Journal of Informatics and Mathematical Sciences
Volume 1 (2009), Number 1, pp. 75–80

A Curious Strong Resemblance between the Goldbach Conjecture and Fermat Last Assertion

Ikorong Anouk Gilbert Nemron

Abstract. The Goldbach conjecture (see [2] or [3] or [4]) states that every even integer $e \geq 4$ is of the form $e = p + q$, where (p, q) is a couple of prime(s). The Fermat last assertion [solved by A. Wiles (see [1])] stipulates that when n is an integer ≥ 3, the equation $x^n + y^n = z^n$ has not solution in integers ≥ 1. In this paper, via two simple Theorems, we present a curious strong resemblance between the Goldbach conjecture and the Fermat last assertion.

1. Preliminary

This paper is an original investigation around *the Fermat last assertion* and the *Goldbach conjecture*, and is divided into two simple Sections (namely Section 2 and Section 3). In Section 2, we introduce definitions that are not standard, and some elementary properties. In Section 3, using definitions of Section 2, we show, via two simple Theorems, a strong resemblance between the *Fermat last assertion* and the *Goldbach conjecture*. Here, all properties are original, and therefore, are completely different from all the investigations that have been done on the Fermat last assertion and the Goldbach conjecture in the past.

2. Non-standard definitions and simple properties

In this section, we introduce definitions that are not standard. These definitions are determining for the final two Theorems.

We say that e is *goldbach*, if e is an even integer ≥ 4 and is of the form $e = p+q$, where (p, q) is a couple of prime(s). The Goldbach conjecture (see Abstract) states that every even integer $e \geq 4$ is goldbach [**Example 1.** 100 is *goldbach*, because 100 is an even integer ≥ 4 and is of the form $100 = 53 + 47$, where 53 and 47 are prime]. We say that e is *goldbachian*, if e is an even integer ≥ 4, and if every even integer v with $4 \leq v \leq e$ is goldbach [**Example 2.** 100 is *golbachian*. Indeed, 100 is an even integer ≥ 4, and it is easy to check that every even integer v of

---

2000 *Mathematics Subject Classification.* 05xx; 11xx.
Key words and phrases. Goldbach; Goldbachian; Wiles; Wilian's.

the form $4 \leq v \leq 100$ is *goldbach*. Note that *goldbachian* implies *goldbach*; so there is not confusion between *goldbachian* and *goldbach*]. Now, for every integer $n \geq 2$, we define $\mathscr{G}(n)$ and g_n as follows: $\mathscr{G}(n) = \{g; 1 < g \leq 2n$, and g is *goldbachian*$\}$, and $g_n = \max_{g \in \mathscr{G}(n)} g$. From the definition of $\mathscr{G}(n)$ and g_n, we immediately deduce: $\mathscr{G}(n+1) = \{g; 1 < g \leq 2n+2$, and g is goldbachian$\}$, and $g_{n+1} = \max_{g \in \mathscr{G}(n+1)} g$. In Section 3, g_{n+1} will help us to give the surgical reformulation of the Goldbach conjecture. From the definition of g_n and g_{n+1}, it is immediate to see.

Proposition 2.1. *Let n be an integer ≥ 2. We have the following two simple properties.*

(2.1.1) *g_n and g_{n+1} are even; $g_n \leq 2n$; and $g_n \leq g_{n+1} \leq 2n+2$.*
(2.1.2) *$g_{n+1} = 2n+2$ if and only if $2n+2$ is goldbachian.*

Proof. Properties (2.1.1) and (2.1.2) are immediate [it suffices to use the definition of g_n and g_{n+1}]. □

Proposition 2.2. *Let n be an integer ≥ 2. Then we have the following three simple properties.*

(2.2.1) *If $2n+2$ is not goldbachian, then $n > 2$ and for every integer $u \geq n$, $2u+2$ is not goldbachian.*
(2.2.2) *If $g_{n+1} < 2n+2$ [i.e., if $2n+2$ is not goldbachian], then $n > 2$ and for every integer $u \geq n$, we have $g_{u+1} = g_{n+1} = g_n$.*
(2.2.3) *If $g_{n+1} = Z$, where $Z < 2n+2$, then $\lim_{y \to +\infty} g_{n+1+y} = Z$.*

Proof. Property (2.2.1) is immediate [it suffices to apply the definition of goldbachian]; property (2.2.2) is an immediate reformulation of Property (2.2.1), and property (2.2.3) is an obvious consequence of property (2.2.2). □

That being so, we say that e is *wiles*, if e is an integer ≥ 3 and if the equation $x^e + y^e = z^e$ has not solution in integers ≥ 1. The Fermat last assertion [solved by Wiles (see Abstract)] states that every integer $e \geq 3$ is wiles [**Example 3.** It is known that 3, 4, 5 and 6 are all *wiles*]. We say that e is *Wilian's*, if e is an integer ≥ 3, and if every integer v with $3 \leq v \leq e$ is wiles [**Example 4.** 6 is *Wilian's*. Indeed, 6 is an integer ≥ 3, and using Example 3, we see that every integer v of the form $3 \leq v \leq 6$ is *wiles*. Note that *Wilian's* implies *wiles;* so there is not confusion between *Wilian's* and *wiles*]. Now, for every integer $n \geq 3$, we define $\mathscr{W}(n)$ and w_n as follows: $\mathscr{W}(n) = \{w; 2 < w \leq n$, and w is *Wilian's*$\}$, and $w_n = 2 \max_{w \in \mathscr{W}(n)} w$. From the definition of $\mathscr{W}(n)$ and w_n, we immediately deduce: $\mathscr{W}(n+1) = \{w; 2 < w \leq n+1$, and w is *Wilian's*$\}$ and $w_{n+1} = 2 \max_{w \in \mathscr{W}(n+1)} w$. In Section 3, w_{n+1} will help us to give the surgical reformulation of the Fermat last assertion. From the definition of w_n and w_{n+1}, it is immediate to see.

Proposition 2.3. *Let n be an integer ≥ 3. We have the following two simple properties.*

(2.3.1) *w_n and w_{n+1} are even; $w_n \leq 2n$; and $w_n \leq w_{n+1} \leq 2n+2$.*
(2.3.2) *$w_{n+1} = 2n+2$ if and only if $n+1$ is Wilian's.*

Proof. Properties (2.3.1) and (2.3.2) are immediate [it suffices to use the definition of w_n and w_{n+1}]. □

Observe that Proposition 2.3 resembles to Proposition 2.1.

Proposition 2.4. *Let n be an integer ≥ 3. Then we have the following three simple properties.*

(2.4.1) *If $n+1$ is not Wilian's, then $n > 3$ and for every integer $u \geq n$, $u+1$ is not Wilian's.*
(2.4.2) *If $w_{n+1} < 2n+2$ [i.e., if $n+1$ is not Wilian's], then $n > 3$ and for every integer $u \geq n$, we have $w_{u+1} = w_{n+1} = w_n$.*
(2.4.3) *If $w_{n+1} = Z$, where $Z < 2n+2$, then $\lim_{y \to +\infty} w_{n+1+y} = Z$.*

Proof. Property (2.4.1) is immediate [it suffices to apply the definition of Wilian's and to observe that, if $n = 3$, then $n+1$ is Wilian's]; property (2.4.2) is an immediate reformulation of Property (2.4.1), and property (2.4.3) is an obvious consequence of property (2.4.2). □

Observe that Proposition 2.4 resembles to Proposition 2.2. Now for every integer $n \geq 2$, we define $\mathscr{P}(n)$ and p_n as follows: $\mathscr{P}(n) = \{p; p$ is prime and $1 < p < 2n\}$, and $p_n = \max_{p \in \mathscr{P}(n)} p$. Using the definition of p_n, it is known:

Theorem 2.5 (The Postulate of Bertrand or Erdos Theorem). *Let n be an integer ≥ 1, then there exists a prime between n and $2n$.*

Corollary 1. *For every integer $n \geq 2$, $p_n \geq n$.*

Proof. Use definition of p_n and Theorem 2.5. □

3. **The strong resemblance between the Goldbach conjecture and the Fermat last assertion**

In this section, we show that the *Goldbach conjecture* and the *Fermat last assertion* can be restated in ways that resemble each other. More precisely, we prove two Theorems which show that the *Goldbach conjecture* and the *Fermat last assertion* are clearly resembling. Using the definition of *goldbachian*, then the following first Theorem is the surgical reformulation of the Goldbach conjecture.

Theorem 3.1 (The surgical reformulation of the Goldbach conjectuire). *The following are equivalent.*

(1) *The Goldbach conjecture holds [i.e. every even integer $e \geq 4$ is of the form $e = p + q$, where (p, q) is a couple of prime(s)].*
(2) *For every integer $n \geq 2$, $2n + 2$ is goldbachian.*
(3) *For every integer $n \geq 2$, $g_{n+1} = 2n + 2$.*
(4) *For every integer $n \geq 2$, $g_n \geq p_n$.*
(5) *For every integer $n \geq 2$, $g_n \geq n$.*
(6) $\lim_{n \to +\infty} g_{n+1} = +\infty$.

To prove easily Theorem 3.1, let us remark.

Remark 1. The following are equivalent.

(i) The Goldbach conjecture holds.
(ii) $\lim_{n \to +\infty} g_{n+1} = +\infty$.

Proof. (i)⇒(ii): Immediate [it suffices to use the definition of the Goldbach conjecture].

(ii)⇒(i): Otherwise, let M be a finite integer such that $2M + 2$ is not goldbachian [such a M clearly exists, since the Goldbach conjecture is false]; since $2M + 2$ is not goldbachian, clearly

$$g_{M+1} \neq 2M + 2 \tag{3.1}$$

Observing [by using the definition of g_{M+1}] that

$$g_{M+1} \leq 2M + 2 \tag{3.2}$$

then, using (3.1) and (3.2), we immediately deduce that

$$g_{M+1} < 2M + 2 \tag{3.3}$$

Inequality (3.3) clearly says that

$$g_{M+1} = Z, \quad \text{where } Z < 2M + 2 \tag{3.4}$$

it is immediate that Z is a *finite integer*, since M is a finite integer. Now using (3.4) and property (2.2.3) of Proposition 2.2, then we immediately deduce that $\lim_{n \to +\infty} g_{M+1+n} = Z$, where Z is a finite integer; the previous immediately implies that $\lim_{n \to +\infty} g_{n+1} = Z$, where Z is a *finite integer*, and this is absurd, since $\lim_{n \to +\infty} g_{n+1} = +\infty$ [by the hypothesis]. So, assuming that the Goldbach conjecture is false gives rise to a serious contradiction; so the Goldbach conjecture holds. □

***Proof of Theorem* 3.1.** (1)⇒(2): Immediate [it suffices to use the definition of the Goldbach conjecture and the definition of goldbachian]; (2)⇒(3): Immediate [it suffices to use the definition of goldbachian and the definition of g_{n+1}]; (3)⇒(4): Immediate [it suffices to observe (by using the definition of p_n) that $p_n \leq 2n - 1$]; (4)⇒(5): Immediate [it suffices to use Corollary 1]; (5)⇒(6): Indeed, observing [by the hypothesis] that for every integer $n \geq 2$ we have $g_{n+1} \geq n$, clearly

$\lim_{n\to+\infty} g_{n+1} = +\infty$; (6)⇒(1): Observing [by the hypothesis] that $\lim_{n\to+\infty} g_{n+1} = +\infty$, then, using Remark 1, we immediately deduce that the Goldbach conjecture holds. □

Now using the definition of *Wilian's*, then the following Theorem is the corresponding surgical reformulation of the Fermat last assertion.

Theorem 3.2 (The surgical reformulation of the Fermat last assertion). *The following are equivalent.*

(1) *The Fermat last assertion holds [i.e. for every integer $n \geq 3$, the equation $x^n + y^n = z^n$ has not solution in integers ≥ 1].*
(2) *For every integer $n \geq 3$, $n+1$ is Wilian's.*
(3) *For every integer $n \geq 3$, $w_{n+1} = 2n+2$.*
(4) *For every integer $n \geq 3$, $w_{n+1} \geq p_n$.*
(5) *For every integer $n \geq 3$, $w_{n+1} \geq n$.*
(6) $\lim_{n\to+\infty} w_{n+1} = +\infty$.

To prove easily Theorem 3.2, let us remark.

Remark 2. The following are equivalent.

(i) The Fermat last assertion holds.
(ii) $\lim_{n\to+\infty} w_{n+1} = +\infty$.

Proof. (i)⇒(ii): Immediate [it suffices to use the definition of the Fermat last assertion].

(ii)⇒(i): Otherwise, let M be a finite integer such that $M+1$ is not Wilian's[such a M exists, since the Fermat last assertion is false]; since $M+1$ is not Wilian's, clearly

$$w_{M+1} \neq 2M+2 \tag{3.5}$$

Observing [by using the definition of w_{M+1}] that

$$w_{M+1} \leq 2M+2 \tag{3.6}$$

then, using (3.5) and (3.6), we immediately deduce that

$$w_{M+1} < 2M+2 \tag{3.7}$$

Inequality (3.7) clearly says that

$$w_{M+1} = Z, \quad \text{where } Z < 2M+2 \tag{3.8}$$

it is immediate that Z is a *finite integer*, since M is a finite integer. Now using (3.8) and property (2.4.3) of Proposition 2.4, then we immediately deduce that $\lim_{n\to+\infty} w_{M+1+n} = Z$, where Z is a finite integer; the previous immediately implies that $\lim_{n\to+\infty} w_{n+1} = Z$, where Z is a *finite integer*, and this is absurd, since

$\lim_{n\to+\infty} w_{n+1} = +\infty$ [by the hypothesis]. So, assuming that the Fermat last assertion is false gives rise to a serious contradiction; so the Fermat last assertion holds. $\square$

Observe that Remark 2 and Remark 1 are resembling.

***Proof of Theorem* 3.2.** (1)⇒(2): Immediate [it suffices to use the definition of the Fermat last assertion and the definition of Wilian's]; (2)⇒(3): Immediate [it suffices to use the definition of Wilian's and the definition of w_{n+1}]; (3)⇒(4): Immediate [it suffices to observe (by using the definition of p_n) that $p_n \leq 2n-1$]; (4)⇒(5): Immediate [it suffices to use Corollary 1]; (5)⇒(6): Indeed, observing [by the hypothesis] that for every integer $n \geq 3$ we have $w_{n+1} \geq n$, clearly $\lim_{n\to+\infty} w_{n+1} = +\infty$; (6)⇒ (1): Observing [by the hypothesis] that $\lim_{n\to+\infty} w_{n+1} = +\infty$, then, using Remark 2, we immediately deduce that the Fermat last assertion holds. $\square$

Visibly, Theorem 3.2 and Theorem 3.1 are strongly resembling. Specially, if the Goldbach conjecture and the Fermat last assertion *simultaneously hold*, then Theorem 3.2 and Theorem 3.1 immediately imply that $w_{n+1} = g_{n+1}$, for every integer $n \geq 3$. The resemblance between Theorem 3.1 and Theorem 3.2 helps us to conjecture the following.

Conjecture. *For every integer $n \geq 3$, $g_{n+1} = w_{n+1}$.*

References

[1] A. Wiles, Modular elliptic Curbes and Fermat's last theorem, *Annals of Mathematics* **141** (1995), 443–551.

[2] Ikorong Anouk Gilbert Nemron, An alternative reformulation of the Goldbach conjecture and the twin primes conjecture, *Mathematicae Notae* **XLIII** (2005), 101–107.

[3] Ikorong Anouk Gilbert Nemron, Around the twin primes conjecture and the Goldbach conjecture I, *Analele Stiintifice Ale Universitatii "Sectiunea Matematica"*, **LIII** (2007), 23–34.

[4] Ikorong Anouk Gilbert, A reformulation of the goldbach conjecture, *Journal of Discrete Mathematical Sciences and Cryptography* **11** (4) (2008), 465–469.

Ikorong Anouk Gilbert Nemron, *Centre de Calcul, d'Enseignement et de Recherche, Univ. Pierre et Marie Curie, France.*
E-mail: `ikorong@ccr.jussieu.fr`

Received April 30, 2009
Accepted July 21, 2009