0,0)$. Then the linearized perturbation equations are

$$\rho\frac{\partial\mathbf{q}}{\partial t} = -\nabla\delta p + \mathbf{g}\delta\rho + \left(\mu+\mu'\frac{\partial}{\partial t}\right)\nabla^2\mathbf{q} + KN_0(\mathbf{q}_d-\mathbf{q}) + \frac{\mu_e}{4\pi}[(\nabla\times\mathbf{h})\times\mathbf{H}], \tag{4.1}$$

$$\nabla\cdot\mathbf{q} = 0, \tag{4.2}$$

$$\frac{\partial}{\partial t}\delta\rho = -w(D\rho), \tag{4.3}$$

$$\left(\frac{m}{K}\frac{\partial}{\partial t}+1\right)\mathbf{q}_d = \mathbf{q}, \tag{4.4}$$

$$\frac{\partial M}{\partial t} + \nabla\cdot\mathbf{q}_d = 0, \tag{4.5}$$

$$\frac{\partial\mathbf{h}}{\partial t} = (\mathbf{H}\cdot\nabla)\mathbf{q} \tag{4.6}$$

and

$$\nabla\cdot\mathbf{h} = 0, \tag{4.7}$$

where $M = N/N_0$ and N, N_0 respectively stands for initial uniform number density and perturbation in number density.

Analyzing the perturbations into normal modes, we seek the solution whose dependence on x, y and t is given by

$$\exp(ik_x x + ik_y y + nt), \tag{4.8}$$

where k_x and k_y are the horizontal components of the wave number, $k = \sqrt{k_x^2+k_y^2}$ is the resultant wave number and n is the growth rate, which is, in general, a complex constant.

With the dependence of physical variables on x, y and t and following the usual procedure, we get

$$n(1+\tau n)[D(\rho Dw)-\rho k^2 w] + n[D(mN_0Dw)-mN_0k^2w] + \frac{gk^2}{n}D\rho(1+\tau n)w - (\mu+\mu' n)(1+\tau n)(D^2-k^2)^2 w + \frac{\mu_e H^2 k_x^2}{4\pi n}(1+\tau n)(D^2-k^2)w = 0, \tag{4.9}$$

where $\tau = m/K$ and $D = d/dz$.

5. Results and discussion

5.1. *Two superposed visco-elastic fluids separated by a horizontal boundary*

In this section, we consider the case of two superposed visco-elastic fluids of uniform density ρ_1 and ρ_2, uniform viscosities μ_1 and μ_2, uniform viscoelasticities μ_1' and μ_2' are separated by a horizontal boundary at $z = 0$. The subscripts 1 and 2 respectively, distinguish the upper and lower fluid. Then in each region of constant ρ, μ and μ', equation (4.9) gives

$$(D^2 - k^2)(D^2 - s^2)w = 0, \tag{5.1}$$

where $s^2 = k^2 + \dfrac{n}{\nu + \nu' n} + \dfrac{mnN_0}{(1+\tau n)(\mu + \mu' n)} + \dfrac{\mu_e H^2 k_x^2}{4\pi n(\mu + \mu' n)}$, and $\nu = \dfrac{\mu}{\rho}$, $\nu' = \dfrac{\mu'}{\rho}$ are respectively the kinematic viscosity and kinematic viscoelasticity.

Since w must vanish both when $z \to +\infty$ (for upper fluid) and $z \to -\infty$ (for lower fluid), the general solution of equation (5.1) is given by

$$w_1 = A_1 e^{kz} + B e^{s_1 z} \quad (z < 0) \tag{5.2}$$

and

$$w_2 = C e^{-kz} + D e^{-s_2 z} \quad (z > 0), \tag{5.3}$$

where A, B, C and D are constant of integration and

$$s_1^2 = k^2 + \frac{n}{\nu_1 + n\nu_1'} + \frac{mnN_0}{\rho_1(\nu_1 + n\nu_1')(1+\tau n)} + \frac{\mu_e H^2 k_x^2}{4\pi n \rho_1(\nu_1 + n\nu_1')} \tag{5.4}$$

and

$$s_2^2 = k^2 + \frac{n}{\nu_2 + n\nu_2'} + \frac{mnN_0}{\rho_2(\nu_2 + n\nu_2')(1+\tau n)} + \frac{\mu_e H^2 k_x^2}{4\pi n \rho_2(\nu_2 + n\nu_2')}. \tag{5.5}$$

Here, it is assumed that s_1 and s_2 are so defined that the real parts of s_1 and s_2 are positive.

The boundary conditions to be satisfied here at $z = 0$ are:

$$w, \tag{5.6}$$

$$Dw \tag{5.7}$$

and

$$(\mu + \mu' n)(D^2 + k^2)w \tag{5.8}$$

must be continuous across the interface between two fluids. Integrating equation (4.9) across the interface at $z = 0$, we get

$$\begin{aligned}
&[\rho_2 D w_2 - \rho_1 D w_1]_{z=0} + \frac{mN_0}{(1+\tau n)}[Dw_2 - Dw_1]_{z=0} \\
&- \frac{1}{n}[(\mu_2 + \mu_2' n)(D^2 - 2k^2)Dw_2 - (\mu_1 + n\mu_1')(D^2 - 2k^2)Dw_1]_{z=0} \\
&+ \frac{gk^2}{n^2}(\rho_2 - \rho_1)w_0 + \frac{\mu_e H^2 k_x^2}{4\pi n^2}[Dw_2 - Dw_1]_{z=0} = 0,
\end{aligned} \tag{5.9}$$

where w_0 is the common value of w_1 and w_2 at $z=0$.

Applying the boundary conditions (5.6) to (5.9) to the solutions (5.2) and (5.3), we get

$$A+B=C+D, \tag{5.10}$$

$$kA+S_1B=-kC-s_2D, \tag{5.11}$$

$$(\mu_1+n\mu_1')[2k^2A+(s_1^2+k^2)B]=(\mu_2+n\mu_2')[2k^2C+(s_2^2+k^2)D] \tag{5.12}$$

and

$$\begin{aligned} &A\left[\frac{gk}{2n^2}(\rho_2-\rho_1)-\rho_1-\frac{mN_0}{(1+\tau n)}-\frac{(\mu_1+n\mu_1')}{n}k^2-\frac{k_x^2V_A^2}{n^2}\right] \\ &+B\left[\frac{gk}{2n^2}(\rho_2-\rho_1)-\frac{(\mu_1+n\mu_1')}{n}ks_1\right] \\ &+C\left[\frac{gk}{2n^2}(\rho_2-\rho_1)-\rho_2-\frac{mN_0}{(1+\tau n)}-\frac{(\mu_2+\mu_2'n)}{n}k^2-\frac{k_x^2V_A^2}{n^2}\right] \\ &+D\left[\frac{gk}{2n^2}(\rho_2-\rho_1)-\frac{(\mu_2+n\mu_2')}{n}ks_2\right]=0, \end{aligned} \tag{5.13}$$

where $V_A^2=\dfrac{\mu_eH^2}{4\pi}$.

On solving the equations (5.10) to (5.13), we get the fourth order determinate

$$\begin{vmatrix} 1 & 1 & -1 & -1 \\ k & s_1 & k & s_2 \\ 2k^2(\mu_1+n\mu_1') & (s_1^2+k^2)(\mu_1+n\mu_1') & -2k^2(\mu_2+n\mu_2') & -(s_2^2+k^2)(\mu_2+n\mu_2') \\ \begin{bmatrix} \frac{gk}{2n^2}(\rho_2-\rho_1)-\rho_1-\frac{mN_0}{(1+\tau n)} \\ -\frac{(\mu_1+n\mu_1')}{n}k^2-\frac{k_x^2V_A^2}{n^2} \end{bmatrix} & \begin{bmatrix} \frac{gk}{2n^2}(\rho_2-\rho_1) \\ -\frac{(\mu_1+n\mu_1')}{n}ks_1 \end{bmatrix} & \begin{bmatrix} \frac{gk}{2n^2}(\rho_2-\rho_1)-\rho_2-\frac{mN_0}{(1+\tau n)} \\ -\frac{(\mu_2+\mu_2'n)}{n}k^2-\frac{k_x^2V_A^2}{n^2} \end{bmatrix} & \begin{bmatrix} \frac{gk}{2n^2}(\rho_2-\rho_1) \\ -\frac{(\mu_2+n\mu_2')}{n}ks_2 \end{bmatrix} \end{vmatrix}=0. \tag{5.14}$$

The above determinate can be reduced into third order determinate by subtracting the first column from the second, third column from the fourth and adding first and third column, we get

$$\begin{vmatrix} s_1-k & 2k & s_2-k \\ \left[n\rho_1+\frac{mnN_0}{(1+\tau n)}+\frac{k_x^2V_A^2}{n}\right] & 2k^2\{\rho_1(\nu_1+n\nu_1')-\rho_2(\nu_2+n\nu_2')\} & -\left[n\rho_2+\frac{mnN_0}{(1+\tau n)}+\frac{k_x^2V_A^2}{n}\right] \\ \begin{bmatrix} \rho_1+\frac{mN_0}{(1+\tau n)} \\ -\frac{(\mu_1+n\mu_1')}{n}k(s_1-k) \\ +\frac{k_x^2V_A^2}{n^2} \end{bmatrix} & \begin{bmatrix} \frac{gk}{2n^2}(\rho_2-\rho_1)-(\rho_1+\rho_2) \\ -\frac{2mN_0}{(1+\tau n)}-\frac{k^2}{n}\{\rho_2(\nu_2+n\nu_2') \\ +\rho_1(\nu_1+n\nu_1')\}-\frac{2k_x^2V_A^2}{n^2} \end{bmatrix} & \begin{bmatrix} \rho_2+\frac{mN_0}{(1+\tau n)} \\ -\frac{(\mu_2+\mu_2'n)}{n}k(s_2-k) \\ +\frac{k_x^2V_A^2}{n^2} \end{bmatrix} \end{vmatrix}=0. \tag{5.15}$$

The determinate (5.15) is quite complicated since the value of s_1 and s_2 involve square roots. For convenience, we assume that the kinematic viscosities and kinematic viscoelasticities of the two fluids are same, i.e., $\nu_1=\nu=\nu_2$, $\nu_1'=\nu'=\nu_2'$ and that the fluids are of high viscosity and high visco-elasticity.

Under the above assumptions, we have

$$s_1 - k = \frac{n}{2k(v+v'n)} + \frac{mnN_0}{2k\rho_1(v+nv')(1+\tau n)} + \frac{k_x^2 V_A^2}{2nk\rho_1(v+nv')}$$

and

$$s_2 - k = \frac{n}{2k(v+v'n)} + \frac{mnN_0}{2k\rho_2(v+nv')(1+\tau n)} + \frac{k_x^2 V_A^2}{2nk\rho_2(v+nv')}.$$

Using the values of $s_1 - k$ and $s_2 - k$ in the determinate (5.15) and other simplification (by Mathematica software), we obtain

$$A_9 n^9 + A_8 n^8 + A_7 n^7 + A_6 n^6 + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0 = 0, \quad (5.16)$$

where

$$\begin{aligned}
A_9 &= 2k^2 v' \tau^3 \rho_1 \rho_2 (\rho_1 + \rho_2), \\
A_8 &= 2k^2 m v' \tau^2 N_0 (\rho_1 + \rho_2)^2 + 2mN_0 \tau^2 \rho_1 \rho_2 \\
&\quad + 2k^2 \tau^2 \rho_1 \rho_2 (\tau v + 3v')(\rho_1 + \rho_2), \\
A_7 &= 2\tau m^2 N_0^2 (1 + k^2 v')(\rho_1 + \rho_2) + 4\tau m N_0 \rho_1 \rho_2 \\
&\quad + \tau^3 \rho_1 \rho_2 \{2k_x^2 V_A^2 + gk(\rho_1 - \rho_2)\} \\
&\quad + 6\tau k^2 \rho_1 \rho_2 (\tau v + v')(\rho_1 + \rho_2) \\
&\quad + 2\tau k^2 (\tau^2 v' k_x^2 V_A^2 + 2v' m N_0 + \tau v m N_0)(\rho_1 + \rho_2)^2, \\
A_6 &= 2\tau^2 k^2 k_x^2 V_A^2 (\tau v + 3v')(\rho_1 + \rho_2)^2 \\
&\quad + 2k^2 m N_0 (2\tau v + v')(\rho_1 + \rho_2)^2 \\
&\quad + 2m^2 N_0^2 (1 + k^2 v')(\rho_1 + \rho_2) + 2k^2 \rho_1 \rho_2 (3\tau v + v')(\rho_1 + \rho_2) \\
&\quad + 2\tau^2 m N_0 k_x^2 V_A^2 (1 + 2k^2 v')(\rho_1 + \rho_2) + 2\tau v k^2 m^2 N_0^2 (\rho_1 + \rho_2) \\
&\quad + 2mN_0 (\rho_1 \rho_2 + m^2 N_0^2) + (3\tau^2 \rho_1 \rho_2 + \tau^2 \\
&\quad \times m N_0 (\rho_1 + \rho_2) \{2k_x^2 V_A^2 + gk(\rho_1 - \rho_2)\}, \\
A_5 &= 2\tau^3 k_x^4 V_A^4 (1 + k^2 v')(\rho_1 + \rho_2) + 4\tau k^2 m N_0 k_x^2 V_A^2 (\tau v + 2v')(\rho_1 + \rho_2) \\
&\quad + 2k^2 v m N_0 (\rho_1 + \rho_2)^2 + 6\tau k^2 k_x^2 V_A^2 (\tau v + v')(\rho_1 + \rho_2)^2 \\
&\quad + gk\tau^3 k_x^2 V_A^2 (\rho_1^2 + \rho_2^2) + 2mN_0 (k^2 v m N_0 + 2\tau k_x^2 V_A^2)(\rho_1 + \rho_2) \\
&\quad + 2k^2 v \rho_1 \rho_2 (\rho_1 + \rho_2) + 4\tau m^2 N_0^2 k_x^2 V_A^2 \\
&\quad + \{2\tau m N_0 (\rho_1 + \rho_2) + 3\tau \rho_1 \rho_2 + \tau m^2 N_0^2\}\{2k_x^2 V_A^2 + gk(\rho_1 - \rho_2)\},
\end{aligned}$$

$$
\begin{aligned}
A_4 &= 4k^2 m N_0 k_x^2 V_A^2 (2\tau\nu + \nu')(\rho_1+\rho_2) + 2k^2 k_x^2 V_A^2 (3\tau\nu + \nu')(\rho_1+\rho_2)^2 \\
&\quad + 2\tau^2 k^2 k_x^4 V_A^4 (\tau\nu + 3\nu')(\rho_1+\rho_2) + 2mN_0 k_x^2 V_A^2 (gk\tau^2+1)(\rho_1+\rho_2) \\
&\quad + 2mN_0 k_x^2 V_A^2 (2mN_0 + 3\tau^2 k_x^2 V_A^2) \\
&\quad + \{mN_0(\rho_1+\rho_2) + \rho_1\rho_2 + m^2N_0^2 + 3\tau^2 k_x^2 V_A^2 (\rho_1+\rho_2)\} \\
&\quad \times \{2k_x^2 V_A^2 + gk(\rho_1-\rho_2)\}, \\
A_3 &= 6\tau k^2 k_x^4 V_A^4 (\tau\nu+\nu')(\rho_1+\rho_2) + 2k^2 \nu k_x^2 V_A^2 (\rho_1+\rho_2)^2 \\
&\quad + 12\tau m N_0 k_x^4 V_A^4 + 4kmN_0 k_x^2 V_A^2 (g\tau + \nu k)(\rho_1+\rho_2) \\
&\quad + \{3\tau k_x^2 V_A^2 (\rho_1+\rho_2) + \tau^3 k_x^4 V_A^4\}\{2k_x^2 V_A^2 + gk(\rho_1-\rho_2)\}, \\
A_2 &= 2k^2 k_x^4 V_A^4 (3\tau\nu+\nu')(\rho_1+\rho_2) + 2mN_0 k_x^4 V_A^4 \\
&\quad + \{k_x^2 V_A^2 (\rho_1+\rho_2) + 3\tau^2 k_x^4 V_A^4 + 2mN_0 k_x^2 V_A^2\}\{2k_x^2 V_A^2 + gk(\rho_1-\rho_2)\}, \\
A_1 &= 2k^2 \nu k_x^4 V_A^4 (\rho_1+\rho_2) + 3\tau k_x^4 V_A^4 \{2k_x^2 V_A^2 + gk(\rho_1-\rho_2)\}, \\
A_0 &= k_x^4 V_A^4 \{2k_x^2 V_A^2 + gk(\rho_1-\rho_2)\}.
\end{aligned}
$$

Theorem 5.1. *For potentially stable configuration ($\rho_1 > \rho_2$), the system is always stable.*

Proof. If $\rho_1 > \rho_2$, equation (5.16) does not involve any change of sign and so does not admit any positive value of n. Therefore, the system is stable for disturbances of all wave numbers.

Theorem 5.2. *For the potentially unstable configuration ($\rho_1 < \rho_2$), the system is stable provided $2k_x^2 V_A^2 > gk(\rho_1-\rho_2)$.*

Proof. If $\rho_1 < \rho_2$ and $2k_x^2 V_A^2 > gk(\rho_1-\rho_2)$, then equation (5.16) does not involve any change of sign and so does not allow any positive value of n. Therefore, the system is stable.

Theorem 5.3. *For potentially unstable configuration ($\rho_1 < \rho_2$), the system is unstable provided $2k_x^2 V_A^2 < gk(\rho_1-\rho_2)$.*

Proof. If $\rho_1 < \rho_2$ and $2k_x^2 V_A^2 < gk(\rho_1-\rho_2)$, then the constant term in equation (5.16) is negative. Therefore allow at least one change of sign and so has at least one positive root. The occurrence of a positive root implies that system is unstable.

5.2. *Case of exponentially varying density, viscosity, viscoelasticity, magnetic field and particle number density*

Let us assume that

$$\rho = \rho_0 e^{\beta z}, \quad N_0 = N_1 e^{\beta z}, \quad \mu = \mu_0 e^{\beta z}, \quad H^2 = H_1^2 e^{\beta z} \quad \text{and} \quad \mu' = \mu' e^{\beta_2}, \tag{5.17}$$

where $\rho_0, N_1, \mu_0, H_1, \mu_0'$ and β are constants. Substituting the values of ρ, N_1, μ, μ' and H in equation (4.9), we obtain

$$\left[n+\frac{mnN_1}{\rho_0(1+\tau n)}-(v_0+v_0'n)(D^2-k^2)+\frac{k_x^2V_A^2}{n}\right](D^2-k^2)w$$
$$+\frac{g\beta k^2}{n}w=0\,, \tag{5.18}$$

where $v_0=\dfrac{\mu_0}{\rho_0}$ and $v_0'=\dfrac{\mu_0'}{\rho_0}$.

Consider the case of two free boundaries. The boundary conditions for the case of two free surfaces are

$$w=0,\;\; D^2w=0 \;\text{ at }\; z=0 \;\text{ and }\; z=d\,. \tag{5.19}$$

The proper solution of equation (5.18) satisfying equation (5.19) is given by

$$w=A\sin\frac{m\pi z}{d}\,, \tag{5.20}$$

where A is a constant and m is any integer. Using equation (5.20), equation (5.18) gives

$$n^3[\tau(1+v_0'L)]+n^2\left[\left(1+\frac{mN_1}{\rho_0}+(v_0'+\tau v_0)L\right)\right]$$
$$+n\left[v_0L+\tau\left(k_x^2V_A^2-\frac{g\beta k^2}{L}\right)\right]+\left[k_x^2V_A^2-\frac{g\beta k^2}{L}\right]=0\,. \tag{5.21}$$

Theorem 5.4. *For stable density stratification ($\beta<0$), the system is always stable.*

Proof. For stable stratification ($\beta<0$), equation (5.21) does not involve any change of sign and so does not admit any positive value of n. Therefore, the system is stable for disturbances of all wave numbers.

Theorem 5.5. *For $\beta>0$, the system is stable or unstable provided $k_x^2V_A^2>$ or $<\dfrac{g\beta k^2}{L}$.*

Proof. If $\beta>0$ and $k_x^2V_A^2>\dfrac{g\beta k^2}{L}$, then equation (5.21) does not involve any change of sign and so does not admit any positive value of n. Therefore, the system is stable for disturbances of all wave numbers.

On the other hand, if $\beta>0$ and $k_x^2V_A^2<\dfrac{g\beta k^2}{L}$, then the constant term in equation (5.21) is negative. Therefore allow at least one change of sign and so has at least one positive root. The occurrence of a positive root implies that the system is unstable.

5.2.1. *Discussion of oscillatory modes.*

Equation (5.21) can be written as

$$An^3 + Bn^2 + Cn + D = 0, \tag{5.22}$$

where $A = \tau(1 + \nu_0 L)$, $B = 1 + \dfrac{mN_1}{\rho_0} + (\tau\nu_0 + \nu_0')L$, $V = \nu_0 L + \tau\left(k_x^2 V_A^2 - \dfrac{g\beta k^2}{L}\right)$ and $D = k_x^2 V_A^2 - \dfrac{g\beta k^2}{L}$.

After dividing by n, the real and imaginary parts of equation (5.22) are

$$A(n_r^2 - n_i^2) + Bn_r + C + \frac{Dn_r}{|n|^2} = 0 \tag{5.23}$$

and

$$n_i\left[2An_r + B - \frac{D}{|n|^2}\right] = 0. \tag{5.24}$$

Theorem 5.6. *For $\beta < 0$, the estimate of n for the growth rate of oscillatory stable modes is given by $|n|^2 > \dfrac{D}{B}$.*

Proof. If $\beta < 0$, then the value of B and D are definite positive. Since modes are oscillatory ($n_i \neq 0$) and if n_r is negative (for stable mode), then for the consistency of equation (5.24), we must have $|n|^2 > \dfrac{D}{B}$. Hence, for $\beta < 0$, the estimate of n for the growth rate of oscillatory stable modes is given by $|n|^2 > D|B$.

Theorem 5.7. *For $\beta < 0$, the estimate of n for the growth rate of oscillatory unstable modes is given by $|n|^2 < \dfrac{D}{B}$.*

Proof. If $\beta < 0$, then the value of B and D are definite positive. Since the modes are oscillatory ($n_i \neq 0$) and if n_r is positive (for unstable mode), then for the consistency of equation (5.24), we must have $|n|^2 < \dfrac{D}{B}$. Hence, for $\beta < 0$, the estimate of n for the growth rate of oscillatory unstable modes is given by $|n|^2 < \dfrac{D}{B}$.

Theorem 5.8. *For $\beta > 0$ and $k_x^2 V_A^2 > \dfrac{g\beta k^2}{L}$, the estimate of n for the growth rate of oscillatory stable or unstable modes are respectively given by $|n|^2 > \dfrac{D}{B}$ or $|n|^2 < \dfrac{D}{B}$.*

Proof. If $\beta > 0$ and $k_x^2 V_A^2 > \dfrac{g\beta k^2}{L}$, the value of A, B and D are positive definite. Since modes are oscillatory ($n_i \neq 0$) and stable ($n_r < 0$), then equation (5.24) gives $|n|^2 > \dfrac{D}{B}$. Also, for oscillatory unstable modes ($n_i \neq 0, n_r > 0$), we must have for the consistency of equation (5.24) as $|n|^2 < \dfrac{D}{B}$ under the given conditions.

5.2.2. *Discussion of non-oscillatory modes.*

For non-oscillatory modes, we must have $n_i = 0$, then equation (5.22) becomes

$$An_r^3 + Bn_r^2 + Cn_r + D = 0 \tag{5.25}$$

where $A = \tau(1 + v_0' L)$, $B = 1 + \dfrac{mN_1}{\rho_0} + (\tau v_0 + v_0')L$, $C = v_0 L + \tau \left(k_x^2 V_A^2 - \dfrac{g\beta k^2}{L} \right)$ and $D = k_x^2 V_A^2 - \dfrac{g\beta k^2}{L}$.

Theorem 5.9. *For $\beta < 0$, the non-oscillatory modes are always stable.*

Proof. If $\beta < 0$, then equation (5.25) does not involve any change of sign and therefore does not allow any positive root. Therefore, the non-oscillatory modes are stable for all wave numbers according to the given condition.

Theorem 5.10. *For $\beta > 0$, the non-oscillatory modes are stable provided $k_x^2 V_A^2 > \dfrac{g\beta k^2}{L}$.*

Proof. For $\beta > 0$ and $k_x^2 V_A^2 > \dfrac{g\beta k^2}{L}$, equation (5.25) does not involve any change of sign and therefore does not allow any positive root. Therefore, the non-oscillatory modes are stable.

Theorem 5.11. *For $\beta > 0$, the non-oscillatory modes are unstable provided $k_x^2 V_A^2 < \dfrac{g\beta k^2}{L}$.*

Proof. For $\beta > 0$ and $k_x^2 V_A^2 < \dfrac{g\beta k^2}{L}$, the value of D is negative. Therefore, equation (5.25) involves at least one change of sign so has at least one positive root. Therefore, the non-oscillatory modes are unstable.

Theorem 5.12. *For $\beta > 0$ and $k_x^2 V_A^2 < \dfrac{g\beta k^2}{L}$, there are wave propagating for a given wave number.*

Proof. Let the roots of equation (5.25) are $n_{r_1}, n_{r_2}, n_{r_3}$, then using the theory of equations, we get

$$n_{r_1} \cdot n_{r_2} \cdot n_{r_3} = -\frac{D}{A} > 0$$

and

$$n_{r_1} \cdot n_{r_2} \cdot n_{r_3} = -\frac{B}{A} < 0.$$

Clearly, when $\beta > 0$ and $k_x^2 V_A^2 < \dfrac{g\beta k^2}{L}$, then D is definite negative. Also, A and B are positive definite. So the product of the roots is positive and the sum of the roots is negative. Therefore, the possibility that all the three non-oscillatory modes can be unstable is ruled out. It follows that two waves of propagation are damped and one is amplified for a given wave number.

6. Conclusion

The stability of superposed fluids under varying assumptions of hydromagnetics has been discussed in details by Chandrasekhar [1]. With the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on Rivlin-Ericksen visco-elastic fluid are desirable. In the present paper, the stability of Rivlin-Ericksen visco-elastic dusty fluid in the presence of magnetic field is considered. For potentially stable configuration, the system is found to be stable for disturbances of all wave numbers. The magnetic field succeeds in stabilizing certain wave-number range, for the potentially unstable configuration. The case of exponentially varying density, viscosity, viscoelasticity, magnetic field and particle number density is also considered. For stable density stratification, the system is found to be stable for disturbances of all wave numbers. The magnetic field succeeds in stabilizing the potentially unstable stratifications for a certain wave-number range which were unstable in the absence of the magnetic field. It is also found that for $\beta < 0$, the non-oscillatory modes are always stable and for $\beta > 0$, the non-oscillatory modes are stable or unstable under certain conditions.

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