Communications in Mathematics and Applications
Vol. 9, No. 4, pp. 713–723, 2018
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v9i4.975

Research Article

A Study on Soft S-Metric Spaces

Cigdem Gunduz Aras[1,*], Sadi Bayramov[2] and Vefa Cafarli[2]

[1] *Department of Mathematics, Kocaeli University, Kocaeli, 41380, Turkey*
[2] *Department of Algebra and Geometry, Baku State University, Baku, AZ 1148, Azerbaican*
***Corresponding author:** caras@kocaeli.edu.tr

Abstract. The first aim to this paper is to define soft S-metric space and to investigate some important theorems on sequential compact and totally bounded in soft S-metric space. Moreover, we introduce soft uniformly continuous mapping and examine some of its properties.

Keywords. Soft set; Soft S-metric; Soft sequential compact metric space; Soft uniformly continuous mapping

MSC. 54E45

Received: July 12, 2018 **Revised:** September 5, 2018 **Accepted:** September 9, 2018

This article was submitted to "*5th International Congress on Fundamental and Applied Sciences*" (ICFAS 2018) organized by International University of Balkan, Skopje, Macedonia during June 18-22, 2018.

Academic Editors: Dr. Muhammet Kurulay, Yildiz Technical University, Turkey
Dr. Muttalip Özavşar, Yildiz Technical University, Turkey
Dr. Fatih Demirkale, Yildiz Technical University, Turkey

1. Introduction

Metric space is one of the most useful and important notions in mathematics and applied sciences. Some authors interested and have tried to give generalizations of metric spaces in different ways. For example, concepts of D-metric spaces and 2-metric spaces were introduced [6] and [7], respectively. Partial metric space was introduced by [11]. Mustafa and Sims [14] introduced G-metric space as a new structure of generalized metric spaces. Sedghi *et al*. [16] modified the concepts of D-metric space and gave the concept of D^*-metric space. Later, Sedghi *et al*. initiated the notion of S-metric space which is different from other space as a generalization of a metric space in [17]. Some authors have proved fixed point type theorems in these spaces.

Metric spaces wide area provides a powerful tool to the study of optimization and approximation theory, variational inequalities and so many. After Molodtsov [13] initiated a novel concept of soft set theory as a new mathematical tool for dealing with uncertainties, applications of soft set theory in other disciplines and real life problems was progressing rapidly, the study of soft metric space which is based on soft point of soft sets was initiated by Das and Samanta [5].

Topological structures of soft set have been studied by some authors. Shabir and Naz [18] have initiated the study of soft topological spaces which are defined over an initial universe with a fixed set of parameters and showed that a soft topological space gives a parameterized family of topological spaces. Theoretical studies of soft topological spaces have also been researched by some authors in [3], [4], [8], [9], [12], [15], [19], etc.

The purpose of this paper is to contribute for investigating on soft S-metric space which is based on soft point of soft sets and give some of their properties. Moreover, we introduce the concepts of soft continuous and soft sequentially continuous mapping and examine the connection between them.

We briefly next section give some basic definitions of concepts which serve a background to this work.

2. Preliminaries

Throughout this paper, X denotes initial universe, E denotes the set of all parameters, $P(X)$ denotes the power set of X.

Definition 1 ([13])**.** A pair (F,E) is called a soft set over X, where F is a mapping given by $F: E \to P(X)$.
In other words, the soft set is a parameterized family of subsets of the set X. For $a \in E$, $F(a)$ may be considered as the set of a-elements of the soft set (F,E), or as the set of a-approximate elements of the soft set.

Definition 2 ([1])**.** For two soft sets (F,E) and (G,E) over X, (F,E) is called a soft subset of (G,E) if $\forall\ a \in E$, $F(a) \subseteq G(a)$.
This relationship is denoted by $(F,E)\tilde{\subseteq}(G,E)$.
Similarly, (F,E) is called a soft superset of (G,E) if (G,E) is a soft subset of (F,E). This relationship is denoted by $(F,E)\tilde{\supseteq}(G,E)$. Two soft sets (F,E) and (G,E) over X are called soft equal if (F,E) is a soft subset of (G,E) and (G,E) is a soft subset of (F,E).

Definition 3 ([1])**.** The intersection of two soft sets (F,E) and (G,E) over X is the soft set (H,E), where $\forall\ a \in E$, $H(a) = F(a) \cap G(a)$. This is denoted by $(F,E)\tilde{\cap}(G,E) = (H,E)$.

Definition 4 ([1])**.** The union of two soft sets (F,E) and (G,E) over X is the soft set (H,E), where $\forall a \in E$, $H(a) = F(a) \cup G(a)$. This is denoted by $(F,E)\tilde{\cup}(G,E) = (H,E)$.

Definition 5 ([10]). A soft set (F,E) over X is said to be a null soft set denoted by Φ if for all $a \in E$, $F(a) = \varnothing$.

Definition 6 ([10]). A soft set (F,E) over X is said to be an absolute soft set denoted by $\widetilde{X}$ if for all $a \in E$, $F(a) = X$.

Definition 7 ([18]). The difference (H,E) of two soft sets (F,E) and (G,E) over X, denoted by $(F,E)\widetilde{\setminus}(G,E)$, is defined as $H(a) = F(a)\setminus G(a)$ for all $a \in E$.

Definition 8 ([18]). The complement of a soft set (F,E), denoted by $(F,E)^c$, is defined $(F,E)^c = (F^c,E)$, where $F^c : E \to P(X)$ is a mapping given by $F^c(a) = X\setminus F(a)$, $\forall a \in E$ and F^c is called the soft complement function of F.

Definition 9 ([18]). Let $\tilde{\tau}$ be the collection of soft sets over X, then $\tilde{\tau}$ is said to be a soft topology on X if

(1) $\Phi, \widetilde{X}$ belongs to $\tilde{\tau}$;

(2) the union of any number of soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$;

(3) the intersection of any two soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.

The triplet $(X,\tilde{\tau},E)$ is called a soft topological space over X.

Definition 10 ([18]). Let $(X,\tilde{\tau},E)$ be a softtopological space over X, then members of $\tilde{\tau}$ are said to be a soft open sets in X.

Definition 11 ([18]). Let $(X,\tilde{\tau},E)$ be a soft topological space over X. A soft set (F,E) over X is said to be a soft closed set in X, if its complement $(F,E)^c$ belongs to $\tilde{\tau}$.

Proposition 1 ([18]). *Let $(X,\tilde{\tau},E)$ be a soft topological space over X. Then the collection $\tilde{\tau}_a = \{F(a) : (F,E) \in \tilde{\tau}\}$ for each $a \in E$, defines a topology on X.*

Definition 12 ([18]). Let $(X,\tilde{\tau},E)$ be a soft topological space over X and (F,E) be a soft set over X. Then the soft closure of (F,E), denoted by $\overline{(F,E)}$ is the intersection of all soft closed super sets of (F,E). Clearly $\overline{(F,E)}$ is the smallest soft closed set over X which contains (F,E). Similarly, the soft interior of (F,E), denoted by $(F,E)^0$, is the union of all soft open subsets of (F,E).

Definition 13 ([2]). Let $(X,\tilde{\tau},E)$ be a soft topological space over X and (F,E) be a soft set over X. Then the soft bounded of (F,E), denoted by $\partial(F,E)$, is defined $\partial(F,E) = \overline{(F,E)} \cap \overline{(F,E)^c}$.

Definition 14 ([2], [5]). Let (F,E) be a soft set over X. The soft set (F,E) is called a soft point, denoted by (x_a,E), if for the element $a \in E$, $F(a) = \{x\}$ and $F(a') = \varnothing$ for all $a' \in E - \{a\}$ (briefly denoted by x_a).

It is obvious that each soft set can be expressed as a union of soft points. For this reason, to give the family of all soft sets on X it is sufficient to give only soft points on X.

Definition 15 ([2]). Two soft points x_a and $y_{a'}$ over a common universe X, we say that the soft points are different if $x \neq y$ or $a \neq a'$.

Definition 16 ([2]). The soft point x_a is said to be belonging to the soft set (F,E), denoted by $x_a \tilde{\in} (F,E)$, if $x_a(a) \in F(a)$, i.e., $\{x\} \subseteq F(a)$.

Definition 17 ([2]). Let $(X,\tilde{\tau},E)$ be a soft topological space over X. A soft set $(F,E) \tilde{\subseteq} (X,E)$ is called a soft neighborhood of the soft point $x_a \tilde{\in} (F,E)$ if there exists a soft open set (G,E) such that $x_a \tilde{\in} (G,E) \tilde{\subset} (F,E)$.

Definition 18. [5] Let $\mathbb{R}$ be the set of all real numbers, $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E be taken as a set of parameters. Then a mapping $F: E \to B(\mathbb{R})$ is called a soft real set. It is denoted by (F,E). If (F,E) is a singleton soft set, then it will be called a soft real number and denoted by $\tilde{r},\tilde{s},\tilde{t}$ etc. Here $\tilde{r},\tilde{s},\tilde{t}$ will denote a particular type of soft real numbers such that $\tilde{r}(a) = r$, for all $a \in E$. $\tilde{0}$ and $\tilde{1}$ are the soft real numbers, where $\tilde{0}(a) = 0$, $\tilde{1}(a) = 1$ for all $a \in E$, respectively.

The following definition is about a partial ordering on the set of soft real numbers.

Definition 19 ([5]). Let $\tilde{r},\tilde{s}$ be two soft real numbers, then the following statements hold:

(i) $\tilde{r} \tilde{\leq} \tilde{s}$, if $\tilde{r}(a) \leq \tilde{s}(a)$, for all $a \in E$,

(ii) $\tilde{r} \tilde{\geq} \tilde{s}$, if $\tilde{r}(a) \geq \tilde{s}(a)$, for all $a \in E$,

(iii) $\tilde{r} \tilde{<} \tilde{s}$, if $\tilde{r}(a) < \tilde{s}(a)$, for all $a \in E$,

(iv) $\tilde{r} \tilde{>} \tilde{s}$, if $\tilde{r}(a) > \tilde{s}(a)$, for all $a \in E$.

Definition 20 ([17]). Let X be a nonempty set and $S: X^3 \to [0,\infty)$ be a function satisfying the following conditions for all $x,y,z,t \in X$,

(1) $S(x,y,z) = 0$ if and only if $x = y = z$,

(2) $S(x,y,z) \leq S(x,x,t) + S(y,y,t) + S(z,z,t)$.

Then S is called an S-metric on X and the pair (X,S) is called an S-metric space.

3. Soft S-Metric Spaces

In this section, we introduce soft S-metric spaces and study some important results of its. Let $\tilde{X}$ be the absolute soft set, E be a non-empty set of parameters and $SP(\tilde{X})$ be the collection of all soft points of $\tilde{X}$. Let $\mathbb{R}(E)^*$ denote the set of all non-negative soft real numbers.

Definition 21. A soft S-metric on $SP(\tilde{X})$ is a mapping $S: SP(\tilde{X}) \times SP(\tilde{X}) \times SP(\tilde{X}) \to \mathbb{R}(E)^*$ that satisfies the following conditions, for each soft points $x_a, y_b, z_c, u_d \in SP(\tilde{X})$,

(S1) $S(x_a,y_b,z_c) \geq \tilde{0}$,

(S2) $S(x_a,y_b,z_c) = \tilde{0}$ if and only if $x_a = y_b = z_c$,

(S3) $S(x_a,y_b,z_c) \leq S(x_a,x_a,u_d) + S(y_b,y_b,u_d) + S(z_c,z_c,u_d)$.

Then the soft set $\tilde{X}$ with a soft S-metric is called a soft S-metric space and denoted by $(\tilde{X},S,E)$.

Example 1. Let $E \subset \mathbb{R}$ be a non-empty set of parameters. Let (X,d) be an ordinary metric on X, therefore $d_s(x_a,y_b) = |a-b| + d(x,y)$ is a soft metric. Then let us define a mapping

$$S : SP(\widetilde{X}) \times SP(\widetilde{X}) \times SP(\widetilde{X}) \to \mathbb{R}(E)^* \quad \text{by}$$
$$S(x_a,y_b,z_c) = d_s(x_a,z_c) + d_s(y_b,z_c)$$

for all $x_a,y_b,z_c \in SP(\widetilde{X})$. It is clear that S is a soft S-metric on $SP(\widetilde{X})$. Let us verify only (S3) for soft S-metric space.

$$\begin{aligned} S(x_a,y_b,z_c) &= d_s(x_a,z_c) + d_s(y_b,z_c) \\ &= |a-c| + d(x,z) + |b-c| + d(y,z) \\ &\le |a-d| + |d-c| + d(x,u) + d(u,z) + |b-d| + |d-c| + d(y,u) + d(z,u) \\ &= |a-d| + |b-d| + 2|d-c| + d(x,u) + d(y,u) + 2d(z,u) \\ &\le S(x_a,x_a,u_d) + S(y_b,y_b,u_d) + S(z_c,z_c,u_d). \end{aligned}$$

Thus S is a soft S-metric on $SP(\widetilde{X})$.

Example 2. Let $E \subset \mathbb{C}$ be a non-empty subset of complex numbers and $(X,\|\|)$ be a normed space. Then

$$S : SP(\widetilde{X}) \times SP(\widetilde{X}) \times SP(\widetilde{X}) \to \mathbb{R}(E)^*$$

is defined by

$$S(x_a,y_b,z_c) = |b+c-2a| + |b-c| + \|y+z-2x\| + \|y-z\|,$$

for all $x_a,y_b,z_c \in SP(\widetilde{X})$. Then it can be easily verified that S is a soft S-metric on $SP(\widetilde{X})$. Indeed,

$$\begin{aligned} S(x_a,y_b,z_c) &= |b+c-2a| + |b-c| + \|y+z+2x\| + \|y-z\| \\ &\le |b-a| + |c-a| + |b-c| + \|y-x\| + \|z-x\| + \|y-z\| \\ &\le 2|b-d| + 2|a-d| + 2|c-d| + 2\,\|y-u\| + 2\,\|x-u\| + 2\,\|z-u\| \\ &= S(x_a,x_a,u_d) + S(y_b,y_b,u_d) + S(z_c,z_c,u_d). \end{aligned}$$

Thus S is a soft S-metric on $SP(\widetilde{X})$.

Remark 1. If $(\widetilde{X},S,E)$ is a soft S-metric space, then (X,S_a) is an S-metric space for each $a \in E$. Here S_a stands for the S-metric for only parameter a and (X,S_a) is a crisp S-metric space. It is clear that every soft S-metric space is a family of parameterized S-metric space.

The converse of Remark 1 may not be true in general. This is shown by the following example.

Example 3. Let $E = \mathbb{R}$ be a parameter set and $(X,\widetilde{S})$ be a S-metric space. We define a mapping

$$S : SP(\widetilde{X}) \times SP(\widetilde{X}) \times SP(\widetilde{X}) \to \mathbb{R}(E)^* \quad \text{by}$$
$$S(x_a,y_b,z_c) = \widetilde{S}(x,y,z)^{1+|a-b|+|a-c|},$$

for all $x_a,y_b,z_c \in SP(\widetilde{X})$. Then for all $a \in \mathbb{R}$, S_a is an S-metric on X, but S is not soft S-metric on $SP(\widetilde{X})$.

Lemma 1. *Let $(\widetilde{X},S,E)$ is a soft S-metric space. Then we have*

$$S(x_a,x_a,y_b)=S(y_b,y_b,x_a).$$

Proof. By the third condition of soft S-metric, we get

$$S(x_a,x_a,y_b)\leq 2S(x_a,x_a,x_a)+S(y_b,y_b,x_a)=S(y_b,y_b,x_a) \tag{3.1}$$

and similarly

$$S(y_b,y_b,x_a)\leq 2S(y_b,y_b,y_b)+S(x_a,x_a,y_b)=S(x_a,x_a,y_b). \tag{3.2}$$

Hence, by inequalities (3.1) and (3.2), we obtain

$$S(x_a,x_a,y_b)=S(y_b,y_b,x_a).$$ □

Definition 22. Let $(\widetilde{X},S,E)$ be a soft S-metric space and $\widetilde{r}$ be a non-negative soft real number. For $\widetilde{r}>\widetilde{0}$ and $x_a\in SP(\widetilde{X})$, we define the soft open ball $B_S(x_a,\widetilde{r})$ and soft closed ball $\mathbf{B}_S(x_a,\widetilde{r})$ and with center x_a and a radius $\widetilde{r}$ as follows:

$$B_S(x_a,\widetilde{r})=\{y_b\in SP(\widetilde{X}):S(y_b,y_b,x_a)<\widetilde{r}\},$$
$$\mathbf{B}_S(x_a,\widetilde{r})=\{y_b\in SP(\widetilde{X}):S(y_b,y_b,x_a)\leq\widetilde{r}\}.$$

Example 4. Let $E=\mathbb{Z}$ and $X=\mathbb{R}^2$. Denote

$$S(x_a,y_b,z_c)=|a-c|+|b-c|+d(x,z)+d(y,z)$$

for all $x_a,y_b,z_c\in SP(\widetilde{X})$. Therefore let $\theta=(0,0)\in\mathbb{R}^2$.

$$\begin{aligned}
B_S(\theta_0,\widetilde{6})&=\{y_b\in SP(\widetilde{X}):S(y_b,y_b,\theta_0)<\widetilde{6}\}\\
&=\{y_b\in SP(\widetilde{X}):2|b|+2d(y,\theta)<\widetilde{6}\}\\
&=\{y_b\in SP(\widetilde{X}):d(y,\theta)<\widetilde{3}-|b|\}\\
&=\{y_0\in SP(\widetilde{X}):d(y,\theta)<\widetilde{3}\}\cup\{y_1\in SP(\widetilde{X}):d(y,\theta)<\widetilde{2}\}\\
&\quad\cup\{y_2\in SP(\widetilde{X}):d(y,\theta)<\widetilde{1}\}\cup\{y_{-1}\in SP(\widetilde{X}):d(y,\theta)<\widetilde{2}\}\\
&\quad\cup\{y_{-2}\in SP(\widetilde{X}):d(y,\theta)<\widetilde{1}\},\\
\mathbf{B}_S(\theta_0,\widetilde{6})&=\{y_b\in SP(\widetilde{X}):S(y_b,y_b,\theta_0)\leq\widetilde{6}\}\\
&=\{y_0\in SP(\widetilde{X}):d(y,\theta)\leq\widetilde{3}\}\cup\{y_1\in SP(\widetilde{X}):d(y,\theta)\leq\widetilde{2}\}\\
&\quad\cup\{y_2\in SP(\widetilde{X}):d(y,\theta)\leq\widetilde{1}\}\cup\{y_{-1}\in SP(\widetilde{X}):d(y,\theta)\leq\widetilde{2}\}\\
&\quad\cup\{y_{-2}\in SP(\widetilde{X}):d(y,\theta)\leq\widetilde{1}\}\cup\{\theta_3\}\cup\{\theta_{-3}\}.
\end{aligned}$$

Definition 23. Let $(\widetilde{X},S,E)$ be a soft S-metric space and (F,E) be a soft set.

(a) If for every $x_a\in(F,E)$ there exists $\widetilde{r}>\widetilde{0}$ such that $B_S(x_a,\widetilde{r})\subset(F,E)$, then the soft set (F,E) is called a soft open set in $(\widetilde{X},S,E)$.

(b) The soft set (F,E) is said to be soft S-bounded if there exists $\widetilde{r}>\widetilde{0}$ such that $S(x_a,x_a,y_b)<\widetilde{r}$ for all $x_a,y_b\in(F,E)$.

(c) A soft sequence $\{x^n_{a_n}\}$ in $(\widetilde{X},S,E)$ converges to x_b if and only if $S(x^n_{a_n},x^n_{a_n},x_b)\to\widetilde{0}$ as $n\to\infty$ and we denote this by $\lim\limits_{n\to\infty}x^n_{a_n}=x_b$.

(d) A soft sequence $\{x_{a_n}^n\}$ in $(\widetilde{X},S,E)$ is called a Cauchy sequence if for $\widetilde{\varepsilon} > \widetilde{0}$, there exists $n_0 \in \mathbb{N}$ such that $S(x_{a_n}^n, x_{a_n}^n, x_{a_m}^m) < \widetilde{\varepsilon}$ for each $n,m \geq n_0$.

(e) The soft S-metric space $(\widetilde{X},S,E)$ is said to be complete if every Cauchy sequence is convergent.

Remark 2. It is easily seen that the conditions of base of soft topology is satisfied for soft open balls. This topology is called soft topology induced by the soft S-metric.

Lemma 2. *Let $(\widetilde{X},S,E)$ be a soft S-metric space. If $\widetilde{r} > \widetilde{0}$ and $x_a \in SP(\widetilde{X})$, then the ball $B_S(x_a,\widetilde{r})$ ($\mathbf{B}_S(x_a,\widetilde{r})$) is a soft open (closed) set in $(\widetilde{X},S,E)$.*

Proof. Let $y_b \in B_S(x_a,\widetilde{r})$. Hence $S(y_b,y_b,x_a) < \widetilde{r}$. If we set $\widetilde{d} = S(x_a,x_a,y_b)$ and $\widetilde{r}' = \frac{\widetilde{r}-\widetilde{d}}{2}$, then we prove that $B_S(y_b,\widetilde{r}') \subset B_S(x_a,\widetilde{r})$. Let $z_c \in B_S(y_b,\widetilde{r}')$, therefore, $S(z_c,z_c,y_b) < \widetilde{r}'$. By the third condition of soft S-metric, we have

$$\begin{aligned} S(z_c,z_c,x_a) &\leq S(z_c,z_c,y_b) + S(z_c,z_c,y_b) + S(x_a,x_a,y_b) \\ &< 2\widetilde{r}' + \widetilde{d} = \widetilde{r} \end{aligned}$$

and so $B_S(y_b,\widetilde{r}') \subset B_S(x_a,\widetilde{r})$. □

Lemma 3. *Let $(\widetilde{X},S,E)$ be a soft S-metric space. If the sequence $\{x_{a_n}^n\}$ in $(\widetilde{X},S,E)$ converges to x_b, then x_b is unique.*

Proof. Let $\{x_{a_n}^n\}$ converges to x_b and y_c.Then for each $\widetilde{\varepsilon} > \widetilde{0}$, there exist $n_1,n_2 \in \mathbb{N}$ such that

$$n \geq n_1 \Rightarrow S(x_{a_n}^n, x_{a_n}^n, x_b) < \frac{\widetilde{\varepsilon}}{4}$$

and

$$n \geq n_2 \Rightarrow S(x_{a_n}^n, x_{a_n}^n, y_c) < \frac{\widetilde{\varepsilon}}{2}.$$

If we set $n_0 = \max\{n_1,n_2\}$, therefore for every $n \geq n_0$ and the third condition of S-metric we get

$$\begin{aligned} S(x_b,x_b,y_c) &\leq 2S(x_b,x_b,x_{a_n}^n) + S(y_c,y_c,x_{a_n}^n) \\ &< \frac{\widetilde{\varepsilon}}{2} + \frac{\widetilde{\varepsilon}}{2} = \widetilde{\varepsilon}. \end{aligned}$$

Hence $S(x_b,x_b,y_c) = \widetilde{0}$ and so $x_b = y_c$. □

Lemma 4. *Let $(\widetilde{X},S,E)$ be a soft S-metric space. If the sequence $\{x_{a_n}^n\}$ in $(\widetilde{X},S,E)$ converges to x_b, then $\{x_{a_n}^n\}$ is a Cauchy sequence.*

Proof. Since $\lim_{n\to\infty} x_{a_n}^n = x_b$, then for each $\widetilde{\varepsilon} > \widetilde{0}$ there exist $n_1,n_2 \in \mathbb{N}$ such that

$$n \geq n_1 \Rightarrow S(x_{a_n}^n, x_{a_n}^n, x_b) < \frac{\widetilde{\varepsilon}}{4}$$

and

$$m \geq n_2 \Rightarrow S(x_{a_m}^m, x_{a_m}^m, x_b) < \frac{\widetilde{\varepsilon}}{2}.$$

If we set $n_0 = \max\{n_1,n_2\}$, therefore for every $n,m \geq n_0$ and the third condition of S-metric, we get

$$S(x^n_{a_n},x^n_{a_n},x^m_{a_m}) \le 2S(x^n_{a_n},x^n_{a_n},x_b)+S(x^m_{a_m},x^m_{a_m},x_b) < \frac{\widetilde{\varepsilon}}{2}+\frac{\widetilde{\varepsilon}}{2}=\widetilde{\varepsilon}.$$

Hence, $\{x^n_{a_n}\}$ is a Cauchy sequence. □

Lemma 5. *Let $(\widetilde{X},S,E)$ be a soft S-metric space. If there exist sequences $\{x^n_{a_n}\}$ and $\{y^n_{b_n}\}$ such that $\lim_{n\to\infty} x^n_{a_n}=x_a$ and $\lim_{n\to\infty} y^n_{b_n}=y_b$, then $\lim_{n\to\infty} S(x^n_{a_n},x^n_{a_n},y^n_{b_n})=S(x_a,x_a,y_b)$.*

Proof. Since $\lim_{n\to\infty} x^n_{a_n}=x_a$ and $\lim_{n\to\infty} y^n_{b_n}=y_b$, then for each $\widetilde{\varepsilon}>\widetilde{0}$ there exist $n_1,n_2\in\mathbb{N}$ such that

$$\forall n\ge n_1, S(x^n_{a_n},x^n_{a_n},x_a)<\frac{\widetilde{\varepsilon}}{4}$$

and

$$\forall n\ge n_2, S(y^n_{b_n},y^n_{b_n},y_b)<\frac{\widetilde{\varepsilon}}{4}.$$

If we take $n_0=\max\{n_1,n_2\}$, therefore for every $n\ge n_0$ we get by the third condition of S-metric

$$\begin{aligned}S(x^n_{a_n},x^n_{a_n},y^n_{b_n}) &\le 2S(x^n_{a_n},x^n_{a_n},x_a)+S(y^n_{b_n},y^n_{b_n},x_a)\\ &\le 2S(x^n_{a_n},x^n_{a_n},x_a)+2S(y^n_{b_n},y^n_{b_n},y_b)+S(x_a,x_a,y_b)\\ &< \frac{\widetilde{\varepsilon}}{2}+\frac{\widetilde{\varepsilon}}{2}+S(x_a,x_a,y_b).\end{aligned}$$

Hence we obtain

$$S(x^n_{a_n},x^n_{a_n},y^n_{b_n})-S(x_a,x_a,y_b)<\widetilde{\varepsilon}. \tag{3.3}$$

On the other hand, we get

$$\begin{aligned}S(x_a,x_a,y_b) &\le 2S(x_a,x_a,x^n_{a_n})+S(y_b,y_b,x^n_{a_n})\\ &\le 2S(x_a,x_a,x^n_{a_n})+2S(y_b,y_b,y^n_{b_n})+S(x^n_{a_n},x^n_{a_n},y^n_{b_n})\\ &< \frac{\widetilde{\varepsilon}}{2}+\frac{\widetilde{\varepsilon}}{2}+S(x^n_{a_n},x^n_{a_n},y^n_{b_n}),\end{aligned}$$

that is

$$S(x_a,x_a,y_b)-S(x^n_{a_n},x^n_{a_n},y^n_{b_n})<\widetilde{\varepsilon}. \tag{3.4}$$

By relations (3.3) and (3.4), we have

$$|S(x^n_{a_n},x^n_{a_n},y^n_{b_n})-S(x_a,x_a,y_b)|<\widetilde{\varepsilon}$$

that is $\lim_{n\to\infty} S(x^n_{a_n},x^n_{a_n},y^n_{b_n})=S(x_a,x_a,y_b)$. □

Proposition 2. *Let $(\widetilde{X},S,E)$ be a soft S-metric space and τ_S be a soft topology generated by the soft S-metric S. Then for every $a\in E$ the topology $(\tau_S)_a$ is the topology generated by the S-metric S_a on X.*

Proof. The proof is clear. □

Proposition 3. *Let $(\widetilde{X},S,E)$ be a soft S-metric space and (F,E) be a soft set. Then the following expressions are true:*

(a) $x_a \in \overline{(F,E)} \Leftrightarrow S(x_a,(F,E)) = \inf_{y_b \in (F,E)} S(x_a,y_b,y_b) = \widetilde{0}$,

(b) $x_a \in (F,E)^0 \Leftrightarrow S(x_a,(F,E)^c) \geq \widetilde{0}$,

(c) $x_a \in \partial(F,E) \Leftrightarrow S(x_a,(F,E)) = S(x_a,x_a,(F,E)^c) = \widetilde{0}$.

Proof. The proof is clear. □

Theorem 1. *Every soft S-metric space is a soft normal space.*

Proof. Let (F_1,E) and (F_2,E) be two disjoint soft closed sets in the soft S-metric space $(\widetilde{X},S,E)$. For every soft points $x_a \in (F_1,E)$ and $y_b \in (F_2,E)$, we choose soft open balls $B_S(x_a,\widetilde{\varepsilon}_{x_a})$ and $B_S(y_b,\widetilde{\delta}_{y_b})$ such that $B_S(x_a,\widetilde{\varepsilon}_{x_a}) \cap (F_2,E) = \Phi$ and $B_S(y_b,\widetilde{\delta}_{y_b}) \cap (F_1,E) = \Phi$. Thus, we have

$$(F_1,E) \subset \bigcup_{x_a} B_S\left(x_a,\left(\frac{\widetilde{\varepsilon}}{3}\right)_{x_a}\right) = (U,E)$$

and

$$(F_2,E) \subset \bigcup_{y_b} B_S\left(y_b,\left(\frac{\widetilde{\delta}}{3}\right)_{y_b}\right) = (V,E).$$

We want to show that $(U,E) \cap (V,E) = \Phi$. Assume that $(U,E) \cap (V,E) \neq \Phi$. Then there exists a soft point z_c such that $z_c \in (U,E) \cap (V,E)$. Therefore, there exist soft open balls $B_S\left(x_a,\left(\frac{\widetilde{\varepsilon}}{6}\right)_{x_a}\right)$ and $B_S\left(y_b,\left(\frac{\widetilde{\delta}}{6}\right)_{y_b}\right)$ such that $z_c \in B_S\left(x_a,\left(\frac{\widetilde{\varepsilon}}{6}\right)_{x_a}\right)$ and $z_c \in B_S\left(y_b,\left(\frac{\widetilde{\delta}}{6}\right)_{y_b}\right)$. Here, we have $S(x_a,x_a,z_c) < \left(\frac{\widetilde{\varepsilon}}{6}\right)_{x_a}$ and $S(y_b,y_b,z_c) < \left(\frac{\widetilde{\delta}}{3}\right)_{y_b}$. If we get $\max\left\{\left(\frac{\widetilde{\varepsilon}}{6}\right)_{x_a},\left(\frac{\widetilde{\delta}}{3}\right)_{y_b}\right\} = \left(\frac{\widetilde{\varepsilon}}{6}\right)_{x_a}$, then we have

$$\begin{aligned} S(x_a,x_a,y_b) &\leq 2S(x_a,x_a,z_c) + S(y_b,y_b,z_c) \\ &< \left(\frac{\widetilde{\varepsilon}}{3}\right)_{x_a} + \left(\frac{\widetilde{\varepsilon}}{6}\right)_{x_a} < \widetilde{\varepsilon}_{x_a} \end{aligned}$$

and also $y_b \in B_S(x_a,\widetilde{\varepsilon}_{x_a})$ and which contradicts with our assumption.
Therefore, $(U,E) \cap (V,E) = \Phi$. □

Definition 24. Let $(\widetilde{X},S,E)$ and $(\widetilde{Y},S',E')$be two soft S-metric spaces and $(f,\varphi) : (\widetilde{X},S,E) \to (\widetilde{Y},S',E')$ be a soft mapping. The mapping $(f,\varphi) : (\widetilde{X},S,E) \to (\widetilde{Y},S',E')$ is a soft continuous mapping at the soft point $x_a \in SP(\widetilde{X})$ if for every soft open ball $B_S(f(x)_{\varphi(a)},\widetilde{\varepsilon})$ of $(\widetilde{Y},S',E')$, there exists a soft open ball $B_S(x_a,\widetilde{\delta})$ of $(\widetilde{X},S,E)$ such that $f(B_S(x_a,\widetilde{\delta})) \subset B_S(f(x)_{\varphi(a)},\widetilde{\varepsilon})$.

If (f,φ) is a soft continuous mapping at every soft point x_a of $(\widetilde{X},S,E)$, then it is said to be soft continuous mapping on $(\widetilde{X},S,E)$.

Definition 25. The soft mapping $(f,\varphi) : (\widetilde{X},S,E) \to (\widetilde{Y},S',E')$ is said to be soft sequentially continuous at the soft point $x_a \in SP(\widetilde{X})$ if and only if for every sequence of soft point $\{x^n_{a_n}\}$ converging to the soft point x_a in the soft S-metric space $(\widetilde{X},S,E)$, the sequence $(f,\varphi)(\{x^n_{a_n}\})$ in $(\widetilde{Y},S',E')$ converges to a soft point $(f,\varphi)(x_a) \in SP(\widetilde{Y})$.

Theorem 2. *Soft continuity is equivalent to soft sequential continuity in soft S-metric spaces.*

Proof. Let $(f,\varphi):(\widetilde{X},S,E)\to(\widetilde{Y},S',E')$ be a soft continuous mapping and $\{x^n_{a_n}\}$ be any sequence of soft points converging to the soft point $x_a \in SP(\widetilde{X})$. Let $B_S(f(x)_{\varphi(a)},\widetilde{\varepsilon})$ be a soft open ball in $(\widetilde{Y},S',E')$. By continuity of (f,φ) choose a soft open ball $B_S(x_a,\widetilde{\delta})$ such that $(f,\varphi)(B_S(x_a,\widetilde{\delta}))\subset B_S(f(x)_{\varphi(a)},\widetilde{\varepsilon})$. Since $\{x^n_{a_n}\}$ converges to x_a there exists $n_0\in\mathbb{N}$ such that $\{x^n_{a_n}\}\subset B_S(x_a,\widetilde{\delta})$ for all $n\ge n_0$. Therefore for all $n\ge n_0$, we have

$$(f,\varphi)(\{x^n_{a_n}\})\subset(f,\varphi)(B_S(x_a,\widetilde{\delta}))\subset B_S(f(x)_{\varphi(a)},\widetilde{\varepsilon}),$$

as required.

Conversely, assume for contradiction that $(f,\varphi):(\widetilde{X},S,E)\to(\widetilde{Y},S',E')$ is soft sequential continuous but not soft continuous mapping. Since (f,φ) is not soft continuous at the soft point x_a, there exist $\widetilde{\varepsilon}>\widetilde{0}$ and $y_b\in SP(\widetilde{X})$ for arbitrary $\widetilde{\delta}>\widetilde{0}$ such that $S(x_a,x_a,y_b)<\widetilde{\delta}$ and $S'((f,\varphi)(x_a),(f,\varphi)(x_a),(f,\varphi)(y_b))\ge\widetilde{\varepsilon}$. For $n\ge 1$ $(n\in\mathbb{N})$, define $\widetilde{\delta}_n=\frac{1}{n}$. For $n\ge 1$ we may choose $\{y^n_{b_n}\}$ in $(\widetilde{X},S,E)$ such that

$$S(x_a,x_a,y^n_{b_n})<\widetilde{\delta}_n \quad\text{and}\quad S'((f,\varphi)(x_a),(f,\varphi)(x_a),(f,\varphi)(y^n_{b_n}))\ge\widetilde{\varepsilon}.$$

Therefore, by definition the sequence $\{y^n_{b_n}\}$ converges to x_a. But, the sequence $\{(f,\varphi)(y^n_{b_n})\}$ does not converge to $(f,\varphi)(x_a)$. That is, (f,φ) is not soft sequentially continuous at x_a. □

4. Conclusion

We have introduced soft S-metric space which is based on soft point of soft sets and give some of its properties.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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