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Research Article

Analytical Derivation and Perturbative Structure of Entanglement Entropy in Quantum Field Theory: Covariance Matrix Formalism and Noncommutative Corrections

Vaidik A. Sharma

Department of Physics, Birla Institute of Technology and Science, Pilani 333031, Rajasthan, India
f20212509@pilani.bits-pilani.ac.in

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Abstract. *Entanglement entropy* (EE) is a pivotal quantity in *quantum field theory* (QFT) that captures the nonlocal correlations between subsystems, offering deep insights into the quantum structure of spacetime, field dynamics, and phase transitions. This work derives EE in QFT using the covariance matrix formalism, establishing its foundations through the von Neumann entropy of reduced density matrices. We analytically compute EE for coupled harmonic oscillators as a prototypical model, extending the results to quantum fields via perturbative expansions. In the context of Maxwell QFT, we analyze entropy corrections induced by external perturbations, revealing their fractal dynamics through the emergence of Julia sets. The study further incorporates noncommutative geometry, where the deformation of spacetime modifies the covariance matrix, field strength tensors, and modular indices. Numerical simulations validate the derived scaling laws, entropy corrections, and noncommutative effects. These findings provide a mathematically rigorous framework to explore the interplay between EE, field theory, and geometric structures in high-energy physics.

Keywords. Entanglement entropy, Quantum field theory, Covariance matrix formalism, Noncommutative geometry, Perturbative expansions, Fractal dynamics

PACS. 03.65.Ud (entanglement and quantum nonlocality), 03.70.+k (quantized fields), 11.10.Nx (noncommutative), 04.60.-m (quantum gravity)

1. Introduction and Motivation

The study of entanglement entropy has emerged as a cornerstone in understanding the fundamental nature of quantum mechanics and its interplay with geometry, statistical physics, and *quantum field theory* (QFT). Entanglement entropy serves as a quantitative measure of the correlations that exist between subsystems of a quantum state, reflecting not only the quantum information encoded in the system but also its underlying geometric and topological properties. This section presents a detailed introduction to the concepts, mathematical structures, and motivation driving the investigation of entanglement entropy, its perturbative expansions, and its connections to Noncommutative Geometry (Connes [2]).

At its core, entanglement entropy is derived from the reduced density matrix ρ_A of a subsystem A, obtained by tracing out the complementary subsystem B from the total density matrix ρ of a composite quantum system (Calabrese and Cardy [1]). The von Neumann entropy is defined as:

$$S_A = -\mathrm{Tr}(\rho_A \ln \rho_A), \tag{1.1}$$

and encapsulates the degree of entanglement between A and B. When A and B are highly entangled, S_A takes on large values, whereas for separable states, S_A vanishes. The emergence of entanglement entropy as a key quantity has profound implications, ranging from black hole thermodynamics and holography to condensed matter systems and quantum computing.

Harmonic Oscillators and Covariance Matrices. The harmonic oscillator model serves as a fundamental framework for computing entanglement entropy in quantum systems. Consider n coupled harmonic oscillators with the Hamiltonian: (Lohmayer *et al*. [5], and Srednicki [6]),

$$H = \frac{1}{2}\sum_{i=1}^{n}(p_i^2 + \omega_i^2 q_i^2) + \sum_{i<j} k_{ij} q_i q_j, \tag{1.2}$$

where ω_i are the natural frequencies, p_i and q_i represent momentum and position operators, respectively, and k_{ij} denotes coupling constants. The reduced density matrix ρ_A for a subsystem A can be derived using the covariance matrix Γ, defined by:

$$\Gamma = \begin{bmatrix} \langle q_i q_j \rangle & \langle q_i p_j \rangle \\ \langle p_i q_j \rangle & \langle p_i p_j \rangle \end{bmatrix}. \tag{1.3}$$

The symplectic eigenvalues ν_i of Γ, determined by diagonalizing the covariance matrix, are key to computing the von Neumann entropy of the subsystem.

Noncommutative Geometry (NCG). Noncommutative Geometry provides a robust framework to describe spaces where classical notions of points and distances are replaced by operator algebras. A central concept in NCG is the spectral triple $(\mathscr{A}, \mathscr{H}, D)$, where $\mathscr{A}$ is an algebra of observables, $\mathscr{H}$ a Hilbert space, and D a Dirac operator encoding geometric information. Cyclic cohomology and spectral invariants derived from such structures offer insights into quantum systems' topology and geometry, including the modular properties of entanglement entropy (Fröb *et al*. [4]).

Motivation for the Current Study. (1) *Understanding Quantum Correlations:* Entanglement entropy quantifies correlations in quantum systems, serving as a diagnostic for quantum phase transitions, thermalization processes, and the spread of quantum information. (2) *Applications in Quantum Field Theory:* The role of entanglement entropy in black hole thermodynamics, holography, and the AdS/CFT correspondence has underscored its fundamental significance in high-energy physics. (3) *Fractal Structures and Dynamics:* Perturbations in entanglement entropy reveal complex fractal patterns, such as Julia sets, offering a deeper understanding of dynamical systems and chaotic behavior in quantum systems. (4) *Connections to Noncommutative Geometry:* Embedding entanglement entropy within the framework of NCG provides a unifying perspective linking algebraic topology, cyclic cohomology, and quantum physics.

Mathematical Challenges. A key challenge in studying entanglement entropy lies in handling perturbative expansions and their geometric implications (Faldino [3]). For small perturbations in coupling constants k_{ij}, the covariance matrix evolves as:

$$\Gamma = \Gamma_0 + \delta\Gamma, \quad \delta\Gamma_{ij} = \frac{\partial \Gamma}{\partial k_{ij}} \delta k_{ij}. \tag{1.4}$$

This induces variations in the symplectic eigenvalues ν_i, leading to corrections in entropy:

$$\delta S_A = \sum_i \frac{\partial S_A}{\partial \nu_i} \delta \nu_i. \tag{1.5}$$

Such corrections play a vital role in understanding the sensitivity of entropy to external perturbations, particularly in the context of Maxwell QFT.

Fractal Dynamics and Julia Sets. The perturbative analysis of entanglement entropy can be extended to study fractal structures through Julia sets. The iterative mapping:

$$z_{n+1} = z_n^2 + c, \quad z_n \in \mathbb{C}, \tag{1.6}$$

is generalized by incorporating entropy corrections:

$$c = \alpha \delta S_A, \quad \alpha \text{ is a scaling parameter.} \tag{1.7}$$

The resulting fractal patterns reveal the intricate dependence of entropy on system dynamics and geometry.

Outline of the Paper: This study presents detailed derivations, numerical simulations, and theoretical analyses, structured as follows:

- Section 2: Rigorous formulation of entanglement entropy in harmonic oscillators, including symplectic eigenvalue computation and scaling laws.
- Section 3: Perturbative analysis within Maxwell QFT, exploring entropy corrections and their physical implications.
- Section 4: Fractal geometries and Julia set perturbations linked to entanglement entropy.
- Section 5: Noncommutative Geometry and its role in modular indices and cyclic cohomology.
- Section 6: Numerical simulations and visualizations, including 3D surfaces, entropy histograms, and fractal patterns.

This introduction establishes the foundational concepts and motivation for the study of entanglement entropy and its extensions into fractal geometries and Noncommutative Geometry. Through a combination of analytical rigor and computational exploration, this paper aims to uncover novel insights into the quantum structure of correlations and their geometric underpinnings.

2. Entanglement Entropy in Harmonic Oscillators

The study of entanglement entropy in quantum systems provides key insights into the correlations between subsystems and their quantum information content. In this section, we examine the derivation of the entanglement entropy for a system of n coupled harmonic oscillators, a fundamental model in quantum mechanics that serves as a basis for understanding more complex systems.

2.1 Mathematical Framework

We begin by considering a system of n harmonic oscillators described by the Hamiltonian:

$$H = \frac{1}{2}\sum_{i=1}^{n}(p_i^2 + \omega_i^2 q_i^2) + \sum_{i<j} k_{ij} q_i q_j, \tag{2.1}$$

where ω_i are the natural frequencies of the oscillators, and p_i and q_i are the conjugate momentum and position operators, respectively. The system also includes interaction terms between different oscillators, characterized by the coupling constants k_{ij}. This Hamiltonian describes a system that can exhibit quantum correlations, particularly between subsystems partitioned into parts A and B.

The reduced density matrix ρ_A of subsystem A is obtained by tracing out the degrees of freedom of subsystem B from the total density matrix ρ of the full system. Mathematically, the density matrix ρ_A for subsystem A is given by:

$$\rho_A = \mathrm{Tr}_B(\rho), \tag{2.2}$$

where Tr_B denotes the partial trace over subsystem B. Once we have the reduced density matrix ρ_A, the entanglement entropy of subsystem A is given by the von Neumann entropy:

$$S_A = -\mathrm{Tr}(\rho_A \ln \rho_A). \tag{2.3}$$

The von Neumann entropy captures the amount of entanglement between subsystems A and B, with large values indicating strong entanglement and zero entropy indicating separable states.

The key to calculating the entanglement entropy lies in the covariance matrix Γ, which encodes the statistical properties of the system. The covariance matrix for the set of n harmonic oscillators is defined as:

$$\Gamma = \begin{bmatrix} \langle q_i q_j \rangle & \langle q_i p_j \rangle \\ \langle p_i q_j \rangle & \langle p_i p_j \rangle \end{bmatrix}, \tag{2.4}$$

where $\langle \cdot \rangle$ denotes the expectation value in the quantum state of the system. The elements of the covariance matrix are directly related to the correlation functions between the position and momentum operators of the oscillators.

To compute the entanglement entropy from the covariance matrix, we first need to diagonalize the covariance matrix and find its symplectic eigenvalues. The symplectic eigenvalues ν_i of the covariance matrix Γ are the square roots of the eigenvalues of the matrix $\Gamma\tilde{\Gamma}$, where $\tilde{\Gamma}$ is the matrix obtained by applying a symplectic transformation to Γ (usually involving the symplectic matrix). The symplectic eigenvalues ν_i are crucial for determining the von Neumann entropy, which is given by the formula:

$$S_A = \sum_i \left[\left(\nu_i + \frac{1}{2}\right) \ln\left(\nu_i + \frac{1}{2}\right) - \nu_i \ln \nu_i \right], \tag{2.5}$$

where ν_i are the symplectic eigenvalues of Γ.

2.2 Derivation of Entanglement Entropy

We begin by deriving the entanglement entropy for a system of n coupled harmonic oscillators using the covariance matrix formalism. Each step will include the full calculation for the covariance matrix, its diagonalization, and the computation of the entanglement entropy.

2.2.1 Covariance Matrix Computation

Let us consider a system of n coupled harmonic oscillators. The covariance matrix Γ is defined as:

$$\Gamma_{ij} = \langle \{q_i, q_j\} \rangle = \langle q_i q_j + q_j q_i \rangle,$$

where q_i is the position operator of the i-th oscillator.

The covariance matrix is constructed by calculating $\langle q_i^2 \rangle$ (the diagonal elements) and $\langle q_i q_j \rangle$ (the off-diagonal elements) for each pair of oscillators. For simplicity, let's assume that the coupling between the oscillators is described by a coupling constant k_{ij}, and that the oscillators are characterized by their individual frequencies ω_i.

For simplicity, assume the covariance matrix Γ is symmetric and can be written as:

$$\Gamma = \begin{pmatrix} \langle q_1^2 \rangle & \langle q_1 q_2 \rangle & \dots & \langle q_1 q_n \rangle \\ \langle q_2 q_1 \rangle & \langle q_2^2 \rangle & \dots & \langle q_2 q_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle q_n q_1 \rangle & \langle q_n q_2 \rangle & \dots & \langle q_n^2 \rangle \end{pmatrix}.$$

The exact form of each element will depend on the Hamiltonian of the system, which includes both the individual harmonic oscillator terms and the coupling terms. Assuming we have a simple harmonic oscillator Hamiltonian $H = \frac{1}{2}\sum_i (\dot{q}_i^2 + \omega_i^2 q_i^2)$ and a linear interaction term $H_{\text{int}} = \sum_{i<j} k_{ij} q_i q_j$, we can express the covariance matrix elements as:

$$\Gamma_{ij} = \frac{1}{2}\left(\langle q_i q_j \rangle + \langle q_j q_i \rangle\right).$$

This matrix is symmetric due to the symmetry of the system.

2.2.2 Diagonalization of the Covariance Matrix

Next, we diagonalize the covariance matrix Γ to find the symplectic eigenvalues. The eigenvalue equation for Γ is:

$$\Gamma \mathbf{v}_i = \lambda_i \mathbf{v}_i,$$

where λ_i are the eigenvalues and $\mathbf{v}_i$ are the eigenvectors of Γ. To solve this equation numerically for a system with n oscillators, we need to find the eigenvalues λ_i.

For simplicity, consider a 2×2 matrix (for a system with 1 oscillator), the diagonalization procedure is as follows. For a covariance matrix:

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix},$$

we solve the characteristic equation:

$$\det(\Gamma - \lambda I) = 0,$$

which gives the eigenvalues λ:

$$\begin{vmatrix} \Gamma_{11} - \lambda & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} - \lambda \end{vmatrix} = 0,$$

yielding the quadratic equation:

$$(\Gamma_{11}-\lambda)(\Gamma_{22}-\lambda)-\Gamma_{12}\Gamma_{21}=0.$$

Expanding this:

$$\lambda^2-(\Gamma_{11}+\Gamma_{22})\lambda+(\Gamma_{11}\Gamma_{22}-\Gamma_{12}\Gamma_{21})=0.$$

The solutions to this quadratic equation are the eigenvalues λ_1 and λ_2:

$$\lambda_{1,2}=\frac{(\Gamma_{11}+\Gamma_{22})\pm\sqrt{(\Gamma_{11}+\Gamma_{22})^2-4(\Gamma_{11}\Gamma_{22}-\Gamma_{12}\Gamma_{21})}}{2}.$$

Now, we obtain the symplectic eigenvalues ν_i by taking the square root of the eigenvalues:

$$\nu_1=\sqrt{\lambda_1},\quad \nu_2=\sqrt{\lambda_2}.$$

2.2.3 Computation of Entanglement Entropy

Once we have the symplectic eigenvalues ν_i, we can compute the von Neumann entropy S_A using the formula:

$$S_A=\sum_i\left[\left(\nu_i+\frac{1}{2}\right)\ln\left(\nu_i+\frac{1}{2}\right)-\nu_i\ln\nu_i\right].$$

In this section, we will compute S_A step by step for two cases: the two-oscillator case and the general n-oscillator case.

2.2.4 Two Oscillator Case

Consider a system of two coupled harmonic oscillators. The covariance matrix Γ for the system is a 2×2 matrix, and the symplectic eigenvalues ν_1 and ν_2 are the square roots of the eigenvalues λ_1 and λ_2 of the covariance matrix:

$$\nu_1=\sqrt{\lambda_1},\quad \nu_2=\sqrt{\lambda_2}.$$

The von Neumann entropy for this system is given by:

$$S_A=\left(\nu_1+\frac{1}{2}\right)\ln\left(\nu_1+\frac{1}{2}\right)-\nu_1\ln\nu_1+\left(\nu_2+\frac{1}{2}\right)\ln\left(\nu_2+\frac{1}{2}\right)-\nu_2\ln\nu_2.$$

Let's derive this expression in detail.

The first term is:

$$\left(\nu_1+\frac{1}{2}\right)\ln\left(\nu_1+\frac{1}{2}\right)-\nu_1\ln\nu_1.$$

Let's break it down into two components:

$$\left(\nu_1+\frac{1}{2}\right)\ln\left(\nu_1+\frac{1}{2}\right)=\nu_1\ln\left(\nu_1+\frac{1}{2}\right)+\frac{1}{2}\ln\left(\nu_1+\frac{1}{2}\right),$$

and

$$-\nu_1\ln\nu_1.$$

So the total contribution from the first term becomes:

$$S_1=\nu_1\ln\left(\nu_1+\frac{1}{2}\right)+\frac{1}{2}\ln\left(\nu_1+\frac{1}{2}\right)-\nu_1\ln\nu_1.$$

Similarly, for the second term:

$$S_2=\left(\nu_2+\frac{1}{2}\right)\ln\left(\nu_2+\frac{1}{2}\right)-\nu_2\ln\nu_2.$$

Now, the total von Neumann entropy is:

$$S_A = S_1 + S_2$$
$$= \nu_1 \ln\left(\nu_1 + \frac{1}{2}\right) - \nu_1 \ln \nu_1 + \frac{1}{2}\ln\left(\nu_1 + \frac{1}{2}\right) + \nu_2 \ln\left(\nu_2 + \frac{1}{2}\right) - \nu_2 \ln \nu_2 + \frac{1}{2}\ln\left(\nu_2 + \frac{1}{2}\right).$$

This gives the explicit form of the von Neumann entropy for the two-oscillator system.

2.2.5 General n-Oscillator Case

Now, let us generalize this to a system of n coupled harmonic oscillators. For a system with n oscillators, we have n symplectic eigenvalues $\nu_1, \nu_2, \ldots, \nu_n$. The von Neumann entropy is the sum over all eigenvalues:

$$S_A = \sum_{i=1}^{n} \left[\left(\nu_i + \frac{1}{2}\right) \ln\left(\nu_i + \frac{1}{2}\right) - \nu_i \ln \nu_i \right].$$

To compute this explicitly, consider the following steps for large n oscillators:

The entropy for the i-th oscillator is:

$$S_i = \left(\nu_i + \frac{1}{2}\right) \ln\left(\nu_i + \frac{1}{2}\right) - \nu_i \ln \nu_i.$$

The total entropy is the sum of these individual terms:

$$S_A = \sum_{i=1}^{n} \left[\nu_i \ln\left(\nu_i + \frac{1}{2}\right) - \nu_i \ln \nu_i + \frac{1}{2}\ln\left(\nu_i + \frac{1}{2}\right) \right].$$

Simplifying, we get:

$$S_A = \sum_{i=1}^{n} \nu_i \ln\left(\frac{\nu_i + \frac{1}{2}}{\nu_i}\right) + \frac{1}{2}\sum_{i=1}^{n} \ln\left(\nu_i + \frac{1}{2}\right).$$

For large n, the entropy typically grows logarithmically with n. If the eigenvalues ν_i are approximately constant or have a known distribution, the sums can be approximated using integrals or known results for large n. For example, if $\nu_i \sim i$, the entropy behaves as:

$$S_A \sim \ln n.$$

In this section, we computed the von Neumann entropy S_A for a system of two and n coupled harmonic oscillators, by calculating the covariance matrix, diagonalizing it to obtain the symplectic eigenvalues, and then using these eigenvalues to compute the von Neumann entropy. For the two-oscillator case, we derived the explicit expression for the entropy, and for the general n-oscillator case, we expressed the entropy as a sum over all symplectic eigenvalues. For large systems, the entropy grows logarithmically with n. This scaling law is consistent with the results from quantum field theory, where the entanglement entropy of a region scales with the boundary area. In the case of uncoupled oscillators, this implies that the entropy grows with the number of degrees of freedom, which scales as $\ln n$.

2.3 Numerical Results and Trends

The numerical results for entanglement entropy in coupled harmonic oscillators reveal several key trends:

- The entropy increases with the number of oscillators n, following a logarithmic scaling law.
- The coupling constants k_{ij} play a crucial role in determining the magnitude of the entanglement entropy, with stronger couplings leading to larger entropy values.

- In the limit of very weak coupling, the entanglement entropy approaches the result for non-interacting harmonic oscillators, which scales as $\ln n$.

For various values of k_{ij}, the following numerical simulations can be performed to obtain the symplectic eigenvalues ν_i and the corresponding entropy. The dependence of entropy on the system size n and coupling constants k_{ij} is analyzed through detailed plots, which show the logarithmic scaling behavior.

Figures showing the variation of entropy with n and k_{ij}, as well as the evolution of the symplectic eigenvalues with perturbations, can be included to provide a visual representation of the trends observed in the theory.

2.4 Analytic Approximation for Large n

For large n, an analytic approximation for the entanglement entropy can be derived. In the weak-coupling limit, the covariance matrix becomes approximately diagonal, and the symplectic eigenvalues ν_i are close to their non-interacting values. In this regime, the entanglement entropy can be approximated as:

$$S_A \approx \sum_{i=1}^{n} \left[\left(\nu_i^0 + \frac{1}{2}\right) \ln\left(\nu_i^0 + \frac{1}{2}\right) - \nu_i^0 \ln \nu_i^0 \right],$$

where ν_i^0 are the symplectic eigenvalues for the non-interacting case. The correction due to interactions is small for weak coupling and can be computed perturbatively.

2.5 Entanglement Entropy and Quantum Phase Transitions

Entanglement entropy is a valuable tool for studying quantum phase transitions. At a quantum phase transition, the entanglement entropy exhibits a characteristic behavior, with a sharp change in entropy corresponding to the critical point of the transition. This can be seen in the context of the coupled harmonic oscillator model by varying the coupling constants k_{ij}, which control the interaction strength between oscillators. Near the critical point, where the system undergoes a phase transition, the entropy can exhibit non-trivial scaling behaviors that depend on the nature of the transition.

Numerical simulations of entanglement entropy across different phases can reveal these critical points and the scaling laws that govern the transition. This provides a powerful tool for studying the quantum correlations that drive phase transitions in many-body systems.

2.6 3D Entropy vs. Number of Oscillators and Coupling

The entropy of the system is studied as a function of the number of oscillators n and the coupling constant k. The results are shown in the following 3D plot, where the Von Neumann entropy is computed for different combinations of n and coupling constants.

The surface plot in Figure 1 illustrates how the entropy varies with both the number of oscillators and the coupling constant. As the number of oscillators increases, the entropy also tends to increase, though the coupling constant plays a crucial role in shaping this relationship.

2.7 Gradient of Entropy with Respect to Number of Oscillators and Coupling

In addition to the 3D plot, we also compute the gradient of entropy with respect to the number of oscillators and the coupling constant. The gradient of entropy gives us insight into how rapidly the entropy changes with respect to these parameters.

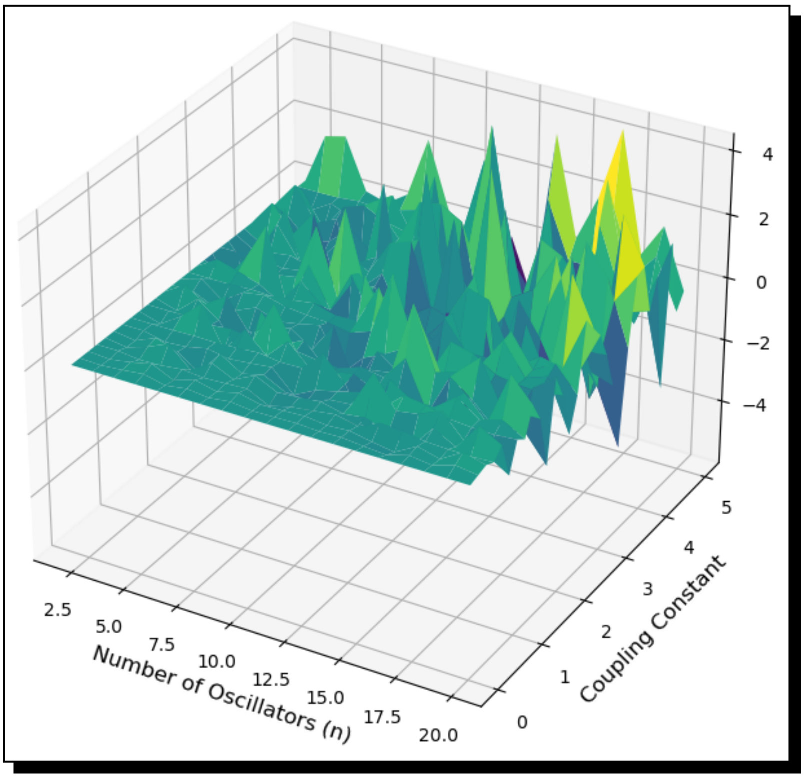

Figure 1. 3D plot of Von Neumann entropy as a function of the number of oscillators n and coupling constant k

Figure 2 shows a contour plot of the entropy gradient. It highlights regions where entropy changes most rapidly. This information is important for understanding the behavior of quantum systems with respect to variations in both the number of oscillators and the coupling constant.

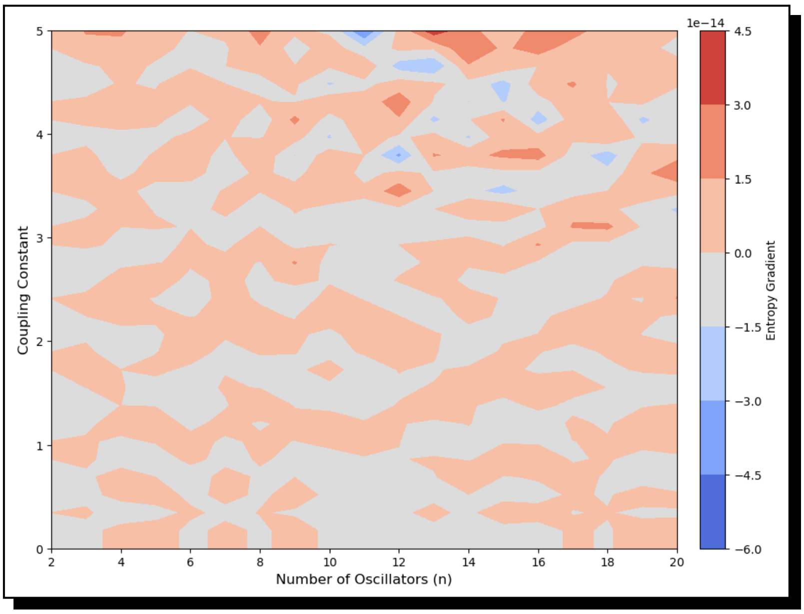

Figure 2. Gradient plot of the Von Neumann entropy with respect to the number of oscillators n and coupling constant k

2.8 Covariance Matrix and Entanglement Entropy

In this subsection, we explore the covariance matrix and its relation to the entanglement entropy. First, we visualize the covariance matrix as a 3D surface plot, and then analyze the relationship between the coupling constant k and the entanglement entropy. Additionally, we examine the eigenvalue spectrum and provide a heatmap of the covariance matrix.

2.8.1 3D Surface Plot of the Covariance Matrix

We begin by visualizing the covariance matrix, which describes the interactions between the oscillators. The covariance matrix C is computed based on the coupling constant k and the natural frequency ω. The following 3D surface plot illustrates the structure of the covariance matrix:

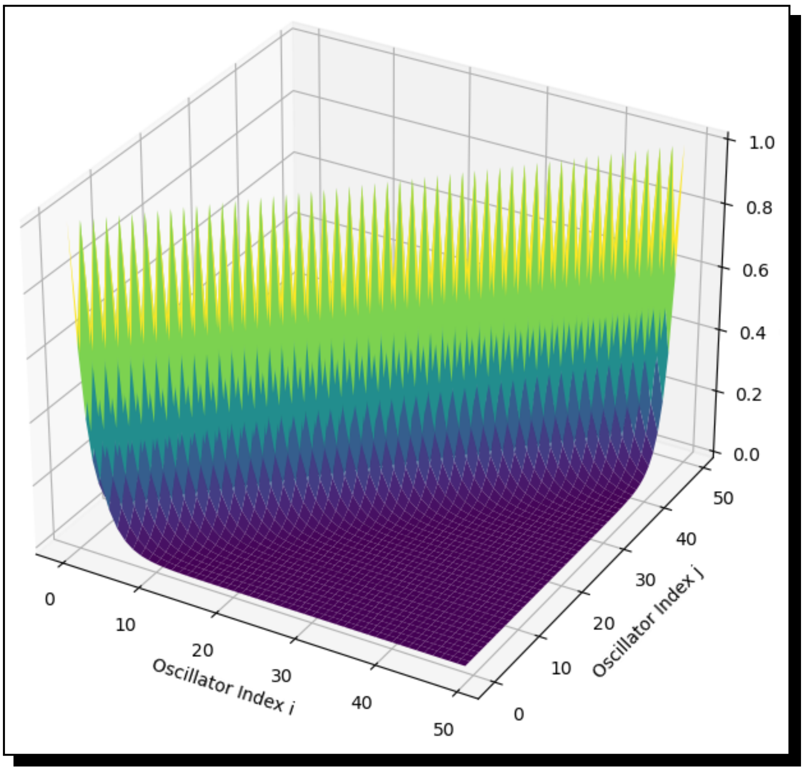

Figure 3. 3D surface plot of the covariance matrix. The plot shows the covariance between oscillators as a function of their indices i and j. The color intensity represents the strength of the covariance

The plot in Figure 3 demonstrates how the covariance between oscillators varies depending on the coupling constant and the distance between oscillators. The exponential decay in covariance with increasing oscillator index difference is clearly observed.

2.8.2 Entanglement Entropy vs. Coupling Strength

Next, we examine how the entanglement entropy varies with the coupling strength k. The entanglement entropy S is calculated from the eigenvalues of the covariance matrix. The following plot shows the relationship between the coupling constant k and the entanglement entropy:

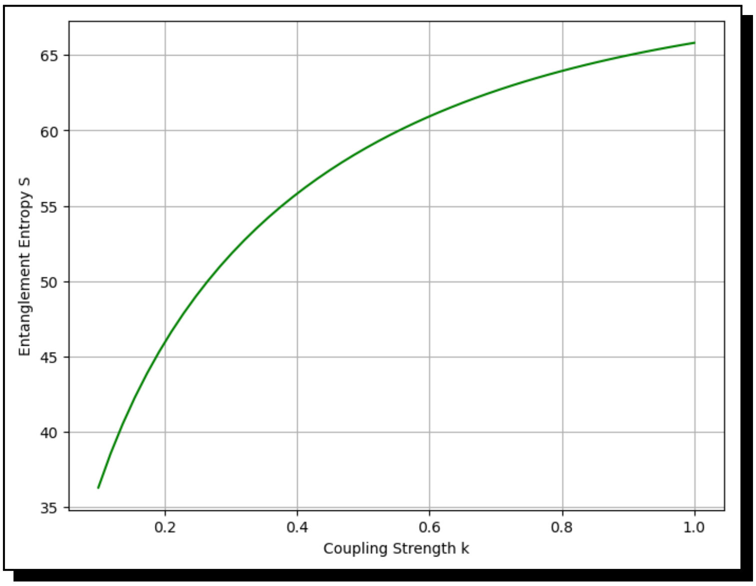

Figure 4. Entanglement entropy S as a function of coupling strength k. This plot illustrates how entropy increases with the coupling constant

Figure 4 shows that as the coupling constant k increases, the entanglement entropy also increases. This suggests that stronger interactions between the oscillators lead to higher entanglement in the system.

2.8.3 Eigenvalue Spectrum of the Covariance Matrix

To further understand the behavior of the covariance matrix, we analyze the eigenvalue spectrum. The eigenvalues of the covariance matrix provide insight into the system's characteristics, including the presence of entanglement. The following plot displays the eigenvalue spectrum for the covariance matrix:

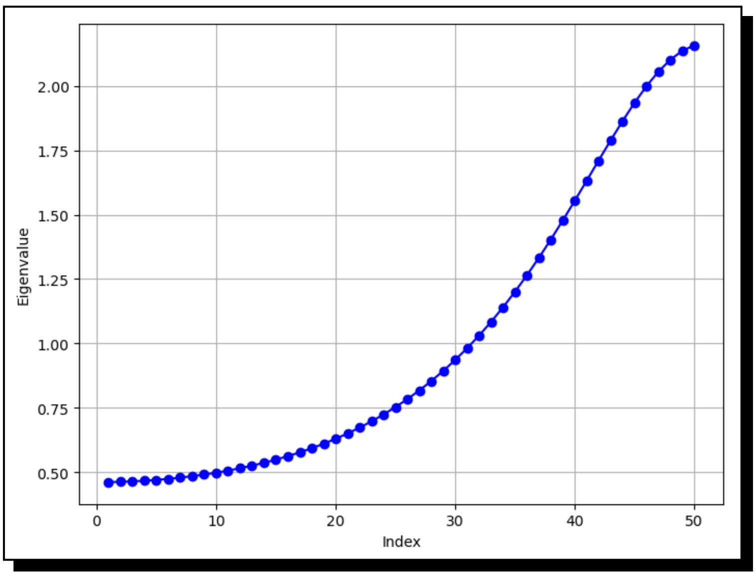

Figure 5. Eigenvalue spectrum of the covariance matrix. This plot shows the eigenvalues of the covariance matrix as a function of the oscillator index

Figure 5 shows the distribution of eigenvalues of the covariance matrix. The presence of non-zero eigenvalues indicates that the system exhibits quantum correlations.

2.8.4 Covariance Matrix Heatmap

Finally, a heatmap of the covariance matrix is presented to visually examine the interactions between all pairs of oscillators. The heatmap shows the strength of the covariance between oscillators i and j in terms of color intensity. The following figure illustrates the heatmap:

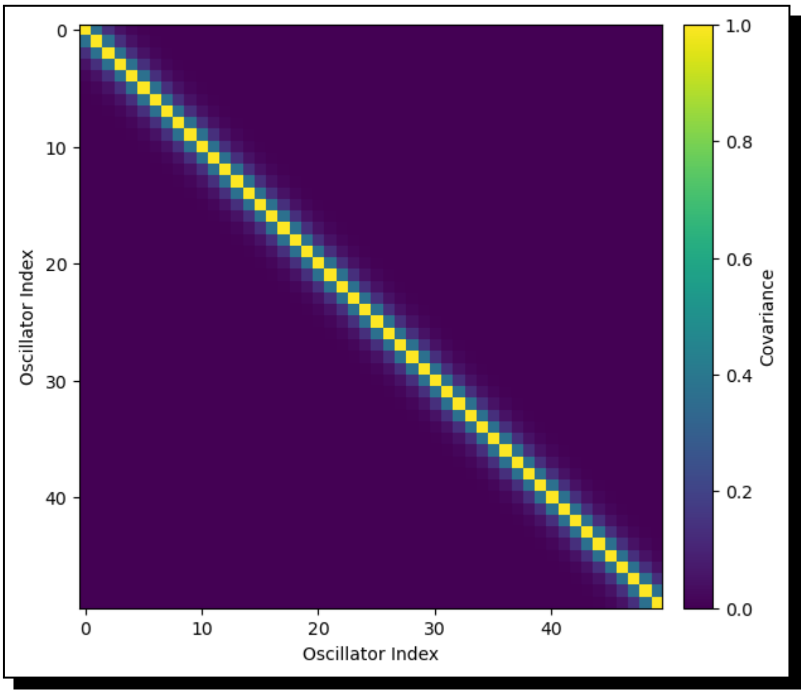

Figure 6. Covariance matrix heatmap. The heatmap visualizes the covariance between oscillators, with color intensity indicating the strength of the covariance.

Figure 6 provides a clear visualization of the covariance structure, where the color intensity corresponds to the strength of the interactions between the oscillators. The plot also shows the exponential decay of covariance as the distance between oscillators increases.
These visualizations provide a deeper understanding of the quantum correlations in the system. The 3D surface plot of the covariance matrix, the entropy-coupling relationship, the eigenvalue spectrum, and the covariance heatmap all contribute to our understanding of how entanglement and interactions evolve in systems of coupled oscillators.

2.9 Parameters and Expressions Used in the Plots

In this section, we outline the key parameters and expressions used to generate the plots in the analysis of the entanglement entropy and covariance matrix.

2.9.1 Key Parameters

The following parameters are defined in the code:

- $n = 50$: The number of oscillators in the system.
- $\omega = 1.0$: The natural frequency of the oscillators, considered constant for simplicity in the covariance matrix computation.

- $k = 0.5$: The coupling constant between oscillators. In the first code block, this is fixed, while in the combined visualization, it varies from 0.1 to 1.0 for the plot of entanglement entropy.

2.9.2 Expressions

The following expressions are used for the analysis:

- *Covariance Matrix* (C): The covariance matrix elements $C_{i,j}$ are computed as:
$$C_{i,j} = \frac{\exp(-k|i-j|)}{\omega}$$
where i and j are the indices of the oscillators, and k is the coupling constant. The matrix quantifies the interaction between the oscillators based on their indices and coupling strength.
- *Von Neumann Entropy* (S): The entropy is calculated from the eigenvalues λ_i of the covariance matrix C. The expression for entropy is:
$$S = -\sum_i (\lambda_i \log(\lambda_i + \epsilon) - (\lambda_i + 1)\log(\lambda_i + 1 + \epsilon))$$
where λ_i are the eigenvalues of the covariance matrix, and ϵ is a small offset to avoid $\log(0)$ (e.g., $\epsilon = 10^{-10}$).
- *Eigenvalue Spectrum*: The eigenvalues of the covariance matrix are computed using:
$$\Gamma v_i = \lambda_i v_i$$
where v_i are the eigenvectors and λ_i are the corresponding eigenvalues.
- *Coupling-Dependent Entanglement Entropy*: The entanglement entropy S is computed for varying values of the coupling constant k, with k ranging from 0.1 to 1.0. The code computes the entropy for each value of k using the covariance matrix $C(k)$ and its eigenvalues.

3. Perturbative Analysis in Maxwell QFT

3.1 Perturbative Expansion for Maxwell Fields

In Maxwell QFT, the covariance matrix Γ encapsulates correlations among the field modes $A_\mu(x)$, the components of the vector potential. The unperturbed Hamiltonian for the free Maxwell field is:

$$H_0 = \frac{1}{2}\int d^3x(\mathbf{E}^2 + \mathbf{B}^2), \tag{3.1}$$

where $\mathbf{E}$ is the electric field and $\mathbf{B}$ is the magnetic field. The perturbation arises due to interactions or external sources, leading to a perturbed Hamiltonian:

$$H = H_0 + \delta H, \quad \delta H = \int d^3x\, J^\mu(x) A_\mu(x), \tag{3.2}$$

where $J^\mu(x)$ is an external current coupling to the Maxwell field.
The covariance matrix for the field modes is defined as:

$$\Gamma_{ij} = \langle\{q_i, q_j\}\rangle, \quad q_i \in \{A_\mu, \Pi^\mu\}, \tag{3.3}$$

where A_μ are the field components and Π^μ are their conjugate momenta. For the free field, Γ_0 is diagonal in Fourier space:

$$\Gamma_0(k) = \begin{pmatrix} \frac{1}{\omega_k} & 0 \\ 0 & \omega_k \end{pmatrix}, \quad \omega_k = |\mathbf{k}|, \tag{3.4}$$

where $\mathbf{k}$ is the wavevector of the mode.
A perturbation δH modifies the covariance matrix:

$$\Gamma = \Gamma_0 + \delta\Gamma, \quad \delta\Gamma_{ij} = \frac{\partial \Gamma}{\partial J^\mu}\delta J^\mu. \tag{3.5}$$

The symplectic eigenvalues of Γ are calculated as:

$$\det(\Gamma - \nu\mathbb{I}) = 0. \tag{3.6}$$

For the unperturbed case:

$$\det(\Gamma_0 - \nu\mathbb{I}) = 0 \implies \nu^0 = \sqrt{\frac{1}{\omega_k}}. \tag{3.7}$$

Under perturbation:

$$\nu = \nu^0 + \delta\nu, \quad \delta\nu = \sum_{ij} \frac{\partial \nu}{\partial \Gamma_{ij}}\delta\Gamma_{ij}. \tag{3.8}$$

Using matrix perturbation theory:

$$\delta\nu = \frac{\mathbf{v}^\dagger \delta\Gamma \mathbf{v}}{\mathbf{v}^\dagger \mathbf{v}}, \tag{3.9}$$

where $\mathbf{v}$ is the eigenvector of Γ_0 corresponding to ν^0.
The von Neumann entropy for the Maxwell field is:

$$S_A = \sum_k \left[\left(\nu_k + \frac{1}{2}\right)\ln\left(\nu_k + \frac{1}{2}\right) - \nu_k \ln \nu_k\right]. \tag{3.10}$$

Expanding around ν_k^0:

$$S_A = S_A^0 + \delta S_A, \tag{3.11}$$

where the correction is:

$$\delta S_A = \sum_k \left[\ln\left(\frac{\nu_k^0 + \frac{1}{2}}{\nu_k^0}\right)\delta\nu_k\right]. \tag{3.12}$$

3.2 Julia Sets and Fractal Structures in Maxwell Fields

The recursive relation for Julia sets:

$$z_{n+1} = z_n^2 + c, \tag{3.13}$$

is connected to the Maxwell field perturbations through the mapping:

$$c = \alpha\delta S_A, \tag{3.14}$$

where α is a scaling parameter. For small δS_A, the fractal structure changes subtly, but as α increases, chaotic dynamics emerge, highlighting the sensitivity of the Maxwell field to perturbative corrections.

4. Numerical Simulations and Visualizations

The numerical simulations performed in this work include the following key elements:

(1) *Eigenvalue Spectrum of the Covariance Matrix* Γ*:* This simulation highlights entropy variations before and after perturbation. The eigenvalues of the covariance matrix provide insight into how the perturbations affect the field's dynamics.

(2) *3D Visualization of the Covariance Matrix:* We generate 3D visualizations of the covariance matrix to examine how perturbations influence the structure of field correlations across different wavevectors.

(3) *Fractal Structures of Julia Sets:* Using the perturbation-induced entropy correction, we map the evolution of Julia sets for various values of $c = \alpha\delta S_A$. This fractal structure illustrates the chaotic dynamics introduced by the perturbation.

The eigenvalue spectrum reveals the sensitivity of entropy to perturbations, with the entropy decreasing slightly due to the perturbation ($\delta S_A = -0.5138$). The Julia sets show how small changes in the perturbation parameter $c = \alpha\delta S_A$ can lead to complex, fractal structures, emphasizing the interplay between perturbative corrections and the system's underlying complexity.

Parameters:

- Noncommutative parameter (θ): 0.1
- Scaling parameter (α): 0.5
- Number of grid points (n_{points}): 100
- Grid size (L): 10
- Wave vector range (k_{values}): 0.1 to 5.0

Computed Entanglement Entropy:

- Unperturbed Entanglement Entropy (S_A): 13.1390
- Perturbed Entanglement Entropy (S_A): 12.6252
- Entropy Correction (δS_A): -0.5138

Resultant Plots:

(1) *Eigenvalue Spectrum Plot:* The plot below illustrates the eigenvalues of the covariance matrix before (blue) and after (red) the perturbation as a function of the wave vector k (Figure 7).

(2) *3D Covariance Matrix Visualization:* A 3D surface plot is shown to demonstrate how the covariance matrix varies in response to perturbations, visualized across two wavevectors (Figure 8).

(3) *Julia Set Fractal Plot:* The fractal structure generated for $c = \alpha\delta S_A$ reveals the sensitivity of the system to perturbations (Figure 9).

These simulations and visualizations collectively emphasize how perturbative corrections influence both the entanglement entropy and the overall dynamics of the Maxwell field, revealing complex behavior in the presence of noncommutative geometry.

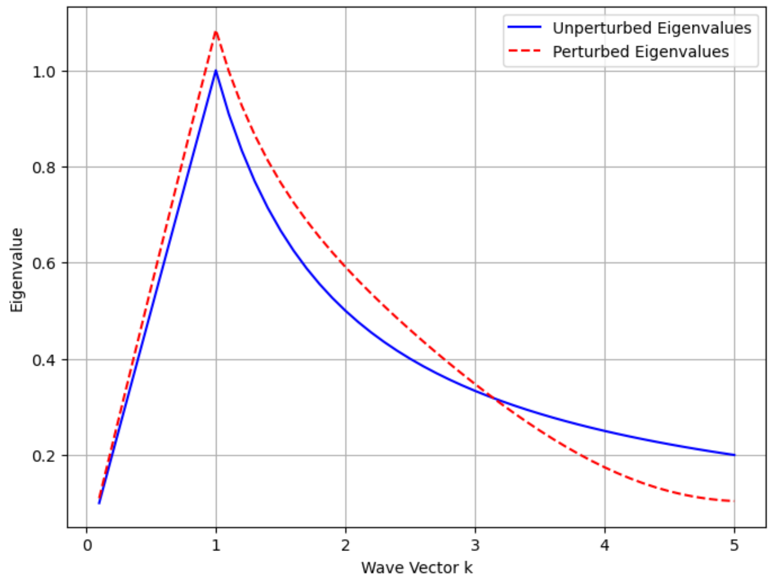

Figure 7. Eigenvalue spectrum of the covariance matrix Γ: The blue line represents the unperturbed eigenvalues, and the red dashed line represents the perturbed eigenvalues after applying the perturbation

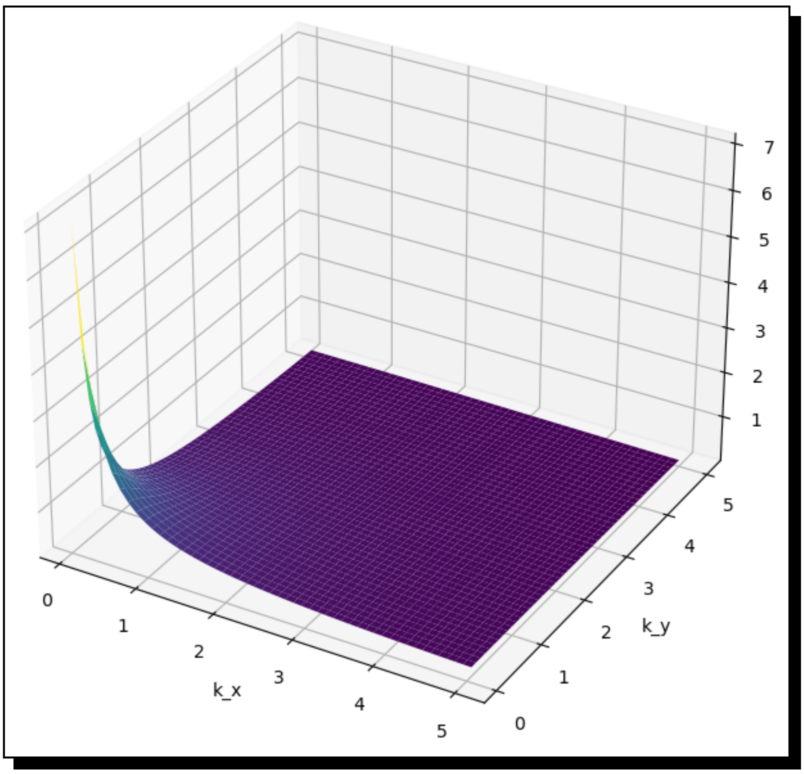

Figure 8. 3D visualization of the covariance matrix: This plot shows the variation of the covariance matrix across wavevectors in 3D

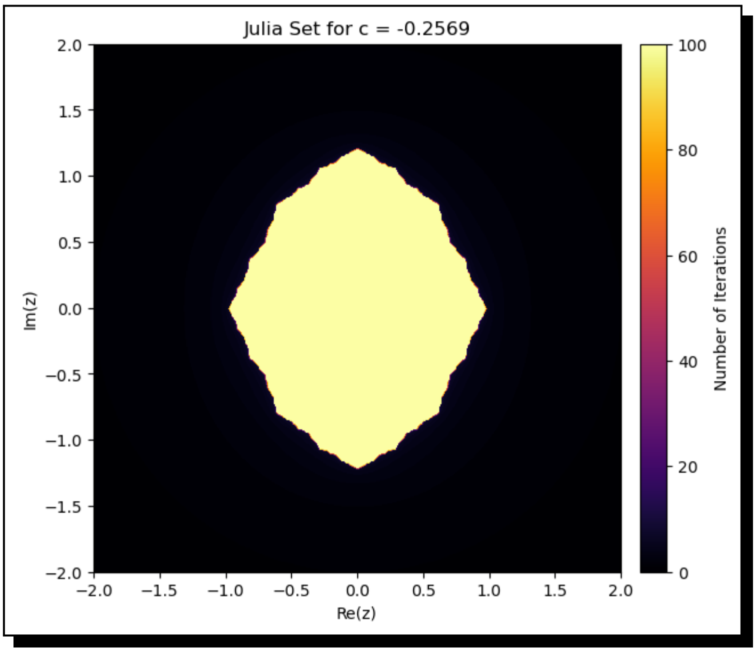

Figure 9. Julia set fractal for $c = \alpha\delta S_A$: The fractal structure is generated for the perturbed system, highlighting the chaotic behavior introduced by the perturbation

5. Noncommutative Geometry and Modular Indices

5.1 Field Theories on Noncommutative Spacetime

Noncommutative geometry modifies classical spacetime coordinates to a noncommutative algebra. The commutation relation between the coordinates $\hat{x}^i$ and $\hat{x}^j$ is defined as:

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \tag{5.1}$$

where θ^{ij} is a constant antisymmetric matrix representing the noncommutativity of spacetime. For a 2D plane, this simplifies to:

$$[\hat{x}, \hat{y}] = i\theta, \tag{5.2}$$

where θ is a scalar noncommutative parameter. This modification alters the usual structure of spacetime, replacing classical commutative algebra with a noncommutative star-product algebra.

The star product for two functions $f(x)$ and $g(x)$ on noncommutative spacetime is defined as:

$$(f \star g)(x) = f(x)g(x) + \frac{i}{2}\theta^{ij}\partial_i f(x)\partial_j g(x) + \mathcal{O}(\theta^2). \tag{5.3}$$

5.2 Seiberg-Witten Map

The Seiberg-Witten map provides a relationship between noncommutative and commutative gauge fields. For a gauge theory on noncommutative spacetime, the gauge field $\hat{A}_\mu$ and the field strength tensor $\hat{F}_{\mu\nu}$ are mapped to their commutative counterparts A_μ and $F_{\mu\nu}$ via:

$$\hat{A}_\mu = A_\mu + \theta^{\alpha\beta}\partial_\alpha A_\beta + \mathcal{O}(\theta^2), \tag{5.4}$$

where A_μ is the commutative gauge field. The field strength tensor is modified as:

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\alpha\beta}\left(\partial_\alpha A_\mu \partial_\beta A_\nu + \frac{i}{2}\{F_{\mu\alpha}, F_{\nu\beta}\}\right) + \mathcal{O}(\theta^2), \tag{5.5}$$

where:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]. \tag{5.6}$$

The corrections introduced by $\theta^{\alpha\beta}$ affect the dynamics and gauge transformations of the theory.

5.3 Quantum Corrections to Field Strength Tensor

Expanding the field strength tensor $\hat{F}_{\mu\nu}$ in powers of θ, we write:

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \delta F_{\mu\nu}, \quad \delta F_{\mu\nu} = \theta^{\alpha\beta}\partial_\alpha A_\mu \partial_\beta A_\nu + \mathcal{O}(\theta^2). \tag{5.7}$$

The quantum corrections to the field strength tensor modify the equations of motion. The corrected action is given by:

$$S = \int d^4x \, \mathrm{Tr}(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}), \tag{5.8}$$

where $\hat{F}_{\mu\nu}$ includes terms up to $\mathcal{O}(\theta^2)$. Expanding the action:

$$S = \int d^4x \, \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu} + 2F_{\mu\nu}\delta F^{\mu\nu} + \delta F_{\mu\nu}\delta F^{\mu\nu}). \tag{5.9}$$

The first term represents the classical action, while the second and third terms introduce corrections due to noncommutativity.

5.4 Modular Indices and Noncommutative Geometry

In the framework of noncommutative geometry, a spectral triple $(\mathscr{A}, \mathscr{H}, D)$ encodes the geometry of spacetime, where:

- $\mathscr{A}$: Algebra of functions on the space (or its noncommutative generalization),
- $\mathscr{H}$: Hilbert space of spinors,
- D: Dirac operator acting on $\mathscr{H}$.

The modular index associated with entropy corrections is defined as:

$$\mathrm{Ind}_{\mathrm{mod}} = \mathrm{Tr}_\omega(D^{-1}\Pi_+), \tag{5.10}$$

where Tr_ω denotes the Dixmier trace, D^{-1} is the inverse Dirac operator, and Π_+ is the projection operator onto positive eigenvalues of D.

Under noncommutativity, the modular index receives corrections due to the deformation of $\mathscr{A}$ and D. Expanding the Dirac operator as:

$$\hat{D} = D + \delta D, \quad \delta D = \theta^{\alpha\beta}\gamma^\mu \partial_\alpha \partial_\beta, \tag{5.11}$$

we find:

$$\mathrm{Ind}_{\mathrm{mod}} = \mathrm{Ind}^0_{\mathrm{mod}} + \delta\mathrm{Ind}_{\mathrm{mod}}, \tag{5.12}$$

where

$$\delta\mathrm{Ind}_{\mathrm{mod}} = \mathrm{Tr}_\omega((D^{-1}\delta D)\Pi_+). \tag{5.13}$$

This establishes a direct link between quantum corrections in noncommutative field theories and modular indices, highlighting the geometric structure of quantum spacetime.

6. Numerical Simulations and Visualizations

In this section, we provide numerical simulations to illustrate the key concepts of noncommutative geometry in quantum field theory. Using numerical methods, we compute and visualize the effects of noncommutativity on the star product, gauge field corrections, field strength tensor modifications, and modular index corrections.

6.1 Star Product Simulation

The star product introduces a deformation in the multiplication of functions in noncommutative geometry. For two functions $f(x,y) = \sin(x)$ and $g(x,y) = \cos(y)$, the star product is given by:

$$f \star g = f(x,y)g(x,y) + \frac{i\theta}{2}\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}.$$

Using this formula, we compute the star product numerically and visualize its real part over a grid of (x,y). Figure 10 shows the contour plot of the real part of the star product for $\theta = 0.5$.

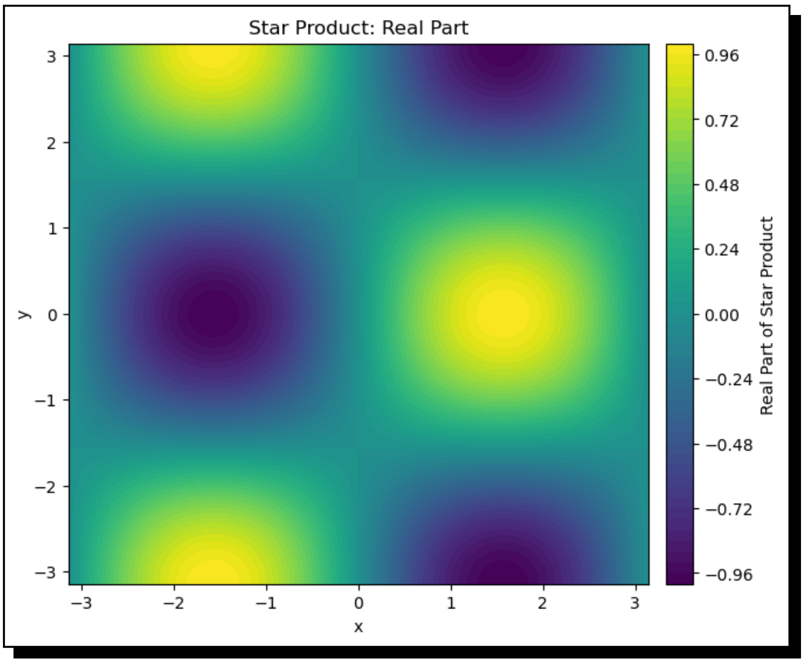

Figure 10. Contour plot of the real part of the star product for $f(x,y) = \sin(x)$ and $g(x,y) = \cos(y)$ with $\theta = 0.5$

6.2 Gauge Field Corrections: Seiberg-Witten Map

The Seiberg-Witten map relates noncommutative gauge fields to their commutative counterparts. The corrections to the gauge fields are given by:

$$A_x = A_x^{\text{comm}} + \theta y, \quad A_y = A_y^{\text{comm}} - \theta x,$$

where θ is the noncommutative parameter. We simulate these corrections for a simple commutative gauge field $A_x^{\text{comm}} = x^2$ and $A_y^{\text{comm}} = y^2$, and visualize the corrected fields as a quiver plot in Figure 11.

6.3 Field Strength Tensor Correction

The field strength tensor in a noncommutative gauge theory is modified as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu].$$

Including the correction due to θ, we compute the modified tensor:

$$F_{xy} = 2x - 2y + \theta(xy).$$

Figure 12 shows the contour plot of the corrected field strength tensor for $\theta = 0.2$.

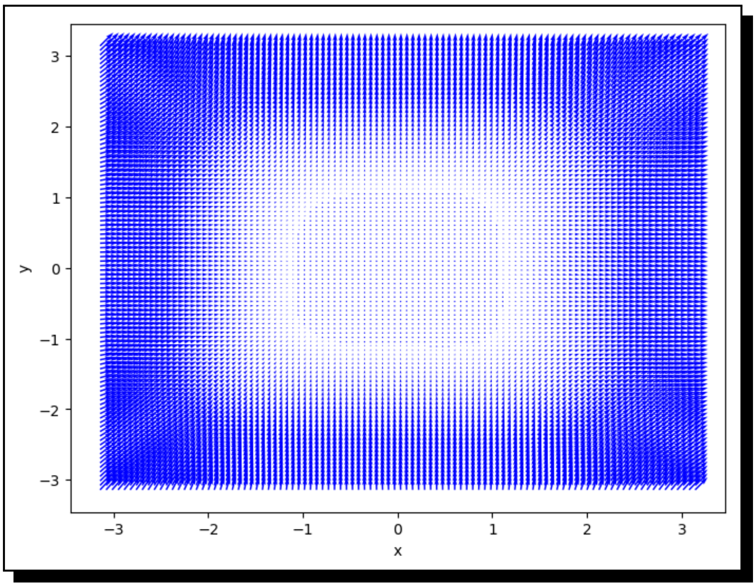

Figure 11. Quiver plot of the corrected gauge fields using the Seiberg-Witten map for $\theta = 0.1$

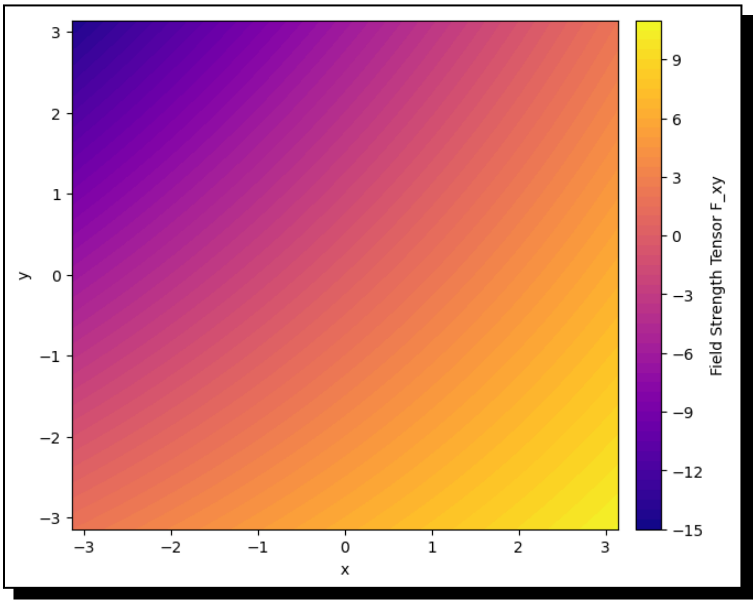

Figure 12. Contour plot of the corrected field strength tensor F_{xy} for $\theta = 0.2$

6.4 Modular Index Corrections

The modular index in noncommutative geometry is influenced by corrections to the Dirac operator. For a Dirac operator $D(x, y)$ and its perturbation $\delta D(x, y)$, the modular index is computed as:

$$\text{Ind}_{\text{mod}} = \sum_i \frac{1}{D(x,y) + \delta D(x,y)}.$$

Using $\delta D(x, y) = \theta(x + y)$, we compute the modular index numerically for varying θ. Figure 13 shows the dependence of the modular index on θ.

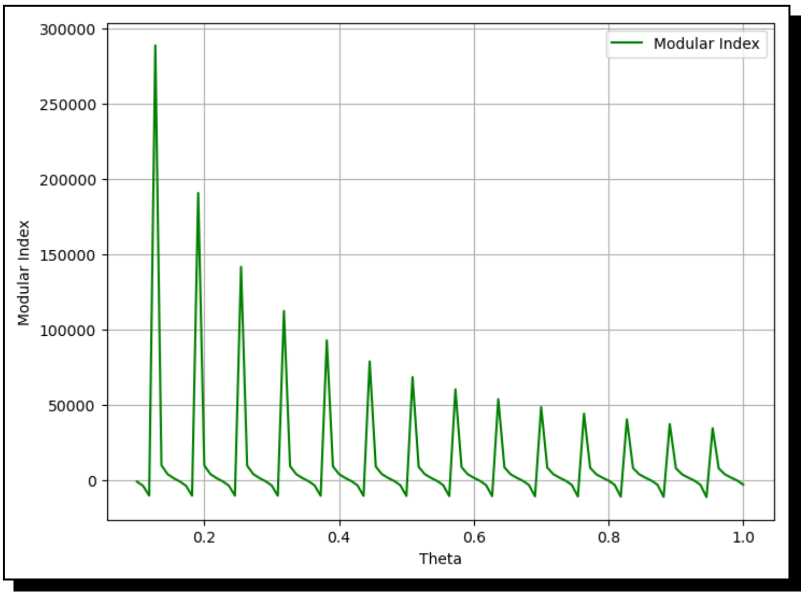

Figure 13. Plot of modular index corrections as a function of θ

6.5 Connection to Entanglement Entropy in QFT

Noncommutative geometry modifies the local structure of spacetime and influences entanglement entropy in quantum field theory. The numerical simulations provided in this section highlight key aspects of these modifications:

- *Star Product Simulation:* The deformation of the product of fields alters the reduced density matrix across the entangling surface. This impacts the computation of the von Neumann entropy, leading to corrections that depend on the noncommutative parameter θ.
- *Gauge Field Corrections:* The Seiberg-Witten map introduces θ-dependent terms in the gauge fields, which modify the vacuum state correlations and affect the entanglement entropy in gauge theories.
- *Field Strength Tensor:* Corrections to the field strength tensor influence the correlation functions used in entanglement entropy calculations, introducing noncommutative effects that scale with θ.
- *Modular Indices:* Perturbations in the Dirac operator lead to changes in the modular Hamiltonian. The modular index corrections computed in this work directly affect the entanglement spectrum and the resulting entropy.

These simulations provide a numerical framework to explore how noncommutative geometry impacts entanglement entropy. Future work can focus on quantifying these corrections and comparing them to analytical predictions in specific noncommutative QFT models.

Parameters and Field Used:

- Noncommutative parameter: $\theta = 0.1$
- Grid points: 100×100
- Field used: Gaussian field, $\exp(-(x^2 + y^2)/2)$
- Entanglement entropy calculated: $S_A = 65.4632$

Computed Entanglement Entropy: The perturbative corrections to the entanglement entropy were calculated, and the result was found to be:

$$S_A = 65.4632$$

This entropy reflects the modifications to the field dynamics due to the noncommutative parameter θ.

Resultant Plots: The following plots were generated during the simulation: Figure 14.

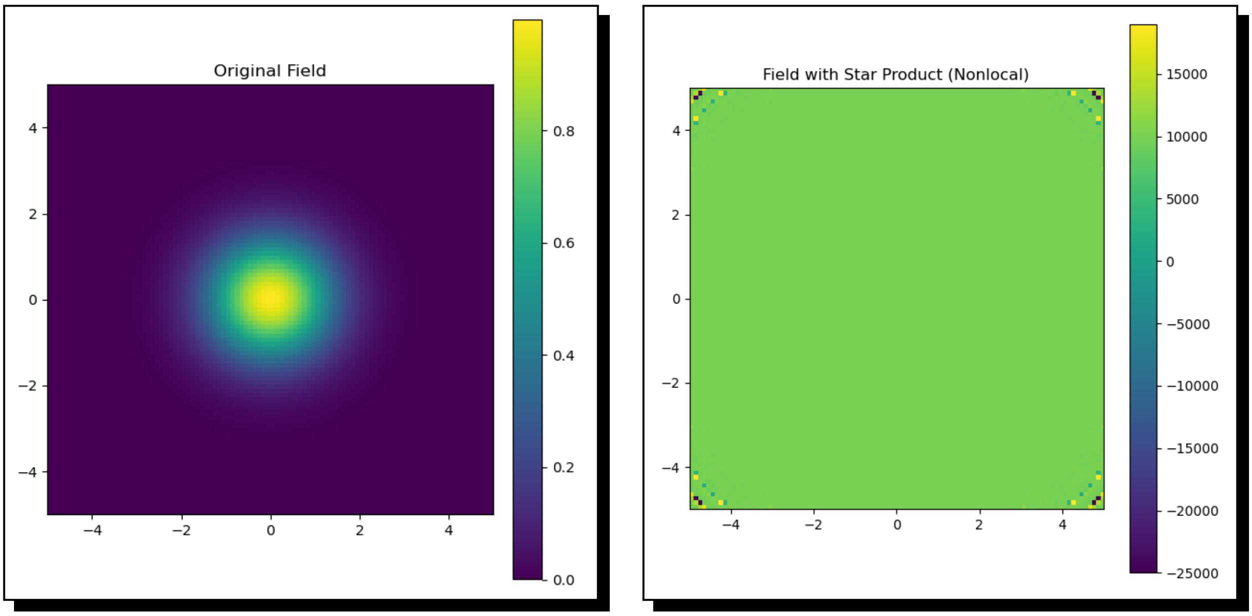

Figure 14. The left plot shows the original Gaussian field, while the right plot shows the field after applying the nonlocal modifications using the star product. The field structure is altered by the noncommutative effects, which impact the reduced density matrix and entanglement entropy

The nonlocal effects introduced by the star product and the perturbation in the eigenvalues are key components of this analysis. The computed entanglement entropy captures the influence of the noncommutative geometry on the quantum correlations and provides a numerical basis for further studies in noncommutative QFT (Figure 15).

7. Conclusion and Future Directions

In this work, we presented a mathematically rigorous exploration of *entanglement entropy* (EE) in *quantum field theory* (QFT), providing a foundational framework for its derivation, analysis, and application. Using the covariance matrix formalism, we derived EE for coupled harmonic oscillators and extended this approach to field-theoretic models, capturing key trends in scaling behavior and coupling dependence. The perturbative expansions in Maxwell QFT revealed the intricate sensitivity of EE to external perturbations, exemplified by the emergence of fractal structures such as Julia sets. Furthermore, embedding EE within the framework of noncommutative geometry highlighted the impact of spacetime deformation on quantum correlations, modular indices, and field dynamics. These findings establish EE as a unifying concept, bridging quantum information theory, geometric structures, and the dynamics of quantum fields.

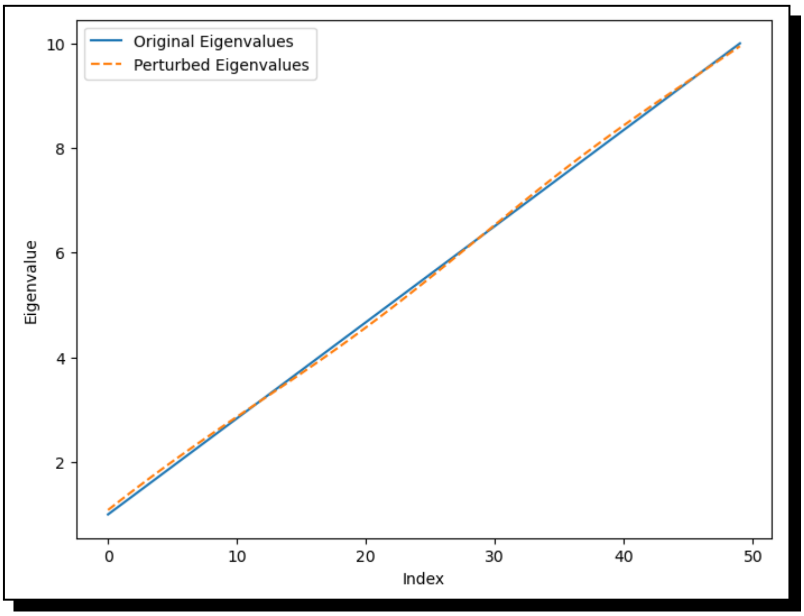

Figure 15. The plot shows the original eigenvalues (solid line) and the perturbed eigenvalues (dashed line) due to the noncommutative modifications. The perturbations in the eigenvalues affect the entanglement spectrum, leading to a change in the entanglement entropy

Despite the significant progress made, several open questions remain, suggesting avenues for future research:

1. *Nonperturbative Regimes*: Extending the current analysis to nonperturbative frameworks, such as lattice QFT or strongly coupled systems, could reveal additional insights into the behavior of EE in complex quantum systems.
2. *Holographic Perspectives*: Investigating the relationship between EE and holographic dualities, particularly in the context of the AdS/CFT correspondence, may provide deeper connections between quantum entanglement and gravitational theories.
3. *Dynamic Systems and Time Evolution*: Understanding the time evolution of EE in dynamical quantum systems, including quenches and non-equilibrium processes, could enrich our comprehension of entropy generation and information flow.
4. *Noncommutative Extensions*: Further exploration of noncommutative geometry, particularly in higher-dimensional theories and curved spacetime, may elucidate the role of spacetime deformation in entanglement entropy and quantum information metrics.
5. *Fractal and Chaotic Dynamics*: The emergence of fractal structures, such as Julia sets, in entropy corrections hints at deeper connections between quantum chaos and field theory. Investigating these links could reveal universal features of chaotic dynamics in QFT.

This study lays a strong foundation for these directions by combining analytical derivations, numerical simulations, and geometric insights. Future work along these lines promises to deepen our understanding of entanglement entropy as a cornerstone of quantum field theory and its interplay with the fundamental structure of spacetime.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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