Journal of Informatics and Mathematical Sciences
Vol. 12, No. 1, pp. 15–39, 2020
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/jims.v12i1.1276

Research Article

A Dynamic Contact Problem Between Elasto-Viscoplastic Piezoelectric Bodies with Adhesion

Laid Maiza[1], Tedjani Hadj Ammar[2*] and Abdelhamid Rehouma[2]

[1] *Department of Mathematics, Laboratory of Applied Mathematics, Kasdi Merbah University, BP 511, Ouargla 30000, Algeria*
[2] *Department of Mathematics, Hamma Lakhdar University, 39000 El Oued, Algeria*
*__Corresponding author:__ hadjammar-tedjani@univ-eloued.dz

Abstract. We consider a dynamic frictionless contact problem between two elasto-viscoplastic piezoelectric bodies with damage. The evolution of the damage is described by an inclusion of parabolic type. The contact is modelled with normal compliance condition. The adhesion of the contact surfaces is considered and is modelled with a surface variable, the bonding field whose evolution is described by a first order differential equation. We derive variational formulation for the model and prove an existence and uniqueness result of the weak solution. The proof is based on arguments of nonlinear evolution equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities, differential equations and fixed-point arguments.

Keywords. Electro-elasto-viscoplastic materials; Normal compliance; Adhesion; Damage; differential equations; Fixed point

MSC. Primary 74M10, 74M15; Secondary 47H10, 49J40

Received: July 16, 2019 **Accepted:** September 21, 2019

1. Introduction

The piezoelectric materials have been used in numerous fields taking advantage of the flexible characteristics of these polymers. Some of the applications of these polymers include Audio device-microphones, high frequency speakers, tone generators and acoustic modems; Pressure switches-position switches, accelerometers, impact detectors, flow meters and load cells; Actuators-electronic fans and high shutters. Since piezoelectric polymers allow their use in a multitude of compositions and geometrical shapes for a large variety of applications from transducers in acoustics, ultrasonic's and hydrophone applications to resonators in band pass filters, power supplies, delay lines, medical scans and some industrial non-destructive testing instruments.

The piezoelectricity was discovered by the brothers Curie in 1880 (Curie and Curie, 1880); it consists on the apparition of electric charges on the surfaces of some crystals after their deformation. The reverse effect was outlined in 1881; it consists on the generation of stress and strain in crystals under the action of electric field on the boundary. A deformable material which undergoes piezoelectric effects is called a piezoelectric material. Different models have been developed to describe the interaction between the electric and mechanical fields (see, e.g., [2, 3, 5, 9, 15, 19, 29]). Therefore there is a need to extend the results on models for contact with deformable bodies which include coupling between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be found in [19, 20] and, more recently, in [4], for viscoelastic piezoelectric materials in [3, 4, 9, 29]. Elasto-viscoplastic piezoelectric materials have been also studied [1, 14].

The importance of this paper is to make the coupling of the piezoelectric problem and a frictionless contact problem with adhesion. The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [1,2,5,6,10,14,15] and recently in the monographs [24]. In all these papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by ζ, it describes the point wise fractional density of adhesion of active bonds on the contact surface, and some times referred to as the intensity of adhesion. Following [16], the bonding field satisfies the restriction $0 \le \zeta \le 1$, when $\zeta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \zeta < 1$ the adhesion is partial and only a fraction ζ of the bonds is active. The novelty of this work lies in the analysis of a system that contains strong couplings in the multivalued boundary conditions: both the normal compliance contact condition and tangential contact condition depend on the adhesion (see (18) and (19)), and the adhesion be written by the differential equation of the general form

$$\dot{\zeta} = H_{ad}\big(\zeta, \alpha_\zeta, R_\nu(u_\nu^1 + u_\nu^2), \boldsymbol{R}_\tau(\boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2)\big).$$

Here, H_{ad} is the adhesion evolution rate function. Then, the adhesion rate function was assumed to depend, in addition to $R_\nu(u_\nu^1 + u_\nu^2)$, $\boldsymbol{R}_\tau(\boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2)$ and α_ζ, where, the truncation operators R_ν, $\boldsymbol{R}_\tau$ are defined by (25), (26), respectively, and $\alpha_\zeta = \int_0^t \zeta(s)ds$. We use it in H_{ad}, since usually when the glue is stretched beyond the limit L it does not contribute more to the bond strength. An example of such a function, used in [6], the following form of the evolution of the bonding field was employed: $\dot{\zeta} = -\zeta_+ \gamma_n R_\nu([u_\nu])^2$, where $[u_\nu] = u_\nu^1 + u_\nu^2$ the stands for the displacements in normal direction, and γ_n the normal rate coefficient and $\gamma_n L$ is the maximal tensile normal traction that the adhesive can provide and $\zeta_+ = \max(0,\zeta)$. We note that in this case, only debonding is allowed. A different rate equation for the the evolution of the bonding field is $\dot{\zeta} = -(\zeta\gamma_n R_\nu([u_\nu])^2 + \zeta\gamma_t |\boldsymbol{R}_\tau([\boldsymbol{u}_\tau])|^2) - \varepsilon_a)_+$, (see, e.g., [1, 7, 14, 15]). Here, $[\boldsymbol{u}_\tau] = \boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2$ the stands for the jump of the displacements in tangential direction, and γ_t the tangential rate coefficient, which may also be interpreted as the tangential stiffness coefficient of the interface when the adhesion is complete ($\zeta = 1$). Another example, in which H_{ad} depends on all variables is $\dot{\zeta} = -\gamma_n \zeta_+ R_\nu([u_\nu])^2 - \gamma_t \beta_+ |\boldsymbol{R}_\tau([\boldsymbol{u}_\tau])|^2 + \gamma_r \frac{\zeta_+(1-\zeta)_+}{1+\alpha_\zeta^2}$, where γ_r is the rebonding rate coefficient. However, the bonding cannot exceed $\zeta = 1$ and, moreover, the rebonding becomes weaker as the process goes on, which is represented by the factor $1 + \alpha_\zeta^2$ in the denominator.

The damage is an extremely important topic in engineering, since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. General models for damage were derived in [12, 13] from the virtual power principle. Mathematical analysis of one-dimensional problems can be found in [11]. The three-dimensional case has been investigated in [18]. In all these papers the damage of the material is described with a damage function ζ^ℓ, restricted to have values between zero and one. When $\zeta^\ell = 1$, there is no damage in the material, when $\zeta^\ell = 0$, the material is completely damaged, when $0 < \zeta^\ell < 1$ there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [11, 22, 23, 26]. In this paper we consider a mathematical frictionless contact problem between two electro-elastic-viscoplastic bodies for rate-type materials of the form (6). The contact is modelled with normal compliance where the adhesion of the contact surfaces is taken into account and is modelled with a surface variable, the bonding field. We model the material's behavior with an electro-elastic-viscoplastic constitutive law with damage.We derive a variational formulation of the problem and prove the existence of a unique weak solution.

We turn now to describe the contact conditions. We assume that the normal stress $\boldsymbol{\sigma}_\nu$ satisfies a general normal compliance response condition with adhesion

$$\sigma_\nu = -p_\nu([u_\nu]) + \gamma_\nu \zeta^2 R_\nu([u_\nu]), \tag{1}$$

where γ_ν is a given adhesion coefficient and p_ν the normal compliance function. It assigns a reactive normal traction or pressure that depends on the interpenetration of the asperities between two bodies. p_ν is a nonnegative prescribed function that vanishes for negative argument. Indeed, when $[u_\nu] < 0$ there is no contact and the normal pressure vanishes. When there is contact, $[u_\nu]$ is positive and is a measure of the interpenetration of the asperities. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu \zeta^2 R_\nu([u_\nu])$, the adhesive traction is tensile and is proportional. A commonly used example of the normal compliance function p_ν, is

$$p_\nu(r) = c_\nu r_+,$$

or, more generally,

$$p_\nu(r) = c_\nu (r_+)^m,$$

where, $c_\nu > 0$ is the surface stiffness coefficient and $m \geq 1$ is the normal compliance exponent. We can also consider the following truncated normal compliance function:

$$p_\nu(r) = \begin{cases} c_\nu r_+ & \text{if } r \leq 0, \\ c_\nu \alpha & \text{if } r > 0, \end{cases}$$

where α is a positive coefficient related to the wear and hardness of the surface. The normal compliance with adhesion contact condition (1) are studied in many publications (see, e.g., [2, 3, 5–7, 14] and references therein).

In this paper we consider a dynamic frictionless contact problem between two electro-elasto-viscoplastic bodies for rate-type materials. The contact is modelled with normal compliance where the adhesion of the contact surfaces is taken into account and is modelled with a surface variable, the bonding field. We model the material's behavior with an electro-elasto-viscoplastic constitutive law with damage. We derive a variational formulation of the problem and prove the existence of a unique weak solution.

The paper is organized as follows. In section 2 we present notation and some preliminaries. The model is described in section 3 where the variational formulation is given. In section 4, we present our main result stated in Theorem 4.1 and its proof which is based on arguments of time-dependent variational inequalities, parabolic inequalities, differential equations and fixed point.

2. Notation and Preliminaries

In this short section, we present the notation we shall use and some preliminary material. For more details, we refer the reader to [21, 26]. We need to introduce some notation and preliminary material. Here and below, $\mathbb{S}^d$ represent the space of second-order symmetric tensors on $\mathbb{R}^d$. We recall that the inner products and the corresponding norms on $\mathbb{S}^d$ and $\mathbb{R}^d$ are given by

$$\boldsymbol{u}^\ell \cdot \boldsymbol{v}^\ell = u_i^\ell \cdot v_i^\ell, \quad |\boldsymbol{v}^\ell| = (\boldsymbol{v}^\ell \cdot \boldsymbol{v}^\ell)^{\frac{1}{2}}, \quad \forall\, \boldsymbol{u}^\ell, \boldsymbol{v}^\ell \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma}^\ell \cdot \boldsymbol{\tau}^\ell = \sigma_{ij}^\ell \cdot \tau_{ij}^\ell, \quad |\boldsymbol{\tau}^\ell| = (\boldsymbol{\tau}^\ell \cdot \boldsymbol{\tau}^\ell)^{\frac{1}{2}}, \quad \forall\, \boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell \in \mathbb{S}^d.$$

Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H^\ell = \{\boldsymbol{v}^\ell = (v_i^\ell);\ v_i^\ell \in L^2(\Omega^\ell)\}, \qquad H_1^\ell = \{\boldsymbol{v}^\ell = (v_i^\ell);\ v_i^\ell \in H^1(\Omega^\ell)\},$$

$$\mathcal{H}^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell);\ \tau_{ij}^\ell = \tau_{ji}^\ell \in L^2(\Omega^\ell)\}, \quad \mathcal{H}_1^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell) \in \mathcal{H}^\ell;\ \operatorname{div}\boldsymbol{\tau}^\ell \in H^\ell\}.$$

The spaces H^ℓ, H_1^ℓ, $\mathcal{H}^\ell$ and $\mathcal{H}_1^\ell$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H^\ell} = \int_{\Omega^\ell} \boldsymbol{u}^\ell \cdot \boldsymbol{v}^\ell dx, \quad (\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H_1^\ell} = \int_{\Omega^\ell} \boldsymbol{u}^\ell \cdot \boldsymbol{v}^\ell dx + \int_{\Omega^\ell} \nabla \boldsymbol{u}^\ell \cdot \nabla \boldsymbol{v}^\ell dx,$$

$$(\boldsymbol{\sigma}^\ell,\boldsymbol{\tau}^\ell)_{\mathcal{H}^\ell}=\int_{\Omega^\ell}\boldsymbol{\sigma}^\ell\cdot\boldsymbol{\tau}^\ell dx,\quad (\boldsymbol{\sigma}^\ell,\boldsymbol{\tau}^\ell)_{\mathcal{H}_1^\ell}=\int_{\Omega^\ell}\boldsymbol{\sigma}^\ell\cdot\boldsymbol{\tau}^\ell dx+\int_{\Omega^\ell}\operatorname{div}\boldsymbol{\sigma}^\ell\cdot\operatorname{Div}\boldsymbol{\tau}^\ell dx$$

and the associated norms $\|\cdot\|_{H^\ell}$, $\|\cdot\|_{H_1^\ell}$, $\|\cdot\|_{\mathcal{H}^\ell}$, and $\|\cdot\|_{\mathcal{H}_1^\ell}$ respectively. Here and below we use the notation

$$\nabla\boldsymbol{u}^\ell=(u_{i,j}^\ell),\ \varepsilon(\boldsymbol{u}^\ell)=(\varepsilon_{ij}(\boldsymbol{u}^\ell)),\quad \varepsilon_{ij}(\boldsymbol{u}^\ell)=\frac{1}{2}(u_{i,j}^\ell+u_{j,i}^\ell),\quad \forall u^\ell\in H_1^\ell,$$

$$\operatorname{Div}\boldsymbol{\sigma}^\ell=(\sigma_{ij,j}^\ell),\quad \forall\boldsymbol{\sigma}^\ell\in\mathcal{H}_1^\ell.$$

For every element $\boldsymbol{v}^\ell\in H_1^\ell$, we also use the notation $\boldsymbol{v}^\ell$ for the trace of $\boldsymbol{v}^\ell$ on Γ^ℓ and we denote by v_ν^ℓ and $\boldsymbol{v}_\tau^\ell$ the *normal* and the *tangential* components of $\boldsymbol{v}^\ell$ on the boundary Γ^ℓ given by

$$v_\nu^\ell=\boldsymbol{v}^\ell\cdot\nu^\ell,\quad \boldsymbol{v}_\tau^\ell=\boldsymbol{v}^\ell-v_\nu^\ell\boldsymbol{\nu}^\ell.$$

Let H'_{Γ^ℓ} be the dual of $H_{\Gamma^\ell}=H^{\frac{1}{2}}(\Gamma^\ell)^d$ and let $(\cdot,\cdot)_{-\frac{1}{2},\frac{1}{2},\Gamma^\ell}$ denote the duality pairing between H'_{Γ^ℓ} and H_{Γ^ℓ}. For every element $\boldsymbol{\sigma}^\ell\in\mathcal{H}_1^\ell$ let $\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell$ be the element of H'_{Γ^ℓ} given by

$$(\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell,\boldsymbol{v}^\ell)_{-\frac{1}{2},\frac{1}{2},\Gamma^\ell}=(\boldsymbol{\sigma}^\ell,\varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell}+(\operatorname{Div}\boldsymbol{\sigma}^\ell,\boldsymbol{v}^\ell)_{H^\ell}\quad \forall\boldsymbol{v}^\ell\in H_1^\ell.$$

Denote by σ_ν^ℓ and $\boldsymbol{\sigma}_\tau^\ell$ the *normal* and the *tangential* traces of $\boldsymbol{\sigma}^\ell\in\mathcal{H}_1^\ell$, respectively. If $\boldsymbol{\sigma}^\ell$ is continuously differentiable on $\Omega^\ell\cup\Gamma^\ell$, then

$$\sigma_\nu^\ell=(\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell).\boldsymbol{\nu}^\ell,\quad \boldsymbol{\sigma}_\tau^\ell=\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell-\sigma_\nu^\ell\boldsymbol{\nu}^\ell,$$

$$(\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell,\boldsymbol{v}^\ell)_{-\frac{1}{2},\frac{1}{2},\Gamma^\ell}=\int_{\Gamma^\ell}\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell.\boldsymbol{v}^\ell da$$

fore all $\boldsymbol{v}^\ell\in H_1^\ell$, where da is the surface measure element.

To obtain the variational formulation of the problem (10)-(24), we introduce for the bonding field the set

$$\mathcal{Z}=\{\alpha\in L^\infty(0,T;L^2(\Gamma_3));\ 0\le\alpha(t)\le 1\ \ \forall t\in[0,T],\ \text{a.e. on }\Gamma_3\},$$

and for the displacement field we need the closed subspace of H_1^ℓ defined by

$$V^\ell=\left\{\boldsymbol{v}^\ell\in H_1^\ell;\ \boldsymbol{v}^\ell=0\text{ on }\Gamma_1^\ell\right\}.$$

Since $meas\Gamma_1^\ell>0$, the following Korn's inequality holds :

$$\|\varepsilon(\boldsymbol{v}^\ell)\|_{\mathcal{H}^\ell}\ge c_K\|\boldsymbol{v}^\ell\|_{H_1^\ell}\quad \forall\boldsymbol{v}^\ell\in V^\ell, \tag{2}$$

where the constant c_K denotes a positive constant which may depends only on Ω^ℓ, Γ_1^ℓ (see [21]). Over the space V^ℓ we consider the inner product given by

$$(\boldsymbol{u}^\ell,\boldsymbol{v}^\ell)_{V^\ell}=(\varepsilon(\boldsymbol{u}^\ell),\varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell},\quad \forall\boldsymbol{u}^\ell,\boldsymbol{v}^\ell\in V^\ell, \tag{3}$$

and let $\|\cdot\|_{V^\ell}$ be the associated norm. It follows from Korn's inequality (2) that the norms $\|\cdot\|_{H_1^\ell}$ and $\|\cdot\|_{V^\ell}$ are equivalent on V^ℓ. Then $(V^\ell,\|\cdot\|_{V^\ell})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3), there exists a constant $c_0>0$, depending only on Ω^ℓ, Γ_1^ℓ and Γ_3 such that

$$\|\boldsymbol{v}^\ell\|_{L^2(\Gamma_3)^d}\le c_0\|\boldsymbol{v}^\ell\|_{V^\ell}\quad \forall\boldsymbol{v}^\ell\in V^\ell. \tag{4}$$

We also introduce the spaces

$$L_0^\ell=L^2(\Omega^\ell),\ L_1^\ell=H^1(\Omega^\ell),\ W^\ell=\left\{\psi^\ell\in L_1^\ell;\ \psi^\ell=0\text{ on }\Gamma_a^\ell\right\},$$

$$\mathcal{W}^\ell = \left\{ \boldsymbol{D}^\ell = (D_i^\ell);\ D_i^\ell \in L^2(\Omega^\ell),\ \mathrm{div}\boldsymbol{D}^\ell \in L^2(\Omega^\ell) \right\}.$$

Since $meas\Gamma_a^\ell > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\psi^\ell\|_{L^2(\Omega^\ell)^d} \geq c_F \|\psi^\ell\|_{H^1(\Omega^\ell)} \quad \forall \psi^\ell \in W^\ell, \tag{5}$$

where $c_F > 0$ is a constant which depends only on Ω^ℓ, Γ_a^ℓ.

Over the space W^ℓ, we consider the inner product given by

$$(\xi^\ell, \psi^\ell)_{W^\ell} = \int_{\Omega^\ell} \nabla\xi^\ell . \nabla\psi^\ell dx$$

and let $\|\cdot\|_{W^\ell}$ be the associated norm. It follows from (5) that $\|\cdot\|_{H^1(\Omega^\ell)}$ and $\|\cdot\|_{W^\ell}$ are equivalent norms on W^ℓ and therefore $(W^\ell, \|\cdot\|_{W^\ell})$ is areal Hilbert space. On the space $\mathcal{W}^\ell$, we use the inner product

$$(\boldsymbol{D}^\ell, \boldsymbol{\Psi}^\ell)_{\mathcal{W}^\ell} = \int_{\Omega^\ell} \boldsymbol{D}^\ell . \boldsymbol{\Psi}^\ell dx + \int_{\Omega^\ell} \mathrm{div}\boldsymbol{D}^\ell . \mathrm{div}\boldsymbol{\Psi}^\ell dx,$$

where $\mathrm{div}\boldsymbol{D}^\ell = (\boldsymbol{D}_{i,i}^\ell)$, and the associated norm $\|\cdot\|_{\mathcal{W}^\ell}$.

In order to simplify the notations, we define the product spaces

$$\boldsymbol{V} = V^1 \times V^2,\ \ H = H^1 \times H^2,\ \ H_1 = H_1^1 \times H_1^2,\ \ \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2,$$

$$\mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2, L_0 = L_0^1 \times L_0^2, L_1 = L_1^1 \times L_1^2, W = W^1 \times W^2, \mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2.$$

The spaces $\boldsymbol{V}$, L_1, W and $\mathcal{W}$ are real Hilbert spaces endowed with the canonical inner products denoted by $(\cdot,\cdot)_{\boldsymbol{V}}$, $(\cdot,\cdot)_{L_1}$, $(\cdot,\cdot)_W$, and $(\cdot,\cdot)_{\mathcal{W}}$. The associate norms will be denoted by $\|\cdot\|_{\boldsymbol{V}}$, $\|\cdot\|_{L_1}$, $\|\cdot\|_W$ and $\|\cdot\|_{\mathcal{W}}$, respectively.

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$, $W^{k,p}(0,T;X)$, where $1 \leq p \leq \infty$, $k \geq 1$. We denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\|f\|_{C(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X,$$

$$\|f\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X + \max_{t\in[0,T]} \|\dot{f}(t)\|_X,$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable and, for areal number r, we use r_+ to represent its positive part, that is $r_+ = \max\{0,r\}$. For the convenience of the reader, we recall the following version of the classical theorem of Cauchy-Lipschitz (see, [26, p.48]).

Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(\cdot,\cdot)_{X_1\times X_2}$.

3. Mechanical and Variational Formulations

We are interested in the deformation of the body on the time interval $[0,T]$. The process is assumed to be electrically static, i.e., all radiation effects are neglected, and mechanically dynamic. We describe the model for the process, we present its variational formulation. The physical setting is the following: Let us consider two electro-elastic-viscoplastics bodies, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d (d = 2,3)$. For each domain Ω^ℓ, the boundary Γ^ℓ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^ℓ, Γ_2^ℓ and Γ_3^ℓ, on one hand, and on two measurable parts Γ_a^ℓ and Γ_b^ℓ, on the other hand, such that $meas\Gamma_1^\ell > 0$, $meas\Gamma_a^\ell > 0$. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The Ω^ℓ body is submitted to $\boldsymbol{f}_0^\ell$ forces and volume electric charges of density q_0^ℓ. The bodies are assumed to be clamped on $\Gamma_1^\ell \times (0,T)$, so the displacement field vanishes there. The surface tractions $\boldsymbol{f}_2^\ell$ act on $\Gamma_2^\ell \times (0,T)$. We also assume that the electrical potential vanishes on $\Gamma_a^\ell \times (0,T)$ and a surface electric charge of density q_2^ℓ is prescribed on $\Gamma_b^\ell \times (0,T)$. The two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$, the process is dynamic, and thus the inertial terms are included in the equation of motion. The bodies is in adhesive contact with an obstacle, over the contact surface Γ_3. We denote by $\boldsymbol{u}^\ell$ the displacement field, by $\boldsymbol{\sigma}^\ell$ the stress field and by $\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell)$ the linearized strain tensor. We use an electro-elastic-viscoplastic constitutive law with damage given by

$$\begin{cases} \boldsymbol{\sigma}^\ell = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) + \mathcal{B}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell) + (\mathcal{E}^\ell)^* \nabla \xi^\ell \\ \qquad + \int_0^t \mathcal{G}^\ell(\boldsymbol{\sigma}^\ell(s) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(s)) - (\mathcal{E}^\ell)^* \nabla \xi^\ell(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(s)), \varsigma^\ell(s))\, ds \\ \boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) - \boldsymbol{\beta}^\ell \nabla \xi^\ell \end{cases} \tag{6}$$

where $\boldsymbol{u}^\ell$ the displacement field, $\boldsymbol{\sigma}^\ell$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell)$ represent the stress and the linearized strain tensor, respectively, $\boldsymbol{D}^\ell$ is the electric displacement vector. Here $\mathcal{A}^\ell$ and $\mathcal{B}^\ell$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, $\mathcal{G}^\ell$ is a nonlinear constitutive function describing the viscoplastic behaviour of the material. We also consider that the viscoplastic function $\mathcal{G}^\ell$ depends on the internal state variable ς^ℓ describing the damage of the material caused by plastic deformations. $\boldsymbol{\beta}^\ell$ denotes the electric permittivity tensor, $E(\xi^\ell) = -\nabla \xi^\ell$ is the electric field, $\mathcal{E}^\ell = (e_{ijk})$ represents the third order piezoelectric tensor, $(\mathcal{E}^\ell)^*$ is its transposition. In (6) and everywhere in this paper the dot above a variable represents derivative with respect to the time variable t. It follows from (6) that at each time moment, the stress tensor $\boldsymbol{\sigma}^\ell(t)$ is split into three parts: $\boldsymbol{\sigma}^\ell(t) = \boldsymbol{\sigma}_V^\ell(t) + \boldsymbol{\sigma}_E^\ell(t) + \boldsymbol{\sigma}_R^\ell(t)$, where $\boldsymbol{\sigma}_V^\ell(t) = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t))$ represents the purely viscous part of the stress, $\boldsymbol{\sigma}_E^\ell(t) = (\mathcal{E}^\ell)^* \nabla \xi^\ell(t)$ represents the electric part of the stress and $\boldsymbol{\sigma}_R^\ell(t)$ satisfies a rate-type elastic-viscoplastic relation

$$\boldsymbol{\sigma}_R^\ell(t) = \mathcal{B}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)) + \int_0^t \mathcal{G}^\ell\big(\boldsymbol{\sigma}_R^\ell(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(s)), \varsigma^\ell(s)\big)\, ds. \tag{7}$$

Various results, examples and mechanical interpretations in the study of elastic-viscoplastic materials of the form (7) can be found in [8, 16] and references therein. Note also that when $\mathcal{G}^\ell = 0$ the constitutive law (6) becomes the Kelvin-Voigt electro-viscoelastic constitutive relation,

$$\boldsymbol{\sigma}^\ell(t) = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)) + \mathcal{B}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)) + (\mathcal{E}^\ell)^* \nabla \xi^\ell(t). \tag{8}$$

Dynamic contact problems with Kelvin-Voigt materials of the form (8) can be found in [9, 25, 29]. The differential inclusion used for the evolution of the damage field is

$$\dot{\varsigma}^\ell - \kappa^\ell \Delta \varsigma^\ell + \partial \psi_{K^\ell}(\varsigma^\ell) \ni \phi^\ell\big(\boldsymbol{\sigma}^\ell - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) - (\mathcal{E}^\ell)^* \nabla \xi^\ell, \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \varsigma^\ell\big),$$

where K^ℓ denotes the set of admissible damage functions defined by

$$K^\ell = \{\omega \in H^1(\Omega^\ell);\ 0 \le \omega \le 1, \text{ a.e. in } \Omega^\ell\}, \tag{9}$$

κ^ℓ is a positive coefficient, $\partial\psi_{K^\ell}$ represents the subdifferential of the indicator function of the set K^ℓ and ϕ^ℓ is a given constitutive function which describes the sources of the damage in the system. General models for damage were derived in [12,13] from the virtual power principle. Mathematical analysis of one-dimensional problems can be found in [11]. The three-dimensional case has been investigated in [18]. In all these papers the damage of the material is described with a damage function ς^ℓ, restricted to have values between zero and one. When $\varsigma^\ell = 1$, there is no damage in the material, when $\varsigma^\ell = 0$, the material is completely damaged, when $0 < \varsigma^\ell < 1$ there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [11,22,23,26].

With these assumptions, the classical formulation of the dynamic problem for frictionless contact problem with normal compliance and adhesion between two electro-elastic-viscoplastics bodies with damage is the following.

Problem P. For $\ell = 1,2$, find a displacement field $\boldsymbol{u}^\ell : \Omega^\ell \times (0,T) \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\ell : \Omega^\ell \times (0,T) \to \mathbb{S}^d$, an electric potential field $\xi^\ell : \Omega^\ell \times (0,T) \to \mathbb{R}$, a damage field $\varsigma^\ell : \Omega^\ell \times (0,T) \to \mathbb{R}$, a bonding field $\zeta : \Gamma_3 \times (0,T) \to \mathbb{R}$ and a electric displacement field $\boldsymbol{D}^\ell : \Omega^\ell \times (0,T) \to \mathbb{R}^d$ such that

$$\begin{aligned}\boldsymbol{\sigma}^\ell &= \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) + \mathcal{B}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell) + (\mathcal{E}^\ell)^* \nabla \xi^\ell \\ &\quad + \int_0^t \mathcal{G}^\ell\Big(\boldsymbol{\sigma}^\ell(s) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(s)) - (\mathcal{E}^\ell)^* \nabla \xi^\ell(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(s)), \varsigma^\ell(s)\Big)\, ds \quad \text{in } \Omega^\ell \times (0,T),\end{aligned} \tag{10}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) - \boldsymbol{\beta}^\ell \nabla \xi^\ell \quad \text{in } \Omega^\ell \times (0,T), \tag{11}$$

$$\dot{\varsigma}^\ell - \kappa^\ell \Delta \varsigma^\ell + \partial\psi_{K^\ell}(\varsigma^\ell) \ni \phi^\ell\big(\boldsymbol{\sigma}^\ell - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) - (\mathcal{E}^\ell)^* \nabla \xi^\ell(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \varsigma^\ell\big) \quad \text{in } \Omega^\ell \times (0,T), \tag{12}$$

$$\rho^\ell \ddot{\boldsymbol{u}}^\ell = \operatorname{Div} \boldsymbol{\sigma}^\ell + \boldsymbol{f}_0^\ell \quad \text{in } \Omega^\ell \times (0,T), \tag{13}$$

$$\operatorname{div} \boldsymbol{D}^\ell - q_0^\ell = 0 \quad \text{in } \Omega^\ell \times (0,T), \tag{14}$$

$$\boldsymbol{u}^\ell = 0 \quad \text{on } \Gamma_1^\ell \times (0,T), \tag{15}$$

$$\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell = \boldsymbol{f}_2^\ell \quad \text{on } \Gamma_2^\ell \times (0,T), \tag{16}$$

$$\dot{\zeta} = H_{ad}(\zeta, \alpha_\zeta, R_\nu([u_\nu]), \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])) \quad \text{on } \Gamma_3 \times (0,T), \tag{17}$$

$$\begin{cases} \sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu, \\ \sigma_\nu = -p_\nu([u_\nu]) + \gamma_\nu \zeta^2 R_\nu([u_\nu]) \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{18}$$

$$\begin{cases} \boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau, \\ \boldsymbol{\sigma}_\tau = p_\tau(\zeta) \boldsymbol{R}_\tau([\boldsymbol{u}_\tau]) \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{19}$$

$$\frac{\partial \varsigma^\ell}{\partial \nu^\ell} = 0 \quad \text{on } \Gamma^\ell \times (0,T), \tag{20}$$

$$\xi^\ell = 0 \quad \text{on } \Gamma_a^\ell \times (0,T), \tag{21}$$

$$\boldsymbol{D}^\ell . \boldsymbol{\nu}^\ell = q_2^\ell \quad \text{on } \Gamma_b^\ell \times (0,T), \tag{22}$$

$$\boldsymbol{u}^\ell(0) = \boldsymbol{u}_0^\ell, \quad \dot{\boldsymbol{u}}^\ell(0) = \boldsymbol{v}_0^\ell, \quad \varsigma^\ell(0) = \varsigma_0^\ell \quad \text{in } \Omega^\ell, \tag{23}$$

$$\zeta(0) = \zeta_0 \quad \text{on } \Gamma_3. \tag{24}$$

First, equations (10) and (11) represent the electro-elastic-viscoplastic constitutive law with damage of the material. The evolution of the damage field is governed by the inclusion of parabolic type given by the relation (12). Equations (13) and (14) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which "Div" and "div" denote the divergence operator for tensor and vector valued functions, respectively. Next, the equations (15) and (16) represent the displacement and traction boundary condition, respectively. Equation (17) represents the ordinary differential equation which describes the evolution of the bonding field. Condition (18) represents the normal compliance conditions with adhesion where γ_ν is a given adhesion coefficient, the adhesive traction is tensile and is proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu \beta^2 L$. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0. \end{cases} \tag{25}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of the operator R_ν, together with the operator $\boldsymbol{R}_\tau$ defined below, is motivated by mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows. Condition (19) represents the adhesive contact condition on the tangential plane, where $[\boldsymbol{u}_\tau] = \boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2$ stands for the jump of the displacements in tangential direction. $\boldsymbol{R}_\tau$ is the truncation operator given by

$$\boldsymbol{R}_\tau(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } |\boldsymbol{v}| \le L, \\ L\frac{\boldsymbol{v}}{|\boldsymbol{v}|} & \text{if } |\boldsymbol{v}| > L. \end{cases} \tag{26}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L. The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted. Boundary condition (20) describes a homogeneous Neumann boundary condition where $\frac{\partial \varsigma^\ell}{\partial \nu^\ell}$ is the normal derivative of ς^ℓ. (21) and (22) represent the electric boundary conditions. (23) represents the initial displacement field, the initial velocity and the initial damage. Finally (24) represents the initial condition in which ζ_0 is the given initial bonding field.

In the study of the mechanical problem (10)–(24), we make the following assumptions.

We assume that the *viscosity operator* $\mathcal{A}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\left\{\begin{array}{l} \text{(a) There exists } C^1_{\mathcal{A}^\ell}, C^2_{\mathcal{A}^\ell} > 0 \text{ such that,} \\ \qquad |\mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\omega})| \le C^1_{\mathcal{A}^\ell}|\boldsymbol{\omega}| + C^2_{\mathcal{A}^\ell} \quad \forall\, \boldsymbol{\omega} \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) There exists } m_{\mathcal{A}^\ell} > 0 \text{ such that} \\ \qquad (\mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\omega}_1) - \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\omega}_2)) \cdot (\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \ge m_{\mathcal{A}^\ell}|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2|^2 \\ \qquad \forall\, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\omega}) \text{ is Lebesgue measurable on } \Omega^\ell, \\ \qquad \text{for any } \boldsymbol{\omega} \in \mathbb{S}^d. \\ \text{(d) The mapping } \boldsymbol{\omega} \mapsto \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\omega}) \text{ is continuous on } \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \end{array}\right. \tag{27}$$

The *elasticity operator* $\mathcal{B}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\left\{\begin{array}{l} \text{(a) There exists } L_{\mathcal{B}^\ell} > 0 \text{ such that} \\ \qquad |\mathcal{B}^\ell(\boldsymbol{x},\boldsymbol{\omega}_1) - \mathcal{B}^\ell(\boldsymbol{x},\boldsymbol{\omega}_2)| \le L_{\mathcal{B}^\ell}|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2| \\ \qquad \forall\, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{B}^\ell(\boldsymbol{x},\boldsymbol{\omega}) \text{ is Lebesgue measurable on } \Omega^\ell, \\ \qquad \text{for any } \boldsymbol{\omega} \in \mathbb{S}^d. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{B}^\ell(\boldsymbol{x},\boldsymbol{0}) \text{ belongs to } \mathcal{H}^\ell. \end{array}\right. \tag{28}$$

The *viscoplasticity operator* $\mathcal{G}^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

$$\left\{\begin{array}{l} \text{(a) There exists } L_{\mathcal{G}^\ell} > 0 \text{ such that} \\ \qquad |\mathcal{G}^\ell(\boldsymbol{x},\boldsymbol{\eta}_1,\boldsymbol{\omega}_1,d_1) - \mathcal{G}^\ell(\boldsymbol{x},\boldsymbol{\eta}_2,\boldsymbol{\omega}_2,d_2)| \le L_{\mathcal{G}^\ell}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2| \\ \qquad + |d_1 - d_2|\big), \quad \forall\, \boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\omega}_1,\boldsymbol{\omega}_2 \in \mathbb{S}^d, \ \forall d_1, d_2 \in \mathbb{R}, \ \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{G}^\ell(\boldsymbol{x},\boldsymbol{\eta},\boldsymbol{\omega},d) \text{ is Lebesgue measurable in } \Omega^\ell, \\ \qquad \text{for any } \boldsymbol{\eta},\boldsymbol{\omega} \in \mathbb{S}^d, \ \ d \in \mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}^\ell(\boldsymbol{x},\boldsymbol{0},\boldsymbol{0},0) \text{ belongs to } \mathcal{H}^\ell. \end{array}\right. \tag{29}$$

The *adhesion rate function* $H_{ad} : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}$ satisfies:

$$\left\{\begin{array}{l} \text{(a) There exists } L_{ad} > 0 \text{ such that} : \forall\, \zeta_1,\zeta_2,\omega_1,\omega_2,r_1,r_2 \in \mathbb{R}, d_1, d_2 \in \mathbb{R}^{d-1}, \\ \quad |H_{ad}(\boldsymbol{x},\zeta_1,\omega_1,r_1,d_1) - H_{ad}(\boldsymbol{x},\zeta_2,\omega_2,r_2,d_2)| \le L_\nu|\zeta_1 - \zeta_2| + |\omega_1 - \omega_2| + \\ \quad |r_1 - r_2| + |d_1 - d_2|, \ \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto H_{ad}(\boldsymbol{x},\zeta,\omega,r,d) \text{ is measurable on } \Gamma_3, \\ \quad \text{for any } \zeta,\omega,r \in \mathbb{R}, \ d \in \mathbb{R}^{d-1}, \\ \text{(c) The mapping } (\zeta,\omega,r,d) \mapsto H_{ad}(\boldsymbol{x},\zeta,\omega,r,d) \text{ is continuous on} \\ \quad \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \\ \text{(d) } H_{ad}(\boldsymbol{x},0,\omega,r,d) = 0, \forall\, \omega, r \in \mathbb{R}, \ d \in \mathbb{R}^{d-1}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \\ \text{(e) } H_{ad}(\boldsymbol{x},\zeta,\omega,r,d) \ge 0, \quad \forall\, \zeta \le 0, \omega, r \in \mathbb{R}, \ d \in \mathbb{R}^{d-1}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \text{ and} \\ \quad H_{ad}(\boldsymbol{x},\zeta,\omega,r,d) \le 0, \quad \forall\, \zeta \ge 1, \omega, r \in \mathbb{R}, \ d \in \mathbb{R}^{d-1}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{array}\right. \tag{30}$$

The *damage source function* $\phi^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$\left\{\begin{array}{l} \text{(a) There exists } L_{\phi^\ell} > 0 \text{ such that} \\ \qquad |\phi^\ell(\boldsymbol{x},\boldsymbol{\eta}_1,\boldsymbol{\omega}_1,\alpha_1) - \phi^\ell(\boldsymbol{x},\boldsymbol{\eta}_2,\boldsymbol{\omega}_2,\alpha_2)| \le L_{\phi^\ell}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2| \\ \qquad + |\alpha_1 - \alpha_2|\big), \quad \forall\, \boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\omega}_1,\boldsymbol{\omega}_2 \in \mathbb{S}^d \text{ and } \alpha_1,\alpha_2 \in \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \phi^\ell(\boldsymbol{x},\boldsymbol{\eta},\boldsymbol{\omega},\alpha) \text{ is Lebesgue measurable on } \Omega^\ell, \\ \qquad \text{for any } \boldsymbol{\eta},\boldsymbol{\omega} \in \mathbb{S}^d \text{ and } \alpha \in \mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \phi^\ell(\boldsymbol{x},\boldsymbol{0},\boldsymbol{0},0) \text{ belongs to } L^2(\Omega^\ell). \end{array}\right. \tag{31}$$

The *piezoelectric tensor* $\mathcal{E}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies:

$$\begin{cases} \text{(a) } \mathcal{E}^\ell(\boldsymbol{x},\tau) = (e^\ell_{ijk}(\boldsymbol{x})\tau_{jk}), \quad \forall \tau = (\tau_{ij}) \in \mathbb{S}^d \ \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) } e^\ell_{ijk} = e^\ell_{ikj} \in L^\infty(\Omega^\ell),\ 1 \le i,j,k \le d. \end{cases} \tag{32}$$

Recall also that the transposed operator $(\mathcal{E}^\ell)^*$ is given by $(\mathcal{E}^\ell)^* = (e^{\ell,*}_{ijk})$ where $e^{\ell,*}_{ijk} = e^\ell_{kij}$ and the following equality hold

$$\mathcal{E}^\ell \sigma.\boldsymbol{v} = \sigma.(\mathcal{E}^\ell)^*\boldsymbol{v} \quad \forall \sigma \in \mathbb{S}^d,\ \ \forall \boldsymbol{v} \in \mathbb{R}^d.$$

The *electric permittivity operator* $\boldsymbol{\beta}^\ell = (\beta^\ell_{ij}) : \Omega^\ell \times \mathbb{R}^d \to \mathbb{R}^d$ verifies:

$$\begin{cases} \text{(a) } \boldsymbol{\beta}^\ell(\boldsymbol{x},\mathbf{E}) = (\beta^\ell_{ij}(\boldsymbol{x})E_j) \quad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d,\ \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) } \beta^\ell_{ij} = \beta^\ell_{ji},\ \beta^\ell_{ij} \in L^\infty(\Omega^\ell), \quad 1 \le i,j \le d. \\ \text{(c) There exists } m_{\boldsymbol{\beta}^\ell} > 0 \text{ such that } \boldsymbol{\beta}^\ell\mathbf{E}.\mathbf{E} \ge m_{\boldsymbol{\beta}^\ell}|\mathbf{E}|^2 \\ \qquad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \end{cases} \tag{33}$$

The *normal compliance functions* $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

$$\begin{cases} \text{(a) } \exists L_\nu > 0 \text{ such that } |p_\nu(\boldsymbol{x},r_1) - p_\nu(\boldsymbol{x},r_2)| \le L_\nu|r_1 - r_2| \\ \qquad \forall r_1,r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x},r) \text{ is measurable on } \Gamma_3,\ \forall r \in \mathbb{R}. \\ \text{(c) } p_\nu(\boldsymbol{x},r) = 0,\ \text{ for all } r \le 0,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{34}$$

The *tangential compliance functions* $p_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

$$\begin{cases} \text{(a) } \exists L_\tau > 0 \text{ such that } |p_\tau(\boldsymbol{x},d_1) - p_\tau(\boldsymbol{x},d_2)| \le L_\tau|d_1 - d_2| \\ \qquad \forall d_1,d_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) } \exists M_\tau > 0 \text{ such that } |p_\tau(\boldsymbol{x},d)| \le M_\tau \ \forall d \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto p_\tau(\boldsymbol{x},d) \text{ is measurable on } \Gamma_3,\ \forall d \in \mathbb{R}. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto p_\tau(\boldsymbol{x},0) \in L^2(\Gamma_3). \end{cases} \tag{35}$$

We suppose that the mass density satisfies

$$\rho^\ell \in L^\infty(\Omega^\ell) \text{ and } \exists \rho_0 > 0 \text{ such that } \rho^\ell(x) \ge \rho_0 \ \text{ a.e. } x \in \Omega^\ell,\ \ell = 1,2. \tag{36}$$

The following regularity is assumed on the density of volume forces, traction, volume electric charges and surface electric charges:

$$\begin{aligned} &\mathbf{f}^\ell_0 \in L^2(0,T;L^2(\Omega^\ell)^d), \quad \mathbf{f}^\ell_2 \in L^2(0,T;L^2(\Gamma^\ell_2)^d), \\ &\quad q^\ell_0 \in C(0,T;L^2(\Omega^\ell)), \quad q^\ell_2 \in C(0,T;L^2(\Gamma^\ell_b)), \end{aligned} \tag{37}$$

$$q^\ell_2(t) = 0 \quad \text{on } \ \Gamma_3 \ \ \forall t \in [0,T]. \tag{38}$$

The adhesion coefficient γ_ν and the limit bound satisfy:

$$\gamma_\nu \in L^\infty(\Gamma_3),\ \gamma_\nu \ge 0,\ \text{a.e. on } \Gamma_3. \tag{39}$$

The microcrack diffusion coefficient verifies

$$\kappa^\ell > 0, \tag{40}$$

and, finally, the initial data satisfy

$$\begin{aligned} &\boldsymbol{u}^\ell_0 \in \boldsymbol{V}^\ell, \quad \boldsymbol{v}^\ell_0 \in H^\ell, \quad \varsigma^\ell_0 \in K^\ell, \quad \ell = 1,2, \\ &\zeta_0 \in L^2(\Gamma_3), \quad 0 \le \zeta_0 \le 1,\ \text{a.e. on } \Gamma_3. \end{aligned} \tag{41}$$

where K^ℓ is the set of admissible damage functions defined in (9).

Let $a : L_1 \times L_1 \to \mathbb{R}$, be the bilinear form

$$a(\varsigma,\omega) = \sum_{\ell=1}^{2} \kappa^\ell \int_{\Omega^\ell} \nabla\varsigma^\ell.\nabla\omega^\ell dx. \tag{42}$$

We will use a modified inner product on H, given by

$$((\boldsymbol{u},\boldsymbol{v}))_H = \sum_{\ell=1}^{2} (\rho^\ell \boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H^\ell}, \quad \forall \boldsymbol{u},\boldsymbol{v} \in H,$$

that is, it is weighted with ρ^ℓ, and we let $|||.|||_H$ be the associated norm, i.e.,

$$|||\boldsymbol{v}|||_H = ((\boldsymbol{v},\boldsymbol{v}))_H^{\frac{1}{2}}, \quad \forall v \in H.$$

It follows from assumption (36) that $|||.|||_H$ and $\|\cdot\|_H$ are equivalent norms on H, and the inclusion mapping of $(\boldsymbol{V},\|\cdot\|_{\boldsymbol{V}})$ into $(H,|||\cdot|||_H)$ is continuous and dense. We denote by $\boldsymbol{V}'$ the dual of $\boldsymbol{V}$. Identifying H with its own dual, we can write the Gelfand triple

$$\boldsymbol{V} \subset H \subset \boldsymbol{V}'.$$

Using the notation $(\cdot,\cdot)_{\boldsymbol{V}'\times\boldsymbol{V}}$ to represent the duality pairing between $\boldsymbol{V}'$ and $\boldsymbol{V}$ we have

$$(\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = ((\boldsymbol{u},\boldsymbol{v}))_H, \quad \forall \boldsymbol{u} \in H, \forall \boldsymbol{v} \in \boldsymbol{V}.$$

Finally, we denote by $\|\cdot\|_{\boldsymbol{V}'}$ the norm on $\boldsymbol{V}'$. Using the Riesz representation theorem, we define the linear mappings $\mathbf{f} : [0,T] \to \boldsymbol{V}'$ and $q : [0,T] \to W$ as follows:

$$(\mathbf{f}(t),\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\ell=1}^{2} \int_{\Omega^\ell} \mathbf{f}_0^\ell(t)\cdot\boldsymbol{v}^\ell\, dx + \sum_{\ell=1}^{2} \int_{\Gamma_2^\ell} \mathbf{f}_2^\ell(t)\cdot\boldsymbol{v}^\ell\, da \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{43}$$

$$(q(t),\varsigma)_W = \sum_{\ell=1}^{2} \int_{\Omega^\ell} q_0^\ell(t)\varsigma^\ell\, dx - \sum_{\ell=1}^{2} \int_{\Gamma_b^\ell} q_2^\ell(t)\varsigma^\ell\, da \quad \forall \varsigma \in W. \tag{44}$$

Next, we denote by $j_{ad} : L^\infty(\Gamma_3) \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ the adhesion functional defined by

$$j_{ad}(\zeta,\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_3} \Big(-\gamma_\nu \zeta^2 R_\nu([u_\nu])[v_\nu] + p_\tau(\zeta)\boldsymbol{R}_\tau([\boldsymbol{u}_\tau])[\boldsymbol{v}_\tau] \Big)\, da. \tag{45}$$

In addition to the functional (45), we need the normal compliance functional

$$j_{\nu c}(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_3} p_\nu([u_\nu])[v_\nu]\, da. \tag{46}$$

Keeping in mind (34) and (35), we observe that the integrals (45) and (46) are well defined and we note that conditions (37) imply

$$\mathbf{f} \in L^2(0,T;\boldsymbol{V}'), \quad q \in C(0,T;W). \tag{47}$$

By a standard procedure based on Green's formula, we derive the following variational formulation of the mechanical (10)–(24).

Problem PV. Find a displacement field $\boldsymbol{u} = (\boldsymbol{u}^1,\boldsymbol{u}^2) : [0,T] \to \boldsymbol{V}$, a stress field $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2) : [0,T] \to \mathcal{H}$, an electric potential field $\xi = (\xi^1,\xi^2) : [0,T] \to W$, a damage field $\varsigma = (\varsigma^1,\varsigma^2) : [0,T] \to L_1$, a bonding field $\zeta : [0,T] \to L^\infty(\Gamma_3)$ and a electric displacement field $\boldsymbol{D} = (\boldsymbol{D}^1,\boldsymbol{D}^2) : [0,T] \to \mathcal{W}$

such that

$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) + \mathcal{B}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell) + (\mathcal{E}^\ell)^* \nabla \xi^\ell + \int_0^t \mathcal{G}^\ell\Big(\boldsymbol{\sigma}^\ell(s) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(s)) - (\mathcal{E}^\ell)^* \nabla \xi^\ell(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(s)), \varsigma^\ell(s)\Big)\, ds, \quad \text{a.e. } (0,T), \tag{48}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) - \boldsymbol{\beta}^\ell \nabla \xi^\ell, \quad \text{a.e. } (0,T), \tag{49}$$

$$(\ddot{u}, v)_{\boldsymbol{V}' \times \boldsymbol{V}} + \sum_{\ell=1}^{2} (\boldsymbol{\sigma}^\ell,\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + j_{ad}(\zeta(t), \boldsymbol{u}(t), \boldsymbol{v}) + j_{vc}(\boldsymbol{u}(t), \boldsymbol{v}) = (\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}' \times \boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{V},\ \text{a.e. } t \in (0,T), \tag{50}$$

$$\varsigma(t) \in K, \quad \sum_{\ell=1}^{2} (\dot{\varsigma}^\ell(t), \omega^\ell - \varsigma^\ell(t))_{L^2(\Omega^\ell)} + a(\varsigma(t), \omega - \varsigma(t)) \geq \sum_{\ell=1}^{2} \Big(\phi^\ell\Big(\boldsymbol{\sigma}^\ell(t) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)) - (\mathcal{E}^\ell)^* \nabla \xi^\ell(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)), \varsigma^\ell(t)\Big), \omega^\ell - \varsigma^\ell(t)\Big)_{L^2(\Omega^\ell)}, \quad \forall \omega \in K,\ \text{a.e. } t \in (0,T), \tag{51}$$

$$\sum_{\ell=1}^{2} (\boldsymbol{\beta}^\ell \nabla \xi^\ell(t), \nabla \phi^\ell)_{H^\ell} - \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell(t)), \nabla \phi^\ell)_{H^\ell} = (q(t), \phi)_W, \quad \forall \phi \in W,\ \text{a.e. } t \in (0,T), \tag{52}$$

$$\dot{\zeta} = H_{ad}(\zeta, \alpha_\zeta, R_\nu([u_\nu]), \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])) \quad \text{a.e. } t \in (0,T), \tag{53}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \quad \varsigma(0) = \varsigma_0, \quad \zeta(0) = \zeta_0. \tag{54}$$

We notice that the variational Problem **PV** is formulated in terms of a displacement field, a stress field, an electrical potential field, a bonding field and a electric displacement field. The existence of the unique solution of Problem **PV** is stated and proved in the next section.

Remark 3.1. We note that, in Problem **P** and in Problem **PV**, we do not need to impose explicitly the restriction $0 \leq \zeta \leq 1$. Indeed, equation (53) guarantees that $\zeta(x,t) \leq \zeta_0(x)$ and, therefore, assumption (41) shows that $\zeta(x,t) \leq 1$ for $t \geq 0$, a.e. $x \in \Gamma_3$. On the other hand, if $\zeta(x,t_0) = 0$ at time t_0, then it follows from (53) that $\dot{\zeta}(x,t) = 0$ for all $t \geq t_0$ and therefore, $\zeta(x,t) = 0$ for all $t \geq t_0$, a.e. $x \in \Gamma_3$. We conclude that $0 \leq \zeta(x,t) \leq 1$ for all $t \in [0,T]$, a.e. $x \in \Gamma_3$.

4. Existence and Uniqueness Result

Now, we propose our existence and uniqueness result

Theorem 4.1. *Assume that* (27)–(41) *hold. Then there exists a unique solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}, \xi, \varsigma, \zeta, \boldsymbol{D}\}$ *to Problem* ***PV****, Moreover, the solution satisfies*

$$\boldsymbol{u} \in H^1(0,T;\boldsymbol{V}) \cap C^1(0,T;H),\ \ddot{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{V}'), \tag{55}$$

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}),\ (\mathrm{Div}\,\boldsymbol{\sigma}^1, \mathrm{Div}\,\boldsymbol{\sigma}^2) \in L^2(0,T;\boldsymbol{V}'), \tag{56}$$

$$\xi \in C(0,T;W), \tag{57}$$

$$\varsigma \in H^1(0,T;L_0) \cap L^2(0,T;L_1), \tag{58}$$

$$\zeta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}, \tag{59}$$

$$\boldsymbol{D} \in C(0,T;\mathcal{W}). \tag{60}$$

The functions $\boldsymbol{u}, \xi, \varsigma$, $\boldsymbol{\sigma}$, $\boldsymbol{D}$ and ζ which satisfy (48)-(54) are called a weak solution of the contact Problem **P**. We conclude that, under the assumptions (27)-(41), the mechanical problem (10)-(24) has a unique weak solution satisfying (55)-(60). We turn now to the proof of Theorem 4.1 which will be carried out in several steps and is based on arguments of nonlinear equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed point arguments. We assume in what follows that assumptions of Theorem 4.1 hold, and we consider that C is a generic positive constant which depends on Ω^ℓ, Γ_1^ℓ, Γ_1^ℓ, Γ_3, p_ν, p_τ, $\mathcal{A}^\ell$, $\boldsymbol{\beta}^\ell$, $\mathcal{B}^\ell$, $\mathcal{G}^\ell$, $\mathcal{E}^\ell$, H_{ad}, γ_ν, ϕ^ℓ, κ^ℓ, and T. but does not depend on t nor of the rest of input data, and whose value may change from place to place. Let a $\eta \in L^2(0,T;\boldsymbol{V}')$ be given. In the first step we consider the following variational problem.

Problem $\mathbf{PV}_\eta^u$. Find a displacement field $\boldsymbol{u}_\eta : [0,T] \to \boldsymbol{V}$ such that

$$\begin{aligned}(\ddot{u}_\eta(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\ell=1}^{2} (\mathcal{A}^\ell \varepsilon(\dot{\boldsymbol{u}}^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + (\eta(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} \\ = (\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{V}, a.e.\ t \in (0,T),\end{aligned} \tag{61}$$

$$\boldsymbol{u}^\ell(0) = \boldsymbol{u}_0^\ell,\ \ \dot{\boldsymbol{u}}^\ell(0) = \boldsymbol{v}_0^\ell \quad \text{in } \Omega^\ell, \tag{62}$$

To solve Problem $\mathbf{PV}_\eta^u$, we apply an abstract existence and uniqueness result which we recall now, for the convenience of the reader. Let $\boldsymbol{V}$ and H denote real Hilbert spaces such that $\boldsymbol{V}$ is dense in H and the inclusion map is continuous, H is identified with its dual and with a subspace of the dual $\boldsymbol{V}'$ of $\boldsymbol{V}$, i.e., $\boldsymbol{V} \subset H \subset \boldsymbol{V}'$, and we say that the inclusions above define a Gelfand triple. The notations $\|\cdot\|_{\boldsymbol{V}}$, $\|\cdot\|_{\boldsymbol{V}'}$ and $(\cdot,\cdot)_{\boldsymbol{V}'\times\boldsymbol{V}}$ represent the norms on $\boldsymbol{V}$ and on $\boldsymbol{V}'$ and the duality pairing between $\boldsymbol{V}'$ them, respectively. The following abstract result may be found in [26, p.48].

Theorem 4.2. *Let $\boldsymbol{V}$, H be as above, and let $A : \boldsymbol{V} \to \boldsymbol{V}'$ be a hemicontinuous and monotone operator which satisfies*

$$(A\boldsymbol{v}, \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \ge \kappa \|\boldsymbol{v}\|_{\boldsymbol{V}}^2 + \lambda \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{63}$$

$$\|A\boldsymbol{v}\|_{\boldsymbol{V}'} \le C(\|\boldsymbol{v}\|_{\boldsymbol{V}} + 1) \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{64}$$

for some constants $\kappa > 0$, $C > 0$ and $\lambda \in \mathbb{R}$. Then, given $\boldsymbol{u}_0 \in H$ and $f \in L^2(0,T;\boldsymbol{V}')$, there exists a unique function $\boldsymbol{u}$ which satisfies

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap C(0,T;H), \quad \dot{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{V}'),$$

$$\dot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) = \mathbf{f}(t) \text{ a.e. } t \in (0,T),$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0$$

We have the following result for the problem.

Lemma 4.1. *There exists a unique solution to Problem* $\boldsymbol{PV}_\eta^u$ *and it has its regularity expressed in* (55).

Proof. We define the operator $A : \boldsymbol{V} \to \boldsymbol{V}'$ by

$$(A\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\ell=1}^{2}(\mathcal{A}^\ell\varepsilon(\boldsymbol{u}^\ell),\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}. \tag{65}$$

Using (65), (3) and (27) it follows that

$$\|A\boldsymbol{u} - A\boldsymbol{v}\|^2_{\boldsymbol{V}'} \le \sum_{\ell=1}^{2} \|\mathcal{A}^\ell\varepsilon(\boldsymbol{u}^\ell) - \mathcal{A}^\ell\varepsilon(\boldsymbol{v}^\ell)\|^2_{\mathcal{H}^\ell} \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V},$$

and keeping in mind the Krasnoselski Theorem (see [17, p.60]), we deduce that $A : \boldsymbol{V} \to \boldsymbol{V}'$ is a continuous operator. Now, by (65), (3) and (27), we find

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \ge m\|\boldsymbol{u} - \boldsymbol{v}\|^2_{\boldsymbol{V}} \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}, \tag{66}$$

where the positive constant $m = \min\{m_{\mathcal{A}^1}, m_{\mathcal{A}^2}\}$. Choosing $\boldsymbol{v} = 0$ in (66) we obtain

$$\begin{aligned}(A\boldsymbol{u},\boldsymbol{u})_{\boldsymbol{V}'\times\boldsymbol{V}} &\ge m\|\boldsymbol{u}\|^2_{\boldsymbol{V}} - \|Ao\|^2_{\boldsymbol{V}'}\|\boldsymbol{u}\|_{\boldsymbol{V}} \\ &\ge \frac{1}{2}m\|\boldsymbol{u}\|^2_{\boldsymbol{V}} - \frac{1}{2m}\|Ao\|^2_{\boldsymbol{V}'} \quad \forall \boldsymbol{u} \in \boldsymbol{V},\end{aligned}$$

which implies that A satisfies condition (63) with $\kappa = \frac{m}{2}$ and $\lambda = -\frac{1}{2m}\|Ao\|^2_{\boldsymbol{V}'}$. Moreover, by (65) and (27) we find

$$\|A\boldsymbol{u}\|_{\boldsymbol{V}'} \le C^1\|u\|_{\boldsymbol{V}} + C^2 \quad \forall \boldsymbol{u} \in \boldsymbol{V}.$$

where $C^1 = \max\{C^1_{\mathcal{A}^1}, C^1_{\mathcal{A}^2}\}$ and $C^2 = \max\{C^2_{\mathcal{A}^1}, C^2_{\mathcal{A}^2}\}$. This inequality and (3) imply that A satisfies condition (64). Finally, we recall that by (37) and (43) we have $\mathbf{f} - \eta \in L^2(0,T;\boldsymbol{V}')$ and $\boldsymbol{v}_0 \in H$.

It follows now from Theorem 4.2 that there exists a unique function $\boldsymbol{v}_\eta$ which satisfies

$$\boldsymbol{v}_\eta \in L^2(0,T;\boldsymbol{V}) \cap C(0,T;H), \quad \dot{\boldsymbol{v}}_\eta \in L^2(0,T;\boldsymbol{V}'), \tag{67}$$

$$\dot{\boldsymbol{v}}_\eta(t) + A\boldsymbol{v}_\eta(t) + \eta(t) = \mathbf{f}(t), \quad a.e.\ t \in [0,T] \tag{68}$$

$$\boldsymbol{v}_\eta(0) = \boldsymbol{v}_0. \tag{69}$$

Let $\boldsymbol{u}_\eta : [0,T] \to \boldsymbol{V}$ be the function defined by

$$\boldsymbol{u}_\eta(t) = \int_0^t \boldsymbol{v}_\eta(s)ds + \boldsymbol{u}_0 \quad \forall t \in [0,T]. \tag{70}$$

It follows from (65) and (67)–(70) that $\boldsymbol{u}_\eta$ is a unique solution of the variational problem $\mathbf{PV}_\eta^u$ and it satisfies the regularity expressed in (55). □

In the second step, let $\eta \in L^2(0,T;\boldsymbol{V}')$, we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.1 and we consider the following variational problem.

Problem $\mathbf{PV}_\eta^\xi$. Find the electric potential field $\xi_\eta : [0,T] \to W$ such that

$$\sum_{\ell=1}^{2}(\boldsymbol{\beta}^\ell\nabla\xi_\eta^\ell(t),\nabla\phi^\ell)_{H^\ell} - \sum_{\ell=1}^{2}(\mathcal{E}^\ell\varepsilon(\boldsymbol{u}_\eta^\ell(t)),\nabla\phi^\ell)_{H^\ell} = (q(t),\phi)_W \tag{71}$$
$$\forall \phi \in W,\ a.e.\ t \in (0,T).$$

We have the following result.

Lemma 4.2. *Problem* $\boldsymbol{PV}_\eta^\xi$ *has a unique solution* ξ_η *which satisfies the regularity* (57).

Proof. We define a bilinear form: $b(\cdot,\cdot): W \times W \to \mathbb{R}$ such that

$$b(\xi,\phi) = \sum_{\ell=1}^{2} (\boldsymbol{\beta}^\ell \nabla \xi^\ell, \nabla \phi^\ell)_{H^\ell} \quad \forall \xi,\phi \in W. \tag{72}$$

We use (5), (33) and (72) to show that the bilinear form $b(\cdot,\cdot)$ is continuous, symmetric and coercive on W, moreover using (44) and the Riesz Representation Theorem we may define an element $q_\eta : [0,T] \to W$ such that

$$(q_\eta(t),\phi)_W = (q(t),\phi)_W + \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(\boldsymbol{u}_\eta^\ell(t)), \nabla \phi^\ell)_{H^\ell} \quad \forall \phi \in W, t \in (0,T).$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\xi_\eta(t) \in W$ such that

$$b(\xi_\eta(t),\phi) = (q_\eta(t),\phi)_W \quad \forall \phi \in W. \tag{73}$$

We conclude that $\xi_\eta(t)$ is a solution of Problem $\mathbf{PV}_\eta^\xi$. Let $t_1, t_2 \in [0,T]$, it follows from (71) that

$$\|\xi_\eta(t_1) - \xi_\eta(t_2)\|_W \le C\big(\|\boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2)\|_{\boldsymbol{V}} + \|q(t_1) - q(t_2)\|_W\big),$$

and the previous inequality, the regularity of $\boldsymbol{u}_\eta$ and q imply that $\varphi_\eta \in C(0,T;W)$. □

In the third step we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma4.1 and we consider the following initial-value problem.

Problem $\mathbf{PV}_\eta^\zeta$. Find the adhesion field $\zeta_\eta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\zeta}_\eta(t) = H_{ad}\big(\zeta_\eta(t), \alpha_{\zeta_\eta}, R_\nu([u_{\eta\nu}](t)), \boldsymbol{R}_\tau([\boldsymbol{u}_{\eta\tau}](t))\big), \tag{74}$$

$$\zeta_\eta(0) = \zeta_0. \tag{75}$$

We have the following result.

Lemma 4.3. *There exists a unique solution* $\zeta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}$ *to Problem* $\boldsymbol{PV}_\eta^\zeta$.

Proof. We consider the mapping $H_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$,

$$H_\eta(t,\zeta) = H_{ad}\big(\zeta_\eta(t), \alpha_{\zeta_\eta}, R_\nu(u_{\eta\nu}(t)), \boldsymbol{R}_\tau(\boldsymbol{u}_{\eta\tau}(t))\big),$$

for all $t \in [0,T]$ and $\zeta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R_ν and $\boldsymbol{R}_\tau$ that H_η is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\zeta \in L^2(\Gamma_3)$, the mapping $t \to H_\eta(t,\zeta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using the Cauchy–Lipschitz theorem given in [27, p. 60], we deduce that there exists a unique function $\zeta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution of the equation (74). Also, the arguments used in Remark 3.1 show that $0 \le \zeta_\eta(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. This completes the proof. □

In the fourth step, we let $\theta \in L^2(0.T;E_0)$ be given and consider the following variational problem for the damage field.

Problem $\mathbf{PV}_\theta^\varsigma$. *Find a damage field* $\varsigma_\theta = (\varsigma_\theta^1, \varsigma_\theta^2) : [0,T] \to L$ such that

$$\begin{aligned}\varsigma_\theta(t) \in K, \ \sum_{\ell=1}^{2} (\dot{\varsigma}_\theta^\ell(t), \omega^\ell - \varsigma_\theta^\ell(t))_{L^2(\Omega^\ell)} + a(\varsigma_\theta(t), \omega - \varsigma_\theta(t)) \\ \geq \sum_{\ell=1}^{2} \left(\theta^\ell(t), \omega^\ell - \varsigma^\ell(t)\right)_{L^2(\Omega^\ell)}, \quad \forall \omega \in K, \text{ a.e. } t \in (0,T),\end{aligned} \tag{76}$$

where $K = K^1 \times K^2$. The following abstract result for parabolic variational inequalities (see, e.g., [26, p.47])

Theorem 4.3. *Let $X \subset Y = Y' \subset X'$ be a Gelfand triple. Let F be a nonempty, closed, and convex set of X. Assume that $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\alpha > 0$ and c_0,*

$$a(v,v) + c_0\|v\|_Y^2 \geq \alpha \|v\|_X^2 \quad \forall v \in X.$$

Then, for every $u_0 \in F$ and $f \in L^2(0,T;Y)$, there exists a unique function $u \in H^1(0,T;Y) \cap L^2(0,T;X)$ such that $u(0) = u_0$, $u(t) \in F \ \ \forall t \in [0,T]$, and

$$(\dot{u}(t), v - u(t))_{X' \times X} + a(u(t), v - u(t)) \geq (f(t), v - u(t))_Y \quad \forall v \in F \text{ a.e. } t \in (0,T).$$

We prove next the unique solvability of Problem $\mathbf{PV}_\theta^\varsigma$.

Lemma 4.4. *There exists a unique solution ς_θ of Problem $\mathbf{PV}_\theta^\varsigma$ and it satisfies*

$$\varsigma_\theta \in H^1(0,T;L_0) \cap L^2(0,T;L_1).$$

Proof. The inclusion mapping of $(L_1, \|\cdot\|_{L_1})$ into $(L_0, \|\cdot\|_{L_0})$ is continuous and its range is dense. We denote by L_1' the dual space of L_1 and, identifying the dual of L_0 with itself, we can write the Gelfand triple

$$L_1 \subset L_0 = L_0' \subset L_1'.$$

We use the notation $(\cdot,\cdot)_{L_1' \times L_1}$ to represent the duality pairing between L' and L_1. We have

$$(\varsigma, \xi)_{L_1' \times L_1} = (\varsigma, \xi)_{L_0} \quad \forall \varsigma \in L_0, \xi \in L_1,$$

and we note that K is a closed convex set in L_1. Then, using (40), (42) and the fact that $\varsigma_0 \in K$ in (41), it is easy to see that Lemma 4.4 is a straight consequence of Theorem 4.3. □

Now we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.1, ξ_η obtained in Lemma 4.2 and ς_θ obtained in Lemma 4.4 to construct the following Cauchy problem for the stress field.

Problem $\mathbf{PV}_{\eta\theta}^\sigma$. *Find a stress field $\boldsymbol{\sigma}_{\eta\theta} = (\boldsymbol{\sigma}_{\eta\theta}^1, \boldsymbol{\sigma}_{\eta\theta}^2) : [0,T] \to \mathcal{H}$ such that*

$$\boldsymbol{\sigma}_{\eta\theta}^\ell(t) = \mathcal{B}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta^\ell(t)) + \int_0^t \mathcal{G}^\ell(\boldsymbol{\sigma}_{\eta\theta}^\ell(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta^\ell(s)), \varsigma_\theta^\ell(s))\,ds, \quad \ell = 1,2, \tag{77}$$

for all $t \in [0,T]$.

In the study of Problem $\mathbf{PV}_{\eta\theta}^\sigma$ we have the following result.

Lemma 4.5. *There exists a unique solution of Problem $\mathbf{PV}_{\eta\theta}^\sigma$ and it satisfies $\boldsymbol{\sigma}_{\eta\theta} \in L^2(0,T;\mathcal{H})$. Moreover, if $\boldsymbol{\sigma}_i$, $\boldsymbol{u}_i$ and ς_i represent the solutions of problems $\mathbf{PV}_{\eta_i\theta_i}^\sigma$, $\mathbf{PV}_{\eta_i}^u$ and $\mathbf{PV}_{\theta_i}^\varsigma$ respectively,* *for* $(\boldsymbol{\eta}_i,\theta_i)\in L^2(0,T;\mathbf{V}'\times E_0)$, $i=1,2$, *then there exists* $c>0$ *such that*

$$\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|^2_{\mathcal{H}}\le c\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\mathbf{V}}+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_{\mathbf{V}}\,ds + \int_0^t\|\varsigma_1(s)-\varsigma_2(s)\|^2_{L_0}\,ds\Big)\quad \forall t\in[0,T]. \tag{78}$$

Proof. Let $\Lambda_{\eta\theta}=(\Lambda^1_{\eta\theta},\Lambda^2_{\eta\theta}):L^2(0,T;\mathcal{H})\to L^2(0,T;\mathcal{H})$ be the operator given by

$$\Lambda^\ell_{\eta\theta}\boldsymbol{\sigma}(t)=\mathcal{B}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_\eta(t))+\int_0^t\mathcal{G}^\ell\big(\boldsymbol{\sigma}^\ell(s),\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_\eta(s)),\varsigma^\ell_\theta\big)\,ds,\quad \ell=1,2 \tag{79}$$

for all $\boldsymbol{\sigma}=(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\in L^2(0,T;\mathcal{H})$ and $t\in[0,T]$. For $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2\in L^2(0,T;\mathcal{H})$ we use (79) and (29) to obtain

$$\|\Lambda_{\eta\theta}\,\boldsymbol{\sigma}_1(t)-\Lambda_{\eta\theta}\,\boldsymbol{\sigma}_2(t)\|_{\mathcal{H}}\le\max(L_{\mathcal{G}^1},L_{\mathcal{G}^2})\int_0^t\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\,ds$$

for all $t\in[0,T]$. It follows from this inequality that for p large enough, a power $\Lambda^p_{\eta\theta}$ of the operator $\Lambda_{\eta\theta}$ is a contraction on the Banach space $L^2(0,T;\mathcal{H})$ and, therefore, there exists a unique element $\boldsymbol{\sigma}_{\eta\theta}\in L^2(0,T;\mathcal{H})$ such that $\Lambda_{\eta\theta}\boldsymbol{\sigma}_{\eta\theta}=\boldsymbol{\sigma}_{\eta\theta}$. Moreover, $\boldsymbol{\sigma}_{\eta\theta}$ is the unique solution of Problem $\mathbf{PV}^\sigma_{\eta\theta}$.

Consider now $(\boldsymbol{\eta}_1,\theta_1)$, $(\boldsymbol{\eta}_2,\theta_2),\in L^2(0,T;\mathbf{V}'\times L_0)$ and, for $i=1,2$, denote $\boldsymbol{u}_{\eta_i}=\boldsymbol{u}_i$, $\boldsymbol{\sigma}_{\eta_i\theta_i}=\boldsymbol{\sigma}_i$ and $\varsigma_{\theta_i}=\varsigma_i$. We have

$$\boldsymbol{\sigma}^\ell_i(t)=\mathcal{B}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_i(t))+\int_0^t\mathcal{G}^\ell\big(\boldsymbol{\sigma}^\ell_i(s),\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_i(s)),\varsigma^\ell_i(s)\big)\,ds,\quad \ell=1,2\ \ \forall t\in[0,T],$$

and, using the properties (28) and (29) of $\mathcal{G}^\ell$, and $\mathcal{B}^\ell$ we find

$$\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|^2_{\mathcal{H}}\le c\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\mathbf{V}}+\int_0^t\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}\,ds + \int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_V\,ds+\int_0^t\|\varsigma_1(s)-\varsigma_2(s)\|^2_{L_0}\,ds\Big)\quad\forall t\in[0,T].$$

Using now a Gronwall argument in the previous inequality we deduce (78), which concludes the proof. □

Finally as a consequence of these results and using the properties of the operator $\mathcal{B}^\ell$, the operator $\mathcal{E}^\ell$, the operator $\mathcal{G}^\ell$, the functional j_{ad}, the function j_{vc} and the function ϕ^ℓ for $t\in[0,T]$, we consider the element

$$\Lambda(\eta,\theta)(t)=\big(\Lambda^1(\eta,\theta)(t),\ \Lambda^2(\eta,\theta)(t)\big)\in\mathbf{V}'\times L_0, \tag{80}$$

defined by the equations

$$\begin{aligned}(\Lambda^1(\eta,\theta)(t),\boldsymbol{v})_{\mathbf{V}'\times\mathbf{V}}&=\sum_{\ell=1}^2\big(\mathcal{B}^\ell\varepsilon(\boldsymbol{u}^\ell_\eta(t)),\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell}+\sum_{\ell=1}^2\big((\mathcal{E}^\ell)^*\nabla\xi^\ell_\eta,\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell}\\&\quad+\sum_{\ell=1}^2\Big(\int_0^t\mathcal{G}^\ell\big(\boldsymbol{\sigma}^\ell_{\eta\theta},\varepsilon(\boldsymbol{u}^\ell_\eta(s)),\varsigma_\theta(s)\big)\,ds,\ \varepsilon(\boldsymbol{v}^\ell)\Big)_{\mathcal{H}^\ell}\\&\quad+j_{ad}(\zeta_\eta(t),\boldsymbol{u}_\eta(t),\boldsymbol{v})+j_{vc}(\boldsymbol{u}_\eta(t),\boldsymbol{v}),\quad\forall\boldsymbol{v}\in\mathbf{V},\end{aligned} \tag{81}$$

$$\Lambda^2(\eta,\theta)(t)=\Big(\phi^1\big(\boldsymbol{\sigma}^1_{\eta\theta}(t),\boldsymbol{\varepsilon}(\boldsymbol{u}^1_\eta(t)),\varsigma^1_\theta(t)\big),\ \phi^2\big(\boldsymbol{\sigma}^2_{\eta\theta}(t),\boldsymbol{\varepsilon}(\boldsymbol{u}^2_\eta(t)),\varsigma^2_\theta(t)\big)\Big). \tag{82}$$

Here, for every $(\eta,\theta) \in L^2(0,T;\boldsymbol{V}' \times L_0)$, $\boldsymbol{u}_\eta$, ξ_η, ζ_η ς_θ, and $\boldsymbol{\sigma}_{\eta\theta}$ represent the displacement field, the stress field, the the potential electric field and bonding field obtained in Lemmas 4.1, 4.2, 4.3, 4.4 and 4.5 respectively. We have the following result.

Lemma 4.6. *The operator Λ has a unique fixed point $(\eta^*,\theta^*) \in L^2(0,T;\boldsymbol{V}' \times L_0)$.*

Proof. We show that, for a positive integer n, the mapping Λ^n is a contraction on $L^2(0,T;\boldsymbol{V}' \times L_0)$. To this end, we suppose that (η_1,θ_1) and (η_2,θ_2) are two functions in $L^2(0,T;\boldsymbol{V}' \times L_0)$ and denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{\eta_i} = \boldsymbol{v}_i$, $\boldsymbol{\sigma}_{\eta_i\theta_i} = \boldsymbol{\sigma}_i$, $\xi_{\eta_i} = \xi_i$, $\varsigma_{\theta_i} = \varsigma_i$ and $\zeta_{\eta_i} = \zeta_i$ for $i = 1,2$. We use (28), (29), (32), (34) and (35), the definition of R_ν, $\boldsymbol{R}_\tau$ and Remark 3.1, we have

$$
\begin{aligned}
&\|\Lambda^1(\eta_1,\theta_1)(t) - \Lambda^1(\eta_2,\theta_1)(t)\|^2_{\boldsymbol{V}'} \\
&\le \sum_{\ell=1}^{2} \|\mathcal{B}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}_1^\ell(t)) - \mathcal{B}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}_2^\ell(t))\|^2_{\mathcal{H}^\ell} + \sum_{\ell=1}^{2} \int_0^t \left\|\mathcal{G}^\ell\left(\boldsymbol{\sigma}_1^\ell(s), \varepsilon(\boldsymbol{u}_1^\ell(s)), \varsigma_1^\ell(s)\right) - \mathcal{G}^\ell\left(\boldsymbol{\sigma}_2^\ell(s), \varepsilon(\boldsymbol{u}_2^\ell(s)), \varsigma_2^\ell(s)\right)\right\|^2_{\mathcal{H}^\ell} ds \\
&\quad + \sum_{\ell=1}^{2} \|(\mathcal{E}^\ell)^* \nabla \xi_1^\ell(t) - (\mathcal{E}^\ell)^* \nabla \xi_2^\ell(t)\|^2_{\mathcal{H}^\ell} + C\|p_\nu([u_{1\nu}(t)]) - p_\nu([u_{2\nu}(t)])\|^2_{L^2(\Gamma_3)} \\
&\quad + C\|\zeta_1^2(t) R_\nu([u_{1\nu}(t)]) - \zeta_2^2(t) R_\nu([u_{2\nu}(t)])\|^2_{L^2(\Gamma_3)} \\
&\quad + C\|p_\tau(\zeta_1(t)) \boldsymbol{R}_\tau([\boldsymbol{u}_{1\tau}(t)]) - p_\tau(\zeta_2(t)) \boldsymbol{R}_\tau([\boldsymbol{u}_{2\tau}(t)])\|^2_{L^2(\Gamma_3)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|\Lambda^1(\eta_1,\theta_1)(t) - \Lambda^1(\eta_2,\theta_1)(t)\|^2_{\boldsymbol{V}'} \\
&\le C\Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}} + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}} ds + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s))\|^2_{\mathcal{H}} ds \\
&\qquad + \int_0^t \|\varsigma_1(s) - \varsigma_2(s))\|^2_{L_0} ds + \|\xi_1(t) - \xi_2(t)\|^2_W + \|\zeta_1(t) - \zeta_2(t)\|^2_{L^2(\Gamma_3)}\Big).
\end{aligned}
$$

We use estimate (78) to obtain

$$
\begin{aligned}
&\|\Lambda^1(\eta_1,\theta_1)(t) - \Lambda^1(\eta_2,\theta_1)(t)\|^2_{\boldsymbol{V}'} \\
&\le C\Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}} + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}} ds + \int_0^t \|\varsigma_1(s) - \varsigma_2(s))\|^2_{L_0} ds \\
&\qquad + \|\xi_1(t) - \xi_2(t)\|^2_W + \|\zeta_1(t) - \zeta_2(t)\|^2_{L^2(\Gamma_3)}\Big).
\end{aligned}
$$

Recall that above $u^\ell_{\eta\nu}$ and $\boldsymbol{u}^\ell_{\eta\tau}$ denote the normal and the tangential component of the function $\boldsymbol{u}^\ell_\eta$ respectively. By similar arguments, from (78), (82) and (31) it follows that

$$
\begin{aligned}
&\|\Lambda^2(\eta_1,\theta_1)(t) - \Lambda^2(\eta_2,\theta_1)(t)\|^2_{L_0} \\
&\le C\Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}} + \|\xi_1(t) - \xi_2(t)\|^2_W + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}} ds + \|\varsigma_1(t) - \varsigma_2(t))\|^2_{L_0} \\
&\qquad + \int_0^t \|\varsigma_1(s) - \varsigma_2(s))\|^2_{L_0} ds\Big).
\end{aligned}
$$

Also, since

$$
\boldsymbol{u}_i^\ell(t) = \int_0^t \boldsymbol{v}_i^\ell(s) ds + \boldsymbol{u}_0^\ell, \quad t \in [0,T], \quad \ell = 1,2,
$$

we have

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_{\boldsymbol{V}} \le \int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|_{\boldsymbol{V}}\,ds$$

which implies

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}} + \int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds \le c\int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|^2_{\boldsymbol{V}}\,ds. \tag{83}$$

Therefore

$$\begin{aligned}\|\Lambda(\eta_1,\theta_1)(t)-\Lambda(\eta_2,\theta_1)(t)\|^2_{\boldsymbol{V}'\times L_0} &\le C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}\\ &+\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds + \|\varsigma_1(t)-\varsigma_2(t))\|^2_{L_0} + \int_0^t \|\varsigma_1(s)-\varsigma_2(s))\|^2_{L_0}\,ds\\ &+\|\xi_1(t)-\xi_2(t)\|^2_W + \|\zeta_1(t)-\zeta_2(t)\|^2_{L^2(\Gamma_3)}\Big).\end{aligned} \tag{84}$$

Moreover, from (61) we obtain

$$(\dot{\boldsymbol{v}}_1-\dot{\boldsymbol{v}}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\ell=1}^{2}(\mathcal{A}^\ell\varepsilon(\boldsymbol{v}_1^\ell)-\mathcal{A}^\ell\varepsilon(\boldsymbol{v}_2^\ell),,\varepsilon(\boldsymbol{v}_1^\ell-\boldsymbol{v}_2^\ell))_{\mathcal{H}^\ell} + (\eta_1-\eta_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}} = 0.$$

We integrate this equality with respect to time, use the initial conditions $\boldsymbol{v}_1(0)=\boldsymbol{v}_2(0)=\boldsymbol{v}_0$ and condition (27) to find

$$\min(m_{\mathcal{A}^1},m_{\mathcal{A}^2})\int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|^2_{\boldsymbol{V}}\,ds \le -\int_0^t (\eta_1(s)-\eta_2(s),\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))_{\boldsymbol{V}'\times\boldsymbol{V}}\,ds,$$

for all $t\in[0,T]$. Then, using the inequality $2ab \le \frac{a^2}{m}+mb^2$ we obtain

$$\int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|^2_{\boldsymbol{V}}\,ds \le C\int_0^t \|\eta_1(s)-\eta_2(s)\|^2_{\boldsymbol{V}'}\,ds \quad \forall t\in[0,T]. \tag{85}$$

On the other hand, from the Cauchy problem (74)–(75) we can write

$$\zeta_i(t)=\zeta_0-\int_0^t H_{ad}\big(\zeta_i(s),\alpha_{\zeta_i}(s),R_\nu([u_{i\nu}](s)),\boldsymbol{R}_\tau([\boldsymbol{u}_{i\tau}](s))\big)ds$$

and then

$$\begin{aligned}\big\|\zeta_1(t)-\zeta_2(t)\big\|_{L^2(\Gamma_3)} &\le C\int_0^t\big\|\zeta_1(s)-\zeta_2(s)\big\|_{L^2(\Gamma_3)}ds + C\int_0^t\big\|R_\nu([u_{1\nu}](s))-R_\nu([u_{2\nu}](s))\big\|_{L^2(\Gamma_3)}ds\\ &+ C\int_0^t \big\|\boldsymbol{R}_\tau([\boldsymbol{u}_{1\tau}](s))-\boldsymbol{R}_\tau([\boldsymbol{u}_{2\tau}](s))\big\|_{L^2(\Gamma_3)}ds.\end{aligned}$$

Using the definition of R_ν and $\boldsymbol{R}_\tau$ and writing $\zeta_1=\zeta_1-\zeta_2+\zeta_2$, we get

$$\big\|\zeta_1(t)-\zeta_2(t)\big\|_{L^2(\Gamma_3)} \le C\Big(\int_0^t \|\zeta_1(s)-\zeta_2(s)\|_{L^2(\Gamma_3)}ds + \int_0^t \big\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\big\|_{L^2(\Gamma_3)^d}ds\Big).$$

Next, we apply Gronwall's inequality to deduce

$$\|\zeta_1(t)-\zeta_2(t)\|_{L^2(\Gamma_3)} \le C\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_{L^2(\Gamma_3)^d}ds.$$

and from the relation (4) we obtain

$$\|\zeta_1(t)-\zeta_2(t)\|^2_{L^2(\Gamma_3)} \le C\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_V ds. \tag{86}$$

We use now (71), (5), (32) and (33) to find

$$\|\xi_1(t)-\xi_2(t)\|^2_W \le C\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_V. \tag{87}$$

We substitute (83), (86) and (87) in (84) to obtain

$$\|\Lambda(\eta_1,\theta_1)(t)-\Lambda(\eta_2,\theta_1)(t)\|^2_{\boldsymbol{V}'\times L_0} \le C\Big(\int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|^2_{\boldsymbol{V}}\,ds+\|\varsigma_1(t)-\varsigma_2(t))\|^2_{L_0}+\int_0^t\|\varsigma_1(s)-\varsigma_2(s))\|^2_{L_0}\,ds\Big). \tag{88}$$

On the other hand, from (76) we deduce that

$$(\dot\varsigma_1-\dot\varsigma_2,\varsigma_1-\varsigma_2)_{L_0}+a(\varsigma_1-\varsigma_2,\varsigma_1-\varsigma_2)\le\big(\theta_1-\theta_2,\varsigma_1-\varsigma_2\big)_{L_0},\ \text{a.e. } t\in(0,T).$$

Integrating the previous inequality with respect to time, using the initial conditions $\varsigma_1(0)=\varsigma_2(0)=\varsigma_0$ and inequality $a(\varsigma_1-\varsigma_2,\varsigma_1-\varsigma_2)\ge 0$, to find

$$\frac{1}{2}\|\varsigma_1(t)-\varsigma_2(t)\|^2_{L_0}\le\int_0^t\big(\theta_1(s)-\theta_2(s),\varsigma_1(s)-\varsigma_2(s)\big)_{L_0}\,ds,$$

which implies that

$$\|\varsigma_1(t)-\varsigma_2(t)\|^2_{L_0}\le\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_{L_0}\,ds+\int_0^t\|\varsigma_1(s)-\varsigma_2(s)\|^2_{L_0}\,ds.$$

This inequality, combined with Gronwall's inequality, leads to

$$\|\varsigma_1(t)-\varsigma_2(t)\|^2_{L_0}\le C\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_{L_0}\,ds\quad\forall\ t\in[0,T]. \tag{89}$$

We substitute (85) and (89) in (88) to obtain

$$\|\Lambda(\eta_1,\theta_1)(t)-\Lambda(\eta_2,\theta_1)(t)\|^2_{\boldsymbol{V}'\times L_0}\le C\int_0^t\|(\eta_1,\theta_1)(s)-(\eta_2,\theta_1)(s)\|^2_{\boldsymbol{V}'\times L_0}\,ds.$$

Reiterating this inequality n times we obtain

$$\|\Lambda^n(\eta_1,\theta_1)-\Lambda^n(\eta_2,\theta_1)\|^2_{L^2(0,T;\boldsymbol{V}'\times L_0)}\le\frac{C^nT^n}{n!}\|(\eta_1,\theta_1)-(\eta_2,\theta_1)\|^2_{L^2(0,T;\boldsymbol{V}'\times L_0)}.$$

Thus, for n sufficiently large, Λ^n is a contraction on the Banach space $L^2(0,T;\boldsymbol{V}'\times L_0)$, and so Λ has a unique fixed point. □

Now, we have all the ingredients to prove Theorem 4.1.

Proof. Existence. Let $(\eta^*,\theta^*)\in L^2(0,T;\boldsymbol{V}'\times L_0)$ be the fixed point of Λ defined by (80)–(82) and denote by

$$\boldsymbol{u}_*=\boldsymbol{u}_{\eta^*},\ \ \xi_*=\xi_{\eta^*},\ \ \varsigma_*=\varsigma_{\theta^*},\ \ \zeta_*=\zeta_{\eta^*}. \tag{90}$$

Let by $\boldsymbol{\sigma}_*=(\boldsymbol{\sigma}^1_*,\boldsymbol{\sigma}^2_*):[0,T]\to\mathcal{H}$ and $\boldsymbol{D}_*=(\boldsymbol{D}^1_*,\boldsymbol{D}^2_*):[0,T]\to H$ the functions defined by

$$\boldsymbol{\sigma}^\ell_*=\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell_*)+(\mathcal{E}^\ell)^*\nabla\xi^\ell_*+\boldsymbol{\sigma}^\ell_{\eta^*\theta^*},\ \ \ell=1,2, \tag{91}$$

$$\boldsymbol{D}^\ell_*=\mathcal{E}^\ell\varepsilon(\boldsymbol{u}^\ell_*)-\boldsymbol{\beta}^\ell\nabla\xi^\ell_*,\quad \ell=1,2. \tag{92}$$

We prove that the $\{\boldsymbol{u}_*,\boldsymbol{\sigma}_*,\xi_*,\varsigma_*,\zeta_*,\boldsymbol{D}_*\}$ satisfies (48)–(54) and the regularites (55)–(60). Indeed, we write (61) for $\eta=\eta^*$ and use (90) to find

$$(\ddot u_*(t),v)_{\boldsymbol{V}'\times\boldsymbol{V}}+\sum_{\ell=1}^{2}(\mathcal{A}^\ell\varepsilon(\dot{\boldsymbol{u}}^\ell_*(t)),\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell}+(\eta^*(t),v)_{\boldsymbol{V}'\times\boldsymbol{V}} =(\mathbf{f}(t),\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}}\quad\forall\boldsymbol{v}\in\boldsymbol{V},a.e.\ t\in[0,T]. \tag{93}$$

We use equalities $\Lambda^1(\eta^*,\theta^*)=\eta^*$ and $\Lambda^2(\eta^*,\theta^*)=\theta^*$ it follows that

$$\begin{aligned}(\eta^*(t),v)_{\boldsymbol{V}'\times\boldsymbol{V}} &= \sum_{\ell=1}^{2}\big(\mathcal{B}^\ell\varepsilon(\boldsymbol{u}_*^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell}+\sum_{\ell=1}^{2}\big((\mathcal{E}^\ell)^*\nabla\xi_*^\ell,\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell}\\ &+\sum_{\ell=1}^{2}\Big(\int_0^t\mathcal{G}^\ell\Big(\boldsymbol{\sigma}_*^\ell(s)-\mathcal{A}^\ell\varepsilon(\dot{\boldsymbol{u}}_*^\ell(s))-(\mathcal{E}^\ell)^*\nabla\xi_*^\ell(s),\varepsilon(\boldsymbol{u}_*^\ell(s)),\varsigma_*^\ell(s)\Big)ds,\ \varepsilon(\boldsymbol{v}^\ell)\Big)_{\mathcal{H}^\ell}\\ &+j_{ad}(\zeta_*(t),\boldsymbol{u}_*(t),\boldsymbol{v})+j_{vc}(\boldsymbol{u}_*(t),\boldsymbol{v}),\quad\forall\boldsymbol{v}\in\boldsymbol{V}.\end{aligned}\tag{94}$$

$$\theta_*^\ell(t)=\phi^\ell\big(\boldsymbol{\sigma}_*^\ell(t)-\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell(t))-(\mathcal{E}^\ell)^*\nabla\xi_*^\ell(t),\boldsymbol{\varepsilon}(\boldsymbol{u}_*^\ell(t)),\varsigma_*^\ell(t)\big),\ \ell=1,2.\tag{95}$$

We now substitute (94) in (93) to obtain

$$\begin{aligned}&(\ddot{\boldsymbol{u}}_*(t),v)_{\boldsymbol{V}'\times\boldsymbol{V}}+\sum_{\ell=1}^{2}(\mathcal{A}^\ell\varepsilon(\dot{\boldsymbol{u}}_*^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell}+\sum_{\ell=1}^{2}\big(\mathcal{B}^\ell\varepsilon(\boldsymbol{u}_*^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell}+\sum_{\ell=1}^{2}\big((\mathcal{E}^\ell)^*\nabla\xi_*^\ell,\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell}\\ &+\sum_{\ell=1}^{2}\Big(\int_0^t\mathcal{G}^\ell\big(\boldsymbol{\sigma}_*^\ell(s)-\mathcal{A}^\ell\varepsilon(\dot{\boldsymbol{u}}_*^\ell(s))-(\mathcal{E}^\ell)^*\nabla\xi_*^\ell(s),\varepsilon(\boldsymbol{u}_*^\ell(s)),\varsigma_*^\ell(s)\big)ds,\ \varepsilon(\boldsymbol{v}^\ell)\Big)_{\mathcal{H}^\ell}\\ &+j_{ad}(\zeta_*(t),\boldsymbol{u}_*(t),\boldsymbol{v})+j_{vc}(\boldsymbol{u}_*(t),\boldsymbol{v})=(\mathbf{f}(t),v)_{\boldsymbol{V}'\times\boldsymbol{V}},\quad\forall\boldsymbol{v}\in\boldsymbol{V},\end{aligned}\tag{96}$$

and we substitute (95) in (76) to have

$$\begin{aligned}&\varsigma_*(t)\in K,\quad\sum_{\ell=1}^{2}(\dot{\varsigma}_*^\ell(t),\omega^\ell-\varsigma_*^\ell(t))_{L^2(\Omega^\ell)}+a(\varsigma(t),\omega-\varsigma(t))\\ &\geq\sum_{\ell=1}^{2}\Big(\phi^\ell\big(\boldsymbol{\sigma}_*^\ell(t)-\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell(t))-(\mathcal{E}^\ell)^*\nabla\xi^\ell(t),\boldsymbol{\varepsilon}(\boldsymbol{u}_*^\ell(t)),\varsigma_*^\ell(t)\big),\omega^\ell-\varsigma_*^\ell(t)\Big)_{L^2(\Omega^\ell)},\\ &\quad\forall\omega\in K,\ \text{a.e. } t\in[0,T].\end{aligned}\tag{97}$$

We write now (71) for $\eta=\eta^*$ and use (90) to see that

$$\sum_{\ell=1}^{2}(\boldsymbol{\beta}^\ell\nabla\xi_*^\ell(t),\nabla\phi^\ell)_{H^\ell}-\sum_{\ell=1}^{2}(\mathcal{E}^\ell\varepsilon(\boldsymbol{u}_*^\ell(t)),\nabla\phi^\ell)_{H^\ell}=(q(t),\phi)_W,\quad\forall\phi\in W,\ \text{a.e. } t\in[0,T].\tag{98}$$

Additionally, we use $\boldsymbol{u}_{\eta^*}$ in (74) and (90) to find

$$\dot{\zeta}_*(t)=H_{ad}\big(\zeta_*(t),\alpha_{\zeta_*},R_\nu([\boldsymbol{u}_{*\nu}](t)),\boldsymbol{R}_\tau([\boldsymbol{u}_{*\tau}](t))\big),\ \text{a.e. } t\in[0,T].\tag{99}$$

The relations (90), (91), (92), (96), (97), (98) and (99) allow us to conclude now that $\{\boldsymbol{u}_*,\boldsymbol{\sigma}_*,\xi_*,\varsigma_*,\zeta_*,\boldsymbol{D}_*\}$ satisfies (48)-(53). Next, (54) and the regularity (55), (57), (58) and (59) follow from Lemmas 4.1, 4.2, 4.4 and 4.3. Since $\boldsymbol{u}_*$ and ξ_* satisfy (55) and (59), it follows from lemma 4.5 and (91) that

$$\sigma_*\in L^2(0,T;\mathcal{H}).\tag{100}$$

We choose $v=(v^1,v^2)$, with $v^\ell\in D(\Omega^\ell)^d$ and $v^{3-\ell}=0$ in (96) and by (90) and (43):

$$\rho^\ell\ddot{\boldsymbol{u}}_*^\ell=\mathrm{Div}\,\boldsymbol{\sigma}_*^\ell+\boldsymbol{f}_0^\ell,\ \text{a.e. } t\in[0,T],\quad\ell=1,2.$$

Also, by (36), (37), (55) and (100) we have:

$$(\mathrm{Div}\,\boldsymbol{\sigma}_*^1,\mathrm{Div}\,\boldsymbol{\sigma}_*^2)\in L^2(0,T;\boldsymbol{V}')$$

Let $t_1, t_2 \in [0,T]$, by (5), (32), (33) and (92), we deduce that

$$\|\boldsymbol{D}_*(t_1) - \boldsymbol{D}_*(t_2)\|_H \le C\left(\|\xi_*(t_1) - \xi_*(t_2)\|_W + \|\boldsymbol{u}_*(t_1) - \boldsymbol{u}_*(t_2)\|_{\boldsymbol{V}}\right).$$

The regularity of $\boldsymbol{u}_*$ and φ_* given by (55) and (57) implies

$$\boldsymbol{D}_* \in C(0,T;H). \tag{101}$$

We choose $\phi = (\phi^1, \phi^2)$ with $\phi^\ell \in D(\Omega^\ell)^d$ and $\phi^{3-\ell} = 0$ in (98) and using (44), (92) we find

$$\div \boldsymbol{D}_*^\ell(t) = q_0^\ell(t) \quad \forall t \in [0,T], \quad \ell = 1,2,$$

and, by (37), (101), we obtain

$$\boldsymbol{D}_* \in C(0,T;\mathcal{W}).$$

Finally we conclude that the weak solution $\{\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \xi_*, \varsigma_*, \zeta_*, \boldsymbol{D}_*\}$ of the piezoelectric contact problem **PV** has the regularity (55)-(60), which concludes the existence part of Theorem 4.1.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ defined by (81)-(82) and the unique solvability of the Problems PV_η^u, PV_η^ξ, PV_η^ζ, $\mathrm{PV}_\theta^\varsigma$ and $\mathrm{PV}_{\eta\theta}^\sigma$. □

5. Conclusion

We presented a model for the dynamic process of frictionless contact between two elasto-viscoplastic piezoelectric bodies with damage. The contact modeled with normal compliance and adhesion. The existence and the uniqueness weak solution for the problem was established by using arguments from the theory of evolutionary variational inequalities, parabolic equalities and fixed point theorem.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] A. Aissaoui and N. Hemici, A frictional contact problem with damage and adhesion for an electro elastic-viscopalastic body, *Electronic Journal of Differential Equations* **11** (2014), 1 – 19.

[2] A. Azeb Ahmed and S. Boutechebak, Analysis of a dynamic thermo-elastic-viscoplastic contact problem, *Electronic Journal of Qualitative Theory of Differential Equations* **71** (2013), 1 – 17, DOI: 10.14232/ejqtde.2013.1.71.

[3] A. Bachmar and S. Boutechebak, T. Variational analysis of a dynamic electroviscoelastic problem with friction, *Journal of Siberian Federal University. Mathematics & Physics* **12** (2019), 68 – 78, `http://elib.sfu-kras.ru/handle/2311/109322`.

[4] R. C. Batra and J. S. Yang, Saint-Venant's principle in linear piezoelectricity, *Journal of Elasticity* **38** (1995), 209 – 218, DOI: 10.1007/BF00042498.

[5] S. Boutechebak and A. A. Ahmed, Analysis of a dynamic contact problem for electro-viscoelastic materials, *Milan Journal of Mathematics* **86** (2018), 105 – 124, DOI: 10.1007/s00032-018-0282-4.

[6] O. Chau, J. R. Fernandez and M. Sofonea, Variational and numerical analysis of a quasistatic viscoelastic contact problem with adhesion, *Journal of Computational and Applied Mathematics* **159** (2003), 431 – 465, DOI: 10.1016/S0377-0427(03)00547-8.

[7] O. Chau, M. Shillor, M. Sofonea and Z. Angew, Dynamic frictionless contact with adhesion, *Journal of Applied Mathematics and Physics* **55** (2004), 32 – 47, DOI: 10.1007/s00033-003-1089-9.

[8] N. Cristescu and I. Suliciu, *Viscoplasticity*, Martinus Nijhoff Publishers, Editura Tehnica, Bucharest (1982).

[9] E. H. Essoufi and M. Kabbaj, Existence of solutions of a dynamic Signorini's problem with nonlocal friction for Viscoelastic Piezoelectric Materials, *Bulletin mathematique de la Societe des Sciences Mathematiques de Roumanie* **48** (2005), 181 – 195.

[10] M. Frémond, Adhérence des solides, *Journal de Mécanique Théorique et Appliquée* **6** (1987), 383 – 407.

[11] M. Frémond and K. L. Kuttler, Shillor Existence and uniqueness of solutions for a one-dimensional damage mode, *Journal of Mathematical Analysis and Applications* **220** (1999), 271 – 294.

[12] M. Frémond and B. Nedjar, Damage in concrete: The unilateral phenomenon, *Nuclear Engineering and Design* **156** (1995), 323 – 335, DOI: 10.1016/0029-5493(94)00970-A.

[13] M. Frémond and B. Nedjar, Gradient of damage and principle of virtual work, *International Journal of Solids and Structures* **33** (1996), 1083 – 1103, DOI: 10.1016/0020-7683(95)00074-7.

[14] T. Hadj Ammar, A. Saïdi and A. Azeb Ahmed, Dynamic contact problem with adhesion and damage between thermo-electro-elasto-viscoplastic bodies, *Comptes Rendus Mécanique* **345** (2017), 329 – 336, DOI: 10.1016/j.crme.2017.03.002.

[15] T. Hadj Ammar, B. Benabderrahmane and S. Drabla, Frictional contact problem for electro-viscoelastic materials with long-term memory, damage, and adhesion, *Electronic Journal of Differential Equations* **222** (2014), 1 – 21.

[16] I. R. Ionescu and M. Sofonea, *Frictional contact problem for electro-viscoelastic materials with long-term memory, damage, and adhesion*, Oxford University Press, Oxford (1994).

[17] O. Kavian, *Introduction á la théorie des points critiques et Applications aux équations elliptiques*, Springer-Verlag, Berlin — Heidelberg (1993).

[18] K. L. Kuttler and M. Shillor, *Existence for models of damage*, preprint (2001).

[19] R. D. Mindlin, Polarisation gradient in elastic dielectrics, *Int. J. Solids Structures* **4** (1968), 637 – 663.

[20] R. D. Mindlin, Elasticity, piezoelectricity and crystal lattice dynamics, *Journal of Elasticity* **2** (1972), 217 – 280, DOI: 10.1007/BF00045712.

[21] J. Nečas and I. Hlaváček, *Mathematical Theory of Elastic and Elastico-Plastic Bodies: An Introduction*, Amsterdam, Oxford, New York (1981).

[22] M. Rochdi, M. Shillor and M. Sofonea, Analysis of a quasistatic viscoelastic problem with friction and damage, *Advances in Mathematical Sciences and Applications* **10** (2000), 173 – 189.

[23] M. Selmani and L. Selmani, A dynamic frictionless contact problem with adhesion and damage, *Bulletin of the Polish Academy of Sciences & Mathematics* **55** (2007), 17 – 34, DOI: 10.4064/ba55-1-3.

[24] M. Shillor, M. Sofonea and J. J. Telega, *Models and Analysis of Quasistatic Contact, Lecture Notes Physics* **655**, Springer, Berlin (2004), DOI: 10.1007/b99799.

[25] M. Sofonea and M. Barboteu, Analysis and numerical approach of a piezoelectric contact problem, *Mathematics and its Applications: Annals of the Academy of Romanian Scientists* **1** (2009), 7 – 30.

[26] M. Sofonea, W. Han and M. Shillor, *Analysis and Approximation of Contact Problems with Adhesion or Damage, Pure and Applied Mathematics*, **276**, Chapman-Hall/CRC Press, New York (2006).

[27] P. Suquet, *Plasticité et homogénéisation*, PhD. thesis, Université Pierre et Marie Curie, Paris 6 (1982).

[28] W. Voigt, *Lehrbuch der Kristall-Physik*, Teubner, Leipzig (1910).

[29] L. Zhor, Z. Zellagui, H. Benseridi and S. Drabla, Variational analysis of an electro viscoelastic contact problem with friction, *Journal of the Association of Arab Universities for Basic and Applied Sciences* **14**(1) (2013), 99 – 100, DOI: 10.1016/j.jaubas.2012.10.001.