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Green's Condition and Green-Kehayopulu Relations on le-Ternary Semigroups

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Abstract. We introduce the concept of the Green-Kehayopulu relations in *le*-ternary semigroups mimics the definition of the Green-Kehayopulu relations in le-semigroups that was introduced in 2002 by Petro and Pasku [5] and investigate the Green-Kehayopulu relations in *le*-ternary semigroups.

1. Introduction

The literature of ternary algebraic system was introduced by Lehmer [4] in 1932. He investigated certain ternary algebraic systems called triplexes which turn out to be ternary groups. The notion of ternary semigroups was known to S. Banach. He showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup. We can see that any semigroup can be reduced to a ternary semigroup. In 2002, Petraq Petro and Elton Pasku [5] introduced the concept of the Green-Kehayopulu relations in le-semigroups and showed that a nonsingleton $\mathscr{H}$-class cannot be a subgroup and an $\mathscr{H}$-class satisfying "Green's condition" need not constitute a subsemigroup.

The main purpose of this paper is to introduce the concept of the Green-Kehayopulu relations in *le*-ternary semigroups and give necessary and sufficient conditions in order that an $\mathscr{H}_t$-class of *le*-ternary semigroup T is a subgroup or a subsemigroup of $\langle T_t, \circ \rangle$.

2. Basic Definitions

We first recall the definition of a ternary semigroup which is important here.

A nonempty set T is called a *ternary semigroup* [4] if there exists a ternary operation $[\ ]\colon T \times T \times T \to T$, written as $(x_1, x_2, x_3) \mapsto [x_1 x_2 x_3]$, satisfying the

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following identity for any $x_1, x_2, x_3, x_4, x_5 \in T$,

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]].$$

A nonempty subset S of a ternary semigroup T is called a *ternary subsemigroup* [1] of T if $[SSS] \subseteq S$.

For any positive integers m and n with $m \leq n$ and any elements $x_1, x_2, \dots, x_{2n}$ and x_{2n+1} of a ternary semigroup T [6], we can write

$$\begin{aligned}[x_1x_2\dots x_{2n+1}] &= [x_1\dots x_m x_{m+1}x_{m+2}\dots x_{2n+1}]\\ &= [x_1\dots[[x_m x_{m+1}x_{m+2}]x_{m+3}x_{m+4}]\dots x_{2n+1}].\end{aligned}$$

Example 1 ([1])**.** Let $T = \{-i, 0, i\}$. Then T is a ternary semigroup under the multiplication over complex number while T is not a semigroup under complex number multiplication.

Example 2 ([1])**.** Let $O = \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$, $I = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $A_1 = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, $A_2 = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, $A_3 = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ and $A_4 = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$. Then $T = \{O, I, A_1, A_2, A_3, A_4\}$ is a ternary semigroup under matrix multiplication.

For any $t \in T$, an element x of a ternary semigroup T is said to be a *t-idempotent* if $[xtx] = x$. For a ternary semigroup T and any $t \in T$, if we define $a \circ b = [atb]$ for all $a, b \in T$, then T becomes a semigroup. We denote this semigroup by T_t.

A ternary semigroup T is called an le-*ternary semigroup* if $\langle T; \vee, \wedge\rangle$ is a lattice with a greatest element (the element is always denoted by e below) [3] and for any $a, b, x, y \in T$,

$$[xy(a \vee b)] = [xya] \vee [xyb] \quad \text{and} \quad [(a \vee b)xy] = [axy] \vee [bxy].$$

Throughout this paper T will stand for an *le*-ternary semigroup. We shall consider the usual order relation $\leq$ on T defined by for any $a, b \in T$, $a \leq b$ if and only if $a \vee b = b$. Then we can show that for any $a, b, x, y \in T$, $a \leq b$ implies $[axy] \leq [bxy]$, $[xay] \leq [xby]$ and $[xya] \leq [xyb]$. Hence we have known that ordered ternary semigroups are a generalization of *le*-ternary semigroups. For any $t \in T$, let the mappings $l_t, r_t : T \to T$ be defined by for any $x \in T$,

$$l_t(x) = [etx] \vee x \quad \text{and} \quad r_t(x) = [xte] \vee x.$$

Then we define equivalence relations on T as follows:

$$\begin{aligned}\mathscr{L}_t &:= \{(x, y) \mid l_t(x) = l_t(y)\},\\ \mathscr{R}_t &:= \{(x, y) \mid r_t(x) = r_t(y)\},\\ \mathscr{H}_t &:= \mathscr{L}_t \cap \mathscr{R}_t.\end{aligned}$$

We shall call the equivalences $\mathscr{L}_t, \mathscr{R}_t$ and $\mathscr{H}_t$ the *Green-Kehayopulu relations* of T. An element x of T is said to be a *t-left ideal* (*t-right ideal*) *element* if $l_t(x) = x$ ($r_t(x) = x$) and a *t-ideal element* if it is both a t-left ideal element and a t-right ideal element; it is called a *t-quasi-ideal element* if $[etx] \wedge [xte] \leq x$. An element x

of T is said to be a *t-regular element* if $x \leq [x[tet]x]$ and a *t-intra-regular element* if $x \leq [[etx]t[xte]]$. An $\mathscr{H}_t$-class H of T satisfying Green's condition if there exist elements a and b of T such that $[atb] \in H$.

3. **Lemmas**

Before the characterizations of the $\mathscr{H}_t$-class of T for the main results, we give auxiliary results which are necessary in what follows.

Lemma 3.1. *For each $x, t \in T$,*

$$l_t(l_t(x)) = l_t(x) \quad \text{and} \quad r_t(r_t(x)) = r_t(x).$$

Proof. From the definition of the mapping l_t it follows that $l_t(l_t(x)) = l_t([etx] \vee x) = [et([etx] \vee x)] \vee [etx] \vee x = [et[etx]] \vee [etx] \vee [etx] \vee x = [et[etx]] \vee [etx] \vee x$. Since e is the greatest element in T, we also have $[ete] \leq e$. Thus $[et[etx]] = [[ete]tx] \leq [etx]$, so $[et[etx]] \vee [etx] = [etx]$. Hence $l_t(l_t(x)) = [etx] \vee x = l_t(x)$. By symmetry, $r_t(r_t(x)) = r_t(x)$. □

Lemma 3.2. *If an element a of T is a t-left ideal element and an element b of T is a t-right ideal element, then $a \wedge b$ is a t-quasi-ideal element.*

Proof. Assume that a is a t-left ideal element and b is a t-right ideal element of T. Then $[eta] \vee a = l_t(a) = a$ and $[bte] \vee b = r_t(b) = b$, so $[eta] \leq a$ and $[bte] \leq b$. Hence $[et(a \wedge b)] \wedge [(a \wedge b)te] \leq [eta] \wedge [bte] \leq a \wedge b$. Therefore $a \wedge b$ is a t-quasi-ideal element. □

Lemma 3.3. *For each $x, t_1, t_2 \in T$,*

$$l_{t_2}(l_{t_2}(x) \wedge r_{t_1}(x)) = l_{t_2}(x) \quad \text{and} \quad r_{t_1}(l_{t_2}(x) \wedge r_{t_1}(x)) = r_{t_1}(x).$$

Proof. Since $x = x \wedge x \leq l_{t_2}(x) \wedge r_{t_1}(x) \leq l_{t_2}(x)$, it follows from Lemma 3.1 that $l_{t_2}(x) \leq l_{t_2}(l_{t_2}(x) \wedge r_{t_1}(x)) \leq l_{t_2}(l_{t_2}(x)) = l_{t_2}(x)$. Hence $l_{t_2}(l_{t_2}(x) \wedge r_{t_1}(x)) = l_{t_2}(x)$. By symmetry, $r_{t_1}(l_{t_2}(x) \wedge r_{t_1}(x)) = r_{t_1}(x)$. □

Lemma 3.4. *Each $\mathscr{H}_t$-class H of T has a greatest element which is equal to $l_t(a) \wedge r_t(a)$ where a is an arbitrary element in H.*

Proof. Let a be an element of the $\mathscr{H}_t$-class H of T. By Lemma 3.3, we have $(l_t(a) \wedge r_t(a), a) \in \mathscr{L}_t$ and $(l_t(a) \wedge r_t(a), a) \in \mathscr{R}_t$. Thus $(l_t(a) \wedge r_t(a), a) \in \mathscr{H}_t$, so $l_t(a) \wedge r_t(a) \in H$. Now let any $x \in H$. Then $(x, a) \in \mathscr{H}_t = \mathscr{L}_t \cap \mathscr{R}_t$, this implies that $x \leq l_t(x) = l_t(a)$ and $x \leq r_t(x) = r_t(a)$. Hence $x \leq l_t(a) \wedge r_t(a)$, so $l_t(a) \wedge r_t(a)$ is a greatest element of H. □

Lemmas 3.1 and 3.2 imply that for each element a of T, the meet $l_t(a) \wedge r_t(a)$ is a t-quasi-ideal element. Lemma 3.4 implies that for each element a of the $\mathscr{H}_t$-class H, $l_t(a) \wedge r_t(a)$ is a greatest element of H. We call the element $l_t(a) \wedge r_t(a)$

the *representative t-quasi-ideal element* of the $\mathscr{H}_t$-class of a; the representative t-quasi-ideal element of an $\mathscr{H}_t$-class H will be denoted by q_H. From Lemma 3.4, the following properties of q_H hold.

(1) $q_H \in H$.
(2) For each $x \in H$, $l_t(x) \wedge r_t(x) = q_H$; in particular, $l_t(q_H) \wedge r_t(q_H) = q_H$.
(3) For each $x \in H$, $x \le q_H$.

Lemma 3.5. *If elements x and y of T are $\mathscr{R}_t$-related (resp. $\mathscr{L}_t$-related), then $[xte] = [yte]$ (resp. $[etx] = [ety]$).*

Proof. Assume that $(x, y) \in \mathscr{R}_t$. Then $r_t(x) = r_t(y)$, so $[xte] \vee x = [yte] \vee y$. This implies that $[[xte]te] \vee [xte] = [([xte] \vee x)te] = [([yte] \vee y)te] = [[yte]te] \vee [yte]$. Since $[ete] \le e$, $[[xte]te] = [xt[ete]] \le [xte]$ and $[[yte]te] = [yt[ete]] \le [yte]$. Hence $[xte] = [[xte]te] \vee [xte] = [[yte]te] \vee [yte] = [yte]$. Similarly, $(x, y) \in \mathscr{L}_t$ implies $[etx] = [ety]$. □

Lemma 3.6. *If H is an $\mathscr{H}_t$-class of T and $x \in H$, then $[etx] \wedge [xte] = [etq_H] \wedge [q_H te]$.*

Proof. Assume that H is an $\mathscr{H}_t$-class of T and $x \in H$. Then $(x, q_H) \in \mathscr{H}_t$. It follows from Lemma 3.5 that $[etx] = [etq_H]$ and $[xte] = [q_H te]$. Hence $[etx] \wedge [xte] = [etq_H] \wedge [q_H te]$. □

4. **Main Results**

In this section, we characterize the relationship between the $\mathscr{H}_t$-classes of T satisfying Green's condition and the semigroup $\langle T_t, \circ \rangle$ and give some conditions which ensure that an $\mathscr{H}_t$-class of T forms a subgroup or a subsemigroup of the semigroup $\langle T_t, \circ \rangle$.

The following theorems collect several properties that hold in every $\mathscr{H}_t$-class of T satisfying Green's condition.

Theorem 4.1. *Let H be an $\mathscr{H}_t$-class of T satisfying Green's condition and $q = q_H$. Then we have the following statements:*

(a) *$[qtq] \in H$ and $q = [etq] \wedge [qte]$.*
(b) *The element q is the only t-quasi-ideal element in H.*
(c) *If $x, y \in H$, then $y \le [etx]$ and $y \le [xte]$.*
(d) *For each integer $n \ge 2$, let $t_1, t_2, \ldots, t_{n-1} \in \{t\}$. Then $[qtq] = [[qte]tq] = [[[[qt_1q]t_2q] \ldots q]t_{n-1}q]$; in particular, $[qtq]$ is a t-idempotent.*
(e) *Every element of H is a t-intra-regular element.*
(f) *The element q is a t-idempotent if and only if q is a t-regular element in which case every element of H is a t-regular element.*

Proof. (a) Since H satisfies Green's condition, there exist $b, c \in H$ such that $[btc] \in H$. Since $b, c \in H$, we have $b \le q$ and $c \le q$. Thus $[btc] \le [qtq] \le$

$[qte]$, this implies that $r_t([btc]) \leq r_t([qtq]) \leq r_t([qte])$. Since $([btc], q) \in \mathscr{H}_t$, $([btc], q) \in \mathscr{R}_t$. Thus $r_t([btc]) = r_t(q)$. On the other hand, since $[ete] \leq e$, we have $r_t([qte]) = [[qte]te] \vee [qte] = [qt[ete]] \vee [qte] = [qte] \leq [qte] \vee q = r_t(q)$. Hence $r_t(q) = r_t([btc]) \leq r_t([qtq]) \leq r_t([qte]) = [qte] \leq r_t(q)$, so $r_t(q) = r_t([qtq]) = [qte]$. By symmetry, $l_t(q) = l_t([qtq]) = [etq]$. Therefore $(q, [qtq]) \in \mathscr{H}_t$, so $[qtq] \in H$. It follows that $q = l_t(q) \wedge r_t(q) = [etq] \wedge [qte]$.

(b) By (a), q is a t-quasi-ideal element in H. Now let t be any t-quasi-ideal element in H. By (a) and Lemma 3.6, we have $t \leq q = [etq] \wedge [qte] = [ett] \wedge [tte] \leq t$. Hence $t = q$, so we conclude that q is the only t-quasi-ideal element in H.

(c) Let any $x, y \in H$. By (a) and Lemma 3.6, we have $y \leq q = [etq] \wedge [qte] = [etx] \wedge [xte]$. Hence $y \leq [etx]$ and $y \leq [xte]$.

(d) By (a), $q = [etq] \wedge [qte] \leq [qte]$. Thus $[qtq] \leq [[qte]tq]$. Since $[etq] \leq e$, $[[qte]tq] = [qt[etq]] \leq [qte]$. Similarly, since $[qte] \leq e$, $[[qte]tq] \leq [etq]$. Thus $[[qte]tq] \leq [etq] \wedge [qte] = q$. Hence $[[[qtq]te]tq] = [[qt[qte]]tq] \leq [qtq]$. By (a), we get $([qtq], q) \in \mathscr{R}_t$. By Lemma 3.5, $[qte] = [[qtq]te]$ and it follows that $[[qte]tq] = [[[qtq]te]tq]$. Hence $[qtq] \leq [[qte]tq]$ and $[[qte]tq] \leq [qtq]$, so $[qtq] = [[qte]tq]$. Now let any integer $k \geq 2$ and $t_1, t_2, \ldots, t_{k-1} \in \{t\}$ be such that $[[[[qt_1q]t_2q]\ldots q]t_{k-1}q] = [qtq]$. Then $[[[[[qt_1q]t_2q]\ldots q]t_{k-1}q]tq] = [[qtq]tq] = [qt[qtq]] = [qt[[qte]tq]] = [[[qtq]te]tq] = [[qte]tq] = [qtq]$. In particular, $[[qtq]t[qtq]] = [qtq]$. Hence $[qtq]$ is a t-idempotent.

(e) Let any $x \in H$. Then $x \leq q$. By (a), we get $q \leq [etq]$ and $q \leq [qte]$. Thus $x \leq [etq] \leq [et[qte]] = [[etq]te]$. By (a), we get $([qtq], q) \in \mathscr{R}_t$. By Lemma 3.5, $[qte] = [[qtq]te]$. This implies that $x \leq [[etq]te] = [et[qte]] = [et[[qtq]te]] = [[etq]t[qte]]$. Since $(x, q) \in \mathscr{H}_t$, it follows from Lemma 3.5 that $[etq] = [etx]$ and $[qte] = [xte]$. Hence $x \leq [[etx]t[xte]]$, so we conclude that x is a t-intra-regular element.

(f) Assume that $q = [qtq]$. By (d), $[qtq] = [[qte]tq]$. Thus $q = [[qte]tq] = [q[tet]q]$, so q is a t-regular element. If $x \in H$, then $x \leq q$. Since $(x, q) \in \mathscr{H}_t$, it follows from Lemma 3.5 that $[etq] = [etx]$ and $[qte] = [xte]$. Hence $x \leq q = [[qte]tq] = [[xte]tq] = [xt[etq]] = [x[tet]x]$. Therefore x is a t-regular element.

Conversely, assume that $q \leq [q[tet]q]$. By (d), $[qtq] = [q[tet]q]$. Thus $q \leq [qtq]$. By (a), $[qtq] \in H$. Thus $[qtq] \leq q$. Hence $q = [qtq]$, so we conclude that q is a t-idempotent.

Therefore we complete the proof of the theorem. □

Using the Theorem 4.1(a) and (d), we have Corollary 4.2.

Corollary 4.2. *An $\mathscr{H}_t$-class H of T satisfies Green's condition if and only if it contains a t-idempotent.*

Theorem 4.3. *An $\mathscr{H}_t$-class H of T is a subgroup of $\langle T_t, \circ\rangle$ if and only if it consists of a single idempotent.*

Proof. Assume that H is a subgroup of T_t and let $q = q_H$. Then $[qtq] = q \circ q \in H$, so $[qtq] \leq q$. Denote by i the identity element of H. Then $i \leq q$, so $q \circ q = [qtq] \leq q = q \circ i = [qti] \leq [qtq] = q \circ q$. Hence $q \circ q = q$, so we conclude that $q = i$. Now let t be an arbitrary element of H. We denote by t^{-1} the inverse element of t in H. Then $t^{-1} \leq q$, so $q = i = t \circ t^{-1} = [ttt^{-1}] \leq [ttq] = t \circ q = t \circ i = t$. On the other hand, $t \leq q$. Therefore $t = q$, so we conclude that H consists of a single idempotent.

The converse is obvious. □

Theorem 4.4. *Let H be an $\mathscr{H}_t$-class of M and $q = q_H$. Then the following statements are equivalent:*

(a) *An $\mathscr{H}_t$-class H is a subsemigroup of $\langle T_t, \circ\rangle$.*
(b) *If $x \in H$, then $[xtx] \in H$.*
(c) *An $\mathscr{H}_t$-class H satisfies Green's condition and $[xtq] = [qtq] = [qtx]$ for every $x \in H$.*

Proof. Since H is a subsemigroup of T_t, we immediately have $[xtx] = x \circ x \in H$ for all $x \in H$. Therefore (a) implies (b). Let any $x \in H$. Then $[xtx] \in H$, so H satisfies Green's condition and $(x, [xtx]) \in \mathscr{H}_t$. By Lemma 3.5, $[etx] = [et[xtx]]$ and $[xte] = [[xtx]te]$. Similarly, since $(x,q) \in \mathscr{H}_t$, we get $[etx] = [etq]$ and $[xte] = [qte]$. By Theorem 4.1(d), $[qtq] = [[qte]tq]$. Hence $[xt[qtq]] = [xt[[qte]tq] = [xt[[xte]tq] = [[[xtx]te]tq] = [[xte]tq] = [[qte]tq] = [qtq]$. Similarly, $[[qtq]tx] = [qtq]$. Since $x, [qtq] \in H$, we have $x \leq q$ and $[qtq] \leq q$. Hence $[qtq] = [xt[qtq]] \leq [xtq] \leq [qtq]$, so we conclude that $[xtq] = [qtq]$. Similarly, $[qtx] = [qtq]$. Thus (b) implies (c). Let any $x, y \in H$. Then $(y,q) \in \mathscr{H}_t$, so $(y,q) \in \mathscr{R}_t$. Thus $r_t(y) = r_t(q)$, so $[yte] \vee y = [qte] \vee q$. Hence $r_t([xty]) = [[xty]te] \vee [xty] = [xt[yte]] \vee [xty] = [xt([yte] \vee y)] = [xt([qte] \vee q)] = [xt[qte]] \vee [xtq] = [[xtq]te] \vee [xtq] = r_t([xtq])$. Since $x \in H$, $[xtq] = [qtq]$. This implies that $r_t([xty]) = r_t([qtq])$. By Theorem 4.1(a), $[qtq] \in H$. It follows that $r_t([qtq]) = r_t(q)$. Hence $r_t([xty]) = r_t(q)$, so $([xty], q) \in \mathscr{R}_t$. Similarly, since $(y,q) \in \mathscr{L}_t$, we have $([xty], q) \in \mathscr{L}_t$. We conclude that $([xty], q) \in \mathscr{H}_t$, so $x \circ y = [xty] \in H$. Therefore H is a subsemigroup of T_t, so we have that (c) implies (a).

Hence the theorem is now completed. □

As a consequence of Theorem 4.4, we immediately have Corollary 4.5.

Corollary 4.5. *If H is an $\mathscr{H}_t$-class of T and $[q_H tx] = q_H = [xtq_H]$ for all $x \in H$, then H is a subsemigroup of $\langle T_t, \circ\rangle$.*

Lemma 4.6. *If H is an $\mathscr{H}_t$-class of T satisfying Green's condition and $q = q_H$ is a t-ideal element, then $[qtx] = q = [xtq]$ for all $x \in H$.*

Proof. Assume that H is an $\mathscr{H}_t$-class of T satisfying Green's condition and $q = q_H$ is a t-ideal element. Then $l_t(q) = q$ and $r_t(q) = q$, so $[etq] \leq q$ and $[qte] \leq q$. By Theorem 4.1(c), we have $q \leq [etq]$ and $q \leq [qte]$. This implies that $[etq] = q = [qte]$. By Theorem 4.1(a), $[qtq] \in H$. Thus $(q, [qtq]) \in \mathscr{L}_t$, it follows from Lemma 3.5 that $[etq] = [et[qtq]]$. Therefore $[[qte]tq] = [[etq]tq] = [et[qtq]] = [etq] = q$. Now let x be an arbitrary element of H. By Lemma 3.5, we have $[etx] = [etq]$ and $[xte] = [qte]$. Hence $[xtq] = [xt[etq]] = [[xte]tq] = [[qte]tq] = q$ and $[qtx] = [[qte]tx] = [qt[etx]] = [qt[etq]] = q$. Therefore $[qtx] = q = [xtq]$ for all $x \in H$.

Hence the proof of the lemma is completed. □

Immediately from Corollary 4.5 and Lemma 4.6, we have Corollary 4.7.

Corollary 4.7. *If H is an $\mathscr{H}_t$-class of T satisfying Green's condition and q_H is a t-ideal element, then H is a subsemigroup of $\langle T_t, \circ \rangle$.*

Corollary 4.8. *An $\mathscr{H}_t$-class H of the greatest element e of T is a subsemigroup of $\langle T_t, \circ \rangle$ if and only if e is a t-idempotent.*

Proof. Assume that an $\mathscr{H}_t$-class H of the greatest element e of T is a subsemigroup of T_t. Then $[ete] = e \circ e \in H$, so H satisfies Green's condition. Since $e \in H$, $e \leq q_H$. Thus $q_H = e$. Since $e \leq [ete] \vee e = l_t(e) = r_t(e) \leq e$, we have $l_t(e) = e = r_t(e)$. Hence e is a t-ideal element. By Lemma 4.6, $[etx] = e = [xte]$ for all $x \in H$. Hence $e = [ete]$, so e is a t-idempotent.

Conversely, assume that e is a t-idempotent in an $\mathscr{H}_t$-class H. Then $[ete] = e \in H$, so H satisfies Green's condition. By the above proof, $q_H = e$ and e is a t-ideal element. It follows from Corollary 4.7 that H is a subsemigroup of T_t.

Hence the proof is completed. □

Theorem 4.9. *Let H be an $\mathscr{H}_t$-class of T such that its representative t-quasi-ideal element $q = q_H$ is minimal in the set of all t-quasi-ideal elements of T. Then $H = \{x \in T \mid x \leq q\}$ is a subsemigroup of $\langle T_t, \circ \rangle$.*

Proof. If $x \in H$, then $x \leq q$. Now assume that x is an element of T such that $x \leq q$. Then $l_t(x) \wedge r_t(x) \leq l_t(q) \wedge r_t(q) = q$. By Lemmas 3.1 and 3.2, $l_t(x) \wedge r_t(x)$ is a t-quasi-ideal element. Since q is a minimal t-quasi-ideal element, $l_t(x) \wedge r_t(x) = q$. Thus $q \leq l_t(x)$ and $q \leq r_t(x)$. By Lemma 3.1, we have $l_t(q) \leq l_t(l_t(x)) = l_t(x)$ and $r_t(q) \leq r_t(r_t(x)) = r_t(x)$. Since $x \leq q$, we have $l_t(x) \leq l_t(q)$ and $r_t(x) \leq r_t(q)$. Hence $l_t(x) = l_t(q)$ and $r_t(x) = r_t(q)$, so $(x, q) \in \mathscr{L}_t \cap \mathscr{R}_t = \mathscr{H}_t$. Therefore $x \in H$, so we conclude that $H = \{x \in T \mid x \leq q\}$. Now let x be an arbitrary element of H. Then $x \leq q$. Since $x \leq e$, we have $[xtx] \leq [etq] \wedge [qte] \leq l_t(q) \wedge r_t(q) = q$. This implies that $[xtx] \in H$. It follows from Theorem 4.4 that H is a subsemigroup of T_t.

Therefore the proof of the theorem is completed. □

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