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Research Article

Embankment Surfaces in Euclidean 3-Space and Their Visualizations

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Abstract. In the present paper, we obtain the parametric representation of an embankment surface and give an example for it. We define the notions of embankmentlike surfaces and tubembankmentlike surfaces. Furthermore, we create some embankmentlike and tubembankmentlike surface examples with the aid of different directrix and draw these directrix and surfaces. Also, we find the Gaussian, mean and second Gaussian curvatures of these surfaces and draw the Gaussian, mean and second Gaussian curvature functions' graphics and the variations of Gaussian, mean and second Gaussian curvatures on related surfaces with the aid of *Mathematica*.

Keywords. Cone; Directrix; Embankment Surface; Gaussian Curvature; Mean curvature

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1. Introduction

A surface of revolution is generated by rotating a planar curve. Moving a line generates a ruled surface. Moving spheres and cones yield canal surfaces and embankment surfaces, respectively. In the literature, there are lots of studies about different geometric properties of surfaces of revolution, ruled surfaces and canal surfaces (see [3], [4], [6], [12], [16], [17] and etc.). We know that, if $\Lambda : (r(t), z(t))$, $t \in [a,b]$, is a parametric curve in $(r-z)$-plane with $r > 0$, then $\Phi := (r(t)\cos\varphi, r(t)\sin\varphi, z(t))$, $t \in [a,b]$, $\varphi \in [0,2\pi]$, is called *parametric surface of revolution* and if $\Lambda : f(r,z) = 0$ is an implicit curve in $(r-z)$-plane with $r > 0$, then $\Phi : f(\sqrt{x^2+y^2}, z)$ is called *implicit surface of revolution*.

If $\Lambda : \mathbf{X} = c(t)$, $t \in [a,b]$, is a regular C^n, $n \geq 1$ curve in E^3 and $r(t)$, $t \in [a,b]$, nonzero vectors of class C^n, then the surface $\Phi : \mathbf{X} = \mathbf{X}(s,t) := c(t) + sr(t)$, $(s,t) \in [a,b] \times [c,d]$, is called *ruled surface*, any line $\mathbf{X} = \mathbf{X}(s, t =$constant) is called *ruling* and Λ is called the *base curve* of the parametrization.

Also, if a one parameter family of regular implicit surfaces $\Phi_c : f(\mathbf{X}, c) = 0$, $c \in [c_1, c_2]$, is given, then the intersection curve of two neighbored surfaces Φ_c and $\Phi_{c+\Delta c}$ fulfills the two equations $f(\mathbf{X}, c) = 0$ and $f(\mathbf{X}, c + \Delta c) = 0$. We consider the limit for $\Delta c \to 0$ and get $f_c(\mathbf{X}, c) = \lim\limits_{\Delta \to 0} \frac{f(\mathbf{X},c) - f(\mathbf{X},c+\Delta c)}{\Delta c} = 0$. The last equation motivates the following definition:

If $\Phi_c : f(\mathbf{X}, c) = 0$, $c \in [c_1, c_2]$, is a one parameter family of regular implicit C^2-surfaces, then the surface which is defined the two equations $f(\mathbf{X}, c) = 0$ and $f_c(\mathbf{X}, c) = 0$ is called *envelope* of the given family of surfaces. With the aid of this definition, one can define the following surfaces:

Let $\Lambda : \mathbf{X} = \alpha(u) = (a(u), b(u), c(u))$ be a regular space curve and $r(t)$ be a C^1-function with $r > 0$ and $|\dot{r}| < \|\dot{c}\|$. The envelope of the one parameter family of spheres

$$f(\mathbf{X}; u) := (x - \alpha(u))^2 - r(u)^2 = 0 \tag{1}$$

is called a *canal surface* and Λ its *directrix*. Also, the parametric representation of canal surfaces can be obtained by

$$\mathbf{X} = \mathbf{X}(u,v) := \alpha(u) - \frac{r(u)\dot{r}(u)}{\|\dot{\alpha}(u)\|^2}\dot{\alpha}(u) + \frac{r(u)\sqrt{\|\dot{\alpha}(u)\|^2 - \dot{r}(u)^2}}{\|\dot{\alpha}(u)\|}(e_1(u)\cos(v) + e_2(u)\sin(v)),$$

with $\{e_1, e_2\}$ an orthonormal base orthogonal to tangent vector $\dot{\alpha}$. In case of a constant radius function, the envelope is called *pipe surface* (see [11]). Canal surfaces (especially pipe surfaces) have been applied to many fields, such as the solid and the surface modeling for CAD/CAM, construction of blending surfaces, shape re-construction and so on. One can see the details for geometric and applied fields of canal surfaces in [1], [7], [13], [14], [15], [18] and etc.

An embankment surface is an envelope of the one parameter family of cones which is stated in the following and this surface type is very important for engineers who draw the embankment constructions' plans (see [8] for details of embankment constructions).

On the other hand, it is important to have the knowledge of the Gaussian and mean curvatures for future structural engineers. For example: Tensile fabric structure (e.g. membrane roof) in a uniform state of tensile prestress behaves like a soap film stretched over a wire which is bent in a shape of a closed space curve. Soap film assumes a form which has the minimal area relative to all other surfaces stretched over the same wire; this surface is therefore called minimal surface. It can be shown that mean curvature vanishes at each point of that surface [9].

In this study, firstly we recall the implicit formulae of an embankment surface and we obtain a parametric representation of embankment surfaces and we give an example for embankment surfaces. Later, we define the notions of embankmentlike surfaces and tubembankmentlike surfaces. Furthermore, we obtain some embankmentlike and tubembankmentlike surface examples with the aid of different directrix and draw these directrix and surfaces. Also, we find the Gaussian and mean (and second Gaussian) curvatures of these surfaces and draw the Gaussian and mean (and second Gaussian) curvature functions' graphics. Besides, we draw graphics which show the variations of Gaussian and mean (and second Gaussian) curvatures on related surfaces. When making these visualizations, we use *Mathematica*.

2. Preliminaries

Here, we recall Frenet-Serret formulae of space curves and Gaussian, mean and second Gaussian curvatures of surfaces in Euclidean 3-space. Also, we recall some basic concepts for quaternions.

Let E^3 be a 3-dimensional Euclidean space with the standard inner product $g: dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is a standard rectangular coordinate system of E^3. Also, let $\alpha : I \to E^3$ be a regular curve with $\dot{\alpha}(t) = \frac{d\alpha}{dt}(t) \neq 0$. If T, N and B are unit tangent vector field, unit principal normal vector field and unit binormal vector field of α, respectively, then $\{T, N, B\}$ is called the *Frenet frame* of α and the Frenet-Serret formulae is given by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2}$$

where

$$g(T,T) = g(N,N) = g(B,B) = 1, \; g(T,N) = g(N,B) = g(B,T) = 0. \tag{3}$$

Here κ and τ are curvature and torsion of α, respectively [5].

Let we denote a surface Γ in E^3 by

$$\Gamma(u,v) = (\Gamma_1(u,v), \Gamma_2(u,v), \Gamma_3(u,v))$$

and N be the standard unit normal vector field on the surface Γ defined by

$$N = \frac{\Gamma_u \times \Gamma_v}{\|\Gamma_u \times \Gamma_v\|},$$

where $\Gamma_u = \frac{\partial \Gamma(u,v)}{\partial u}$. Then the first fundamental form I and the second fundamental form II of the surface Γ are defined by

$$I = g_{11} du^2 + 2 g_{12} du dv + g_{22} dv^2,$$

$$II = L_{11} du^2 + 2 L_{12} du dv + L_{22} dv^2,$$

respectively, where we put

$$g_{11} = g(\Gamma_u, \Gamma_u), \;\; g_{12} = g(\Gamma_u, \Gamma_v), \;\; g_{22} = g(\Gamma_v, \Gamma_v),$$

$$L_{11} = g(\Gamma_{\mathrm{uu}}, N), \;\; L_{12} = g(\Gamma_{uv}, N), \;\; L_{22} = g(\Gamma_{vv}, N).$$

Using classical notation above, the Gaussian curvature K, mean curvature H and second Gaussian curvature K_{II} are defined by

$$K = \frac{L_{11} L_{22} - L_{12}^2}{g_{11} g_{22} - g_{12}^2}, \tag{4}$$

$$H = \frac{g_{22} L_{11} - 2 g_{12} L_{12} + g_{11} L_{22}}{2(g_{11} g_{22} - g_{12}^2)} \tag{5}$$

and

$$K_{II}=\frac{1}{(L_{11}L_{22}-L_{12}^2)^2}\left\{\begin{vmatrix}-\frac{1}{2}L_{11_{vv}}+L_{12_{uv}}-\frac{1}{2}L_{22_{uu}} & \frac{1}{2}L_{11_u} & L_{22_u}-\frac{1}{2}L_{11_v}\\ L_{12_v}-\frac{1}{2}L_{22_u} & L_{11} & L_{12}\\ \frac{1}{2}L_{22_v} & L_{12} & L_{22}\end{vmatrix} - \begin{vmatrix}0 & \frac{1}{2}L_{11_v} & \frac{1}{2}L_{22_u}\\ \frac{1}{2}L_{11_v} & L_{11} & L_{12}\\ \frac{1}{2}L_{22_u} & L_{12} & L_{22}\end{vmatrix}\right\}, \tag{6}$$

respectively [19]-[20]. We know that, a surface is called (X,Y)-Weingarten surface if it satisfies $\Phi(X,Y)=X_uY_v-X_vY_u=0$, where $(X,Y)\in\{(H,K),(H,K_{II}),(K,K_{II})\}$ [17].

Now, let us recall some notions about quaternions. We know that, the algebra $H=\{q=a_01+a_1i+a_2j+a_3k : a_0,a_1,a_2,a_3\in\mathbb{R}\}$ of quaternions is defined as the four dimensional vector space $\mathbb{R}^4$ having a basis $\{1,i,j,k\}$ with the following properties:

$$i^2=j^2=k^2=i\times j\times k=-1,\quad i\times j=-j\times i=k.$$

So, H is an associative and not commutative algebra, 1 is identity element of H and i, j and k are standard orthonormal basis in $\mathbb{R}^3$. A quaternion can also be written as $q=(a_0,w)=Sq+Vq$, where $Sq=a_0\in\mathbb{R}$ is the scalar component and $Vq=w\in\mathbb{R}^3$ is the vector component of q. For two quaternions $q=Sq+Vq$, $p=Sp+Vp$ and $\lambda\in\mathbb{R}$, we have

$$q+p=(Sq+Sp)+(Vq+Vp),$$
$$\lambda q=\lambda Sq+\lambda Vq$$

and also, quaternion product of two quaternions is defined as

$$q\mathbf{x}p=SqSp-g(Vq,Vp)+SqVp+SpVq+Vq\times Vp,$$

where $g(Vq,Vp)$ and $Vq\times Vp$ denote the familiar dot and cross-products, respectively.

If $\|q\|=1$, then the quaternion q is unitary and the unitary quaternion can be written in the trigonometric form as $q=\cos\theta+v\sin\theta$, where $v\in\mathbb{R}^3$ and $\|v\|=1$ (see [1], [2], [10]).

3. Embankment Surfaces in E^3

In this section, we obtain the parametric expression of an embankment surface in Euclidean 3-space and give a characterization for it using quaternions. Also, we create an example for embankment surfaces.

Definition 1. Let $\Lambda:\mathbf{X}=\alpha(u)=(\alpha_1(u),\alpha_2(u),\alpha_3(u))$ be a regular space curve and $m\in\mathbb{R}$, $m>0$ with $|m\dot{\alpha}_3|<\sqrt{\dot{\alpha}_1{}^2+\dot{\alpha}_2{}^2}$. Then the envelope of the one parameter family of cones

$$f(\mathbf{X};u):=(x-\alpha_1(u))^2+(y-\alpha_2(u))^2-m^2(z-\alpha_3(u))^2=0 \tag{7}$$

is called an *embankment surface* and Λ its *directrix* (see [11]).

Let Γ be a parametrization of the envelope of cones defining the embankment surface given by $\Gamma(u,v)=(\Gamma_1(u,v),\Gamma_2(u,v),\Gamma_3(u,v))$; $\alpha(u)=(\alpha_1(u),\alpha_2(u),\alpha_3(u))$ be a regular space curve with non-zero curvature which is called directrix of embankment surface and $m\in\mathbb{R}$, $m>0$ with

$|m\dot{\alpha}_3| < \sqrt{\dot{\alpha}_1{}^2 + \dot{\alpha}_2{}^2}$. Then, from (7), the embankment surface can be written by

$$\begin{aligned}&(\Gamma_1(u,v)-\alpha_1(u))^2+(\Gamma_2(u,v)-\alpha_2(u))^2-m^2(\Gamma_3(u,v)-\alpha_3(u))^2\\&\quad+(\Gamma_3(u,v)-\alpha_3(u))^2-(\Gamma_3(u,v)-\alpha_3(u))^2=0.\end{aligned} \tag{8}$$

From (8), we can write

$$g(\Gamma(u,v)-\alpha(u),\Gamma(u,v)-\alpha(u))=(m^2+1)(\Gamma_3(u,v)-\alpha_3(u))^2. \tag{9}$$

On the other hand, the parametric representation of the embankment surface can be given by

$$\Gamma(u,v)-\alpha(u)=r(u,v)T(u)+s(u,v)N(u)+t(u,v)B(u), \tag{10}$$

where r, s, t are differentiable functions of u and v on the interval I. From (3) and (10), we have

$$g(\Gamma(u,v)-\alpha(u),\Gamma(u,v)-\alpha(u))=r^2(u,v)+s^2(u,v)+t^2(u,v). \tag{11}$$

Thus, from (9) and (11), we get

$$r^2(u,v)+s^2(u,v)+t^2(u,v)=(m^2+1)(\Gamma_3(u,v)-\alpha_3(u))^2. \tag{12}$$

Differentiating (12) with respect to u and v, we have

$$rr_u+ss_u+tt_u=(m^2+1)(\Gamma_3(u,v)-\alpha_3(u))(\Gamma_3(u,v)-\alpha_3(u))_u \tag{13}$$

and

$$rr_v+ss_v+tt_v=(m^2+1)(\Gamma_3(u,v)-\alpha_3(u))\Gamma_3(u,v)_v, \tag{14}$$

respectively. Also, differentiating (10) with respect to u and v, and using (2) we get

$$\Gamma(u,v)_u=(\|\alpha'(u)\|+r_u-s\kappa)T+(r\kappa+s_u-t\tau)N+(s\tau+t_u)B \tag{15}$$

and

$$\Gamma(u,v)_v=r_vT+s_vN+t_vB. \tag{16}$$

Now, let us suppose that

$$g(\Gamma(u,v)-\alpha(u),\Gamma(u,v)_u)=0 \tag{17}$$

satisfies on embankment surface. Then, from (10), (15) and (17), we have

$$\|\alpha'\|.r+rr_u+ss_u+tt_u=0. \tag{18}$$

Thus, from (13) and (18), we obtain

$$r(u,v)=-\frac{(m^2+1)}{\|\alpha'(u)\|}(\Gamma_3(u,v)-\alpha_3(u))(\Gamma_3(u,v)-\alpha_3(u))_u. \tag{19}$$

Using (19) in (12), we get

$$s^2(u,v)+t^2(u,v)=(m^2+1)(\Gamma_3(u,v)-\alpha_3(u))^2\{1-\frac{(m^2+1)}{\|\alpha'(u)\|^2}(\Gamma_3(u,v)-\alpha_3(u))_u^2\} \tag{20}$$

Hence, we can choose

$$\left.\begin{aligned}s(u,v)&=\mp\sqrt{m^2+1}(\Gamma_3(u,v)-\alpha_3(u))\sqrt{1-\tfrac{(m^2+1)}{\|\alpha'(u)\|^2}(\Gamma_3(u,v)-\alpha_3(u))_u^2}.\cos v\\t(u,v)&=\mp\sqrt{m^2+1}(\Gamma_3(u,v)-\alpha_3(u))\sqrt{1-\tfrac{(m^2+1)}{\|\alpha'(u)\|^2}(\Gamma_3(u,v)-\alpha_3(u))_u^2}.\sin v.\end{aligned}\right\} \tag{21}$$

Therefore, from (10), (19) and (21), we can state the following main theorem:

Theorem 1. *Let $\alpha : I \subseteq \mathbb{R} \to E^3$ be a regular space curve with non-zero curvature. Then, the parametrization of embankment surface $\Gamma_e(u,v) = (\Gamma_{e_1}(u,v), \Gamma_{e_2}(u,v), \Gamma_{e_3}(u,v))$ can be given by*

$$\Gamma_e(u,v) = \alpha(u) - \frac{(m^2+1)}{\|\alpha'(u)\|}\Psi_3(u,v)\Psi_3(u,v)_u.T(u)$$
$$\mp \sqrt{m^2+1}\Psi_3(u,v)\sqrt{1-\frac{(m^2+1)}{\|\alpha'(u)\|^2}\Psi_3(u,v)_u^2}.\{\cos v.N(u) + \sin v.B(u)\}, \tag{22}$$

where $\Psi_3(u,v) = \Gamma_{e_3}(u,v) - \alpha_3(u)$, $m \in \mathbb{R}$, $m > 0$ with $|m\dot{\alpha}_3| < \sqrt{\dot{\alpha}_1^2 + \dot{\alpha}_2^2}$.

On the other hand, using the unit quaternion we can give the following characterization for embankment surfaces.

Theorem 2. *Let $\alpha : I \subseteq \mathbb{R} \to E^3$ be a unit speed directrix curve with non-zero curvature of embankment surface Γ_e and $q(u,v) = \cos v + \sin v.T(u)$ be a unit quaternion in $S^3 \subset \mathbb{R}^4$. Then, the embankment surface Γ_e can be given by*

$$\Gamma_e(u,v) = \alpha(u) - (m^2+1)\Psi_3(u,v)\Psi_3(u,v)_u.T(u)$$
$$\mp \sqrt{m^2+1}\Psi_3(u,v)\sqrt{1-(m^2+1)\Psi_3(u,v)_u^2}.q(u,v) \times N(u). \tag{23}$$

Proof. The proof is obvious from quaternion product and the definition of Frenet frame $\{T,N,B\}$ of α. □

Here, let us give an example for embankment surfaces:

Example 1. Let us take directrix as

$$\alpha(u) = (\cos(u), \sin(u), 0), \tag{24}$$

which is a circle in E^3 and let us suppose that $m = \sqrt{3}$. Thus, the embankment surface (22) whose directrix is this circle can be obtained as

$$\Gamma_e(u,v) = (\cos(u) + 4\Gamma_3(u,v)\Gamma_3(u,v)_u \sin(u) - 2\Gamma_3(u,v)\sqrt{1-4\Gamma_3(u,v)_u^2}\cos(u)\cos(v),$$
$$\sin(u) - 4\Gamma_3(u,v)\Gamma_3(u,v)_u \cos(u) - 2\Gamma_3(u,v)\sqrt{1-4\Gamma_3(u,v)_u^2}\sin(u)\cos(v),$$
$$2\Gamma_3(u,v)\sqrt{1-4\Gamma_3(u,v)_u^2}\sin(v)). \tag{25}$$

Since $\Gamma_e(u,v) = (\Gamma_{e_1}(u,v), \Gamma_{e_2}(u,v), \Gamma_{e_3}(u,v))$, we can choose the third component of the surface (25) as

$$\Gamma_{e_3}(u,v) = \frac{\sqrt{4\sin^2(v)-1}}{4\sin(v)}.u. \tag{26}$$

Thus, using (26) in (25), we can write the embankment surface as

$$\Gamma_e(u,v) = \begin{pmatrix} \cos(u) + u.\sin(u).\frac{4\sin^2(v)-1}{4\sin^2(v)} - u.\cos(u)\cos(v).\frac{\sqrt{4\sin^2(v)-1}}{4\sin(v)}, \\ \sin(u) - u.\cos(u).\frac{4\sin^2(v)-1}{4\sin^2(v)} - u.\sin(u)\cos(v).\frac{\sqrt{4\sin^2(v)-1}}{4\sin(v)}, \\ \frac{\sqrt{4\sin^2(v)-1}}{4\sin(v)}.u \end{pmatrix}. \tag{27}$$

One can easily see that,

$$(\Gamma_{e_1}(u,v)-\alpha_1(u))^2+(\Gamma_{e_2}(u,v)-\alpha_2(u))^2-m^2(\Gamma_{e_3}(u,v)-\alpha_3(u))^2=0$$

satisfies for (27) and so, it is an embankment surface. The graphics of the directrix (24) and embankment surface (27) is given by Figure 1.

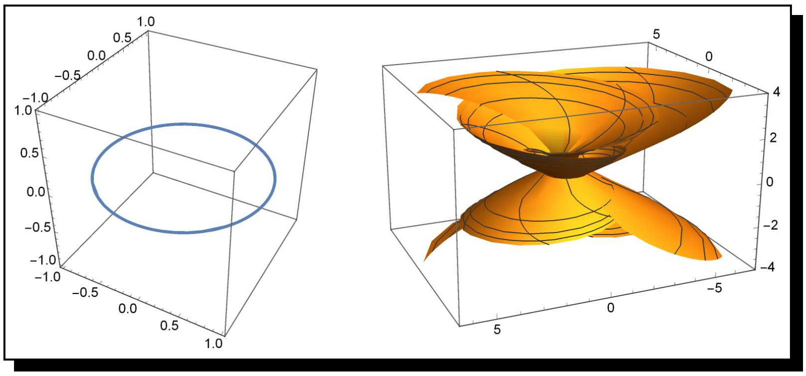

Figure 1. The directrix (24) and embankment surface (27)

Also, in Figure 2, one can see the Gaussian and mean curvatures functions' graphics above and the variations of Gaussian and mean curvatures on embankment surface (27) below.

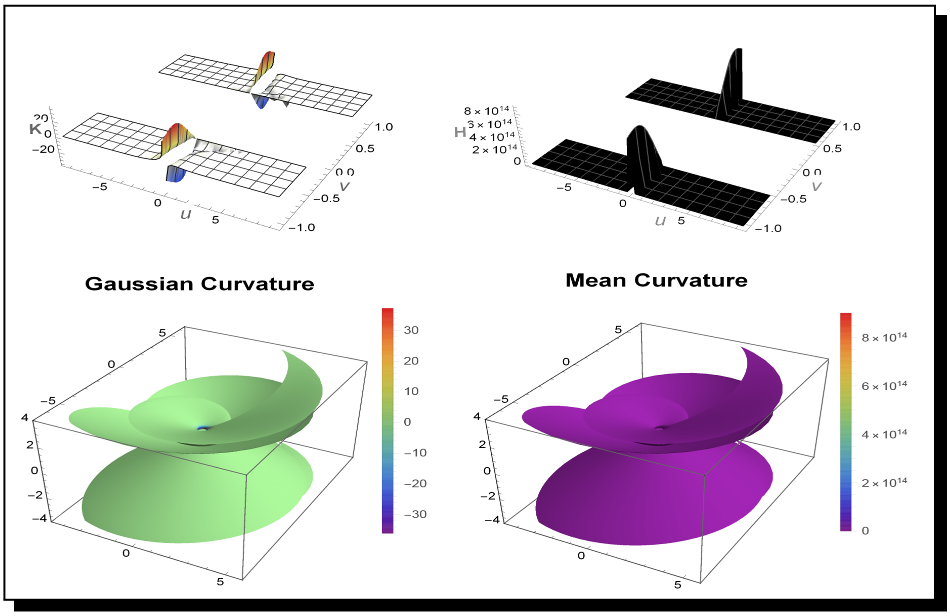

Figure 2. Gaussian and mean curvatures functions' graphics and the variations of Gaussian and mean curvatures on embankment surface (27)

4. Embankmentlike Surfaces in E^3

As one can see from Example 1, finding the third component of the embankment surface is very difficult when we take another directrix curves. Hence, now we define a new surface type using the parametrization (22) of an embankment surface. Here, we take an arbitrary function $\Omega(u,v)$ instead of $\Gamma_{e_3}(u,v)$ in (22) and call it *embankmentlike surface*.

So, we can give the following definition.

Definition 2. Let $\alpha : I \subseteq \mathbb{R} \to E^3$ be a regular space curve with non-zero curvature. Then, the surface $\Gamma_{el}(u,v) = (\Gamma_{el_1}(u,v), \Gamma_{el_2}(u,v), \Gamma_{el_3}(u,v))$ which can be given by

$$\Gamma_{el}(u,v) = \alpha(u) - \frac{(m^2+1)}{\|\alpha'(u)\|}\Psi(u,v)\Psi(u,v)_u.T(u)$$
$$\mp \sqrt{m^2+1}\Psi(u,v)\sqrt{1-\frac{(m^2+1)}{\|\alpha'(u)\|^2}\Psi(u,v)_u^2}.\{\cos v.N(u)+\sin v.B(u)\} \tag{28}$$

is called an *embankmentlike surface*, where $\Psi(u,v) = \Omega(u,v) - \alpha_3(u)$, $m \in \mathbb{R}$, $m > 0$ with $|m\dot{\alpha}_3| < \sqrt{\dot{\alpha}_1^2 + \dot{\alpha}_2^2}$ and $\Omega(u,v)$ is an arbitrary function according to u and v.

Corollary 1. *Let $\alpha : I \subseteq \mathbb{R} \to E^3$ be a unit speed directrix curve with non-zero curvature of embankmentlike surface Γ_{el} and $q(u,v) = \cos v + \sin v.T(u)$ be a unit quaternion in $S^3 \subset \mathbb{R}^4$. Then, the embankmentlike surface Γ_{el} can be written by*

$$\Gamma_{el}(u,v) = \alpha(u) - (m^2+1)\Psi(u,v)\Psi(u,v)_u.T(u)$$
$$\mp \sqrt{m^2+1}\Psi(u,v)\sqrt{1-(m^2+1)\Psi(u,v)_u^2}.q(u,v)\times N(u), \tag{29}$$

where $\Psi(u,v) = \Omega(u,v) - \alpha_3(u)$, $m \in \mathbb{R}$, $m > 0$ with $|m\dot{\alpha}_3| < \sqrt{\dot{\alpha}_1^2 + \dot{\alpha}_2^2}$ and $\Omega(u,v)$ is an arbitrary function according to u and v.

4.1 Visualization for Embankmentlike Surfaces in E^3

In this subsection, we give some visualizations for embankmentlike surfaces in Euclidean 3-space. For this, we create some examples for embankmentlike surfaces with the aid of different directrix and draw these directrix and surfaces.

Also, after computing Gaussian and mean curvatures in each regular point of a surface given by parametric equations, these functions enable us to plot the graphs of the Gaussian and mean curvatures of regular surfaces and to paint surfaces with colours which depend on these curvatures [9].

So, we find the Gaussian and mean curvatures of these surfaces and draw the Gaussian and mean curvature functions' graphics. Furthermore, we draw graphics which show the variations of Gaussian and mean curvatures on related surfaces with the aid of *Mathematica*. For the following visualizations we use the Mathematica colour function "TemperatureMap" with values on a colour-spectrum.

Example 2. Let us take directrix as

$$\alpha(u) = (\cos(u), \sin(u), 0), \tag{30}$$

which is a circle in E^3 and let us suppose that $\Omega(u,v) = 1 - v$, $v \neq 1$ and $m = \sqrt{3}$.
Now, let us obtain the embankmentlike surface (28). Using (30) in (28), we can obtain the embankmentlike surface as

$$\Gamma_{el}(u,v) = (\cos(u) - 2(1-v)\cos(u)\cos(v), \sin(u) - 2(1-v)\sin(u)\cos(v), 2(1-v)\sin(v)). \tag{31}$$

In Figure 3, one can see the directrix (30) and embankmentlike surface (31).

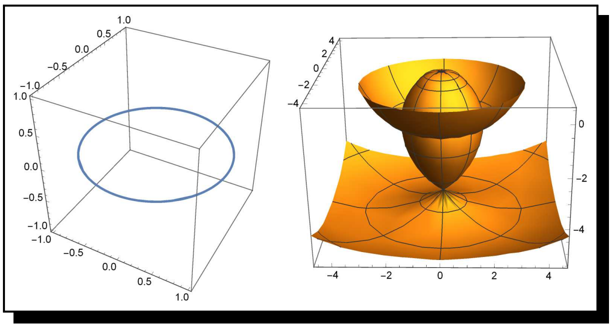

Figure 3. The directrix (30) and embankmentlike surface (31)

Furthermore, we obtain the Gaussian and mean curvatures of embankmentlike surface (31) as

$$K = \frac{(v^2-2v+3)((-1+v)\cos(v)+\sin(v))}{2(v^2-2v+2)^2(1+2(-1+v)\cos(v))}$$

and

$$H = \frac{v^2-2v+3+2(2v^3-6v^2+9v-5)\cos(v)+2(v^2-2v+2)\sin(v)}{4(v^2-2v+2)\sqrt{(v^2-2v+2)(1+2(-1+v)\cos(v))^2}},$$

respectively. In Figure4, one can see the Gaussian and mean curvatures functions' graphics above and the variations of Gaussian and mean curvatures on embankmentlike surface (31) below.

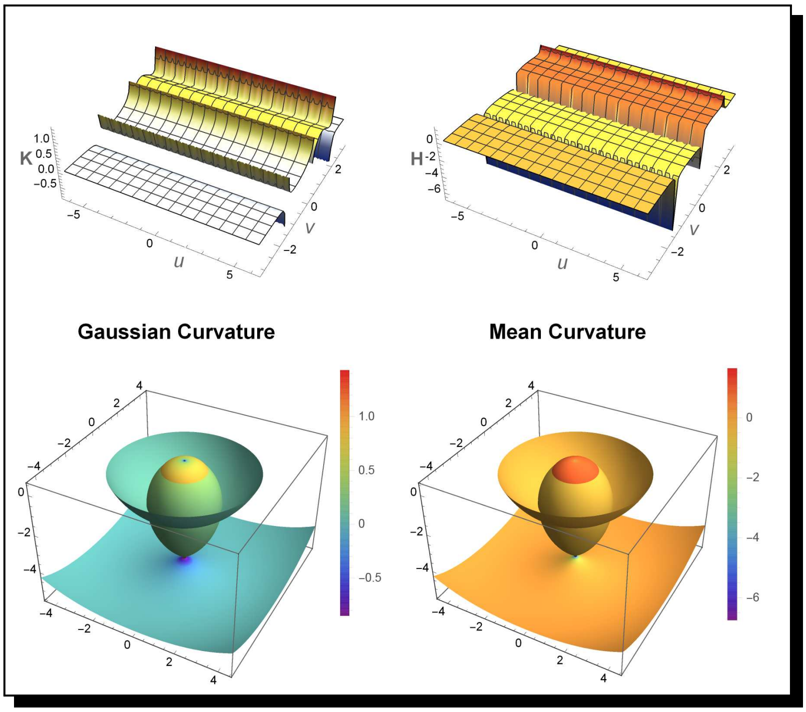

Figure 4. Gaussian and mean curvatures functions' graphics and the variations of Gaussian and mean curvatures on embankmentlike surface (31)

In the following examples, we use the same procedure as Example 2 and don't go into more details.

Example 3. Let us take directrix as

$$\alpha(u) = (2\cos(u), 2\sin(u), u), \tag{32}$$

which is a helix in E^3 and let us suppose that $\Omega(u,v) = u + v$ and $m = 1$. Then, we obtain the embankmentlike surface (28) under these assumptions as

$$\Gamma_{el}(u,v) = \Big(2\cos(u) - \sqrt{2}v\cos(u)\cos(v) + \sqrt{\frac{2}{5}}v\sin(u)\sin(v),$$
$$2\sin(u) - \sqrt{2}v\sin(u)\cos(v) - \sqrt{\frac{2}{5}}v\cos(u)\sin(v), u + \frac{2\sqrt{2}}{\sqrt{5}}v\sin(v)\Big). \tag{33}$$

The directrix (32) and embankmentlike surface (33) can be seen in Figure 5.

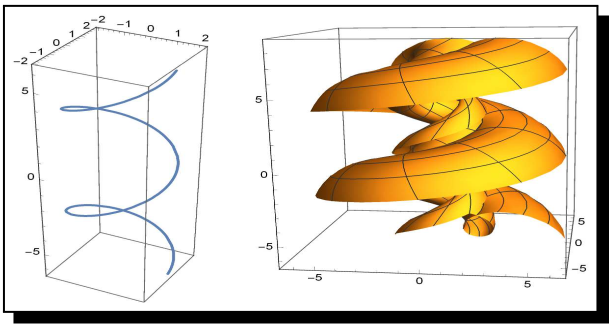

Figure 5. The directrix (32) and embankmentlike surface (33)

In Figure 6 one can see the Gaussian and mean curvatures functions' graphics above and the variations of Gaussian and mean curvatures on embankmentlike surface (33) below.

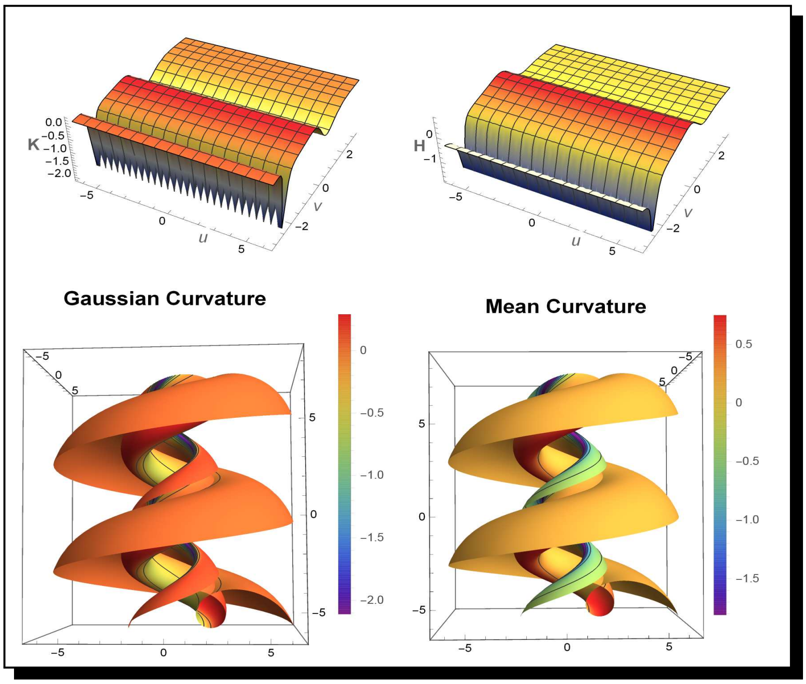

Figure 6. Gaussian and mean curvatures functions' graphics and the variations of Gaussian and mean curvatures on embankmentlike surface (33)

Example 4. Let us take directrix as

$$\alpha(u) = \left(1+\cos(u), \sin(u), 2\sin\left(\frac{u}{2}\right)\right), \tag{34}$$

which is Viviani curve in E^3 and let us suppose that $\Omega(u,v) = 2\left(\sin\left(\frac{u}{2}\right)+\cos\left(\frac{v}{2}\right)\right)$ and $m = 0,5$. Here, one can easily calculate Frenet-Serret apparatus of Viviani curve as

$$\left.\begin{array}{l}
T(u) = \dfrac{1}{\sqrt{3+\cos(u)}}\left(-\sqrt{2}\sin(u), \sqrt{2}\cos(u), \sqrt{2}\cos\left(\dfrac{u}{2}\right)\right) \\
N(u) = \dfrac{1}{2\sqrt{3+\cos(u)}\sqrt{13+3\cos(u)}}\left(-(3+12\cos(u)+\cos(2u)), -2(6+\cos(u))\sin(u), -4\sin\left(\dfrac{u}{2}\right)\right) \\
B(u) = \dfrac{1}{\sqrt{26+6\cos(u)}}\left(3\sin\left(\dfrac{u}{2}\right)+\sin\left(\dfrac{3u}{2}\right), -4\cos\left(\dfrac{u}{2}\right)^3, 4\right) \\
\kappa = \dfrac{\sqrt{13+3\cos(u)}}{(3+\cos(u))^{\frac{3}{2}}}, \\
\tau = \dfrac{6\cos(\frac{u}{2})}{13+3\cos(u)}.
\end{array}\right\} \tag{35}$$

Thus the embankmentlike surface (28) under these assumptions is obtained as

$$\begin{aligned}
\Gamma_{el}(u,v) = \Big(&1+\cos(u)+\lambda_1 N_1(u)+\mu_1 B_1(u), \sin u+\lambda_1 N_2(u)+\mu_1 B_2(u), \\
&2\sin\left(\frac{u}{2}\right)+\lambda_1 N_3(u)+\mu_1 B_3(u)\Big),
\end{aligned} \tag{36}$$

where $\lambda_1 = \sqrt{5}\cos(v)\cos(\frac{v}{2})$, $\mu_1 = \sqrt{5}\sin(v)\cos(\frac{v}{2})$, respectively. In Figure 7, one can see the directrix (34) and embankmentlike surface (36).

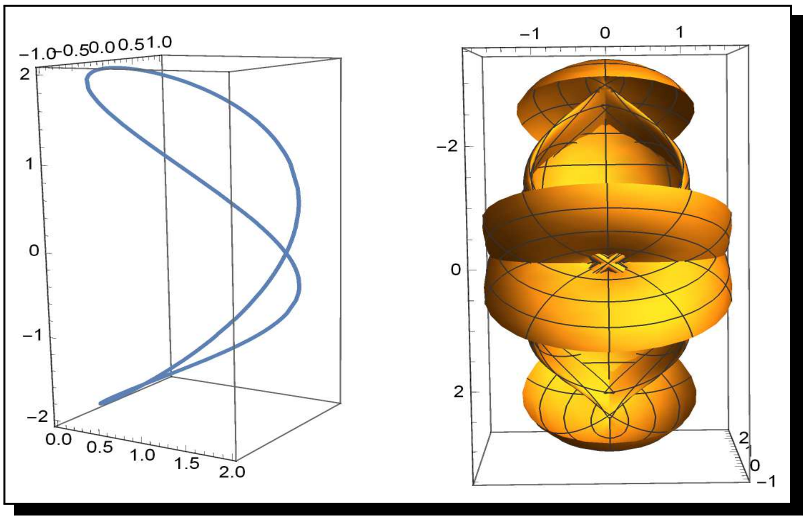

Figure 7. The directrix (34) and embankmentlike surface (36)

In Figure 8 one can see the Gaussian and mean curvatures functions' graphics above and the variations of Gaussian and mean curvatures on embankmentlike surface (36) below.

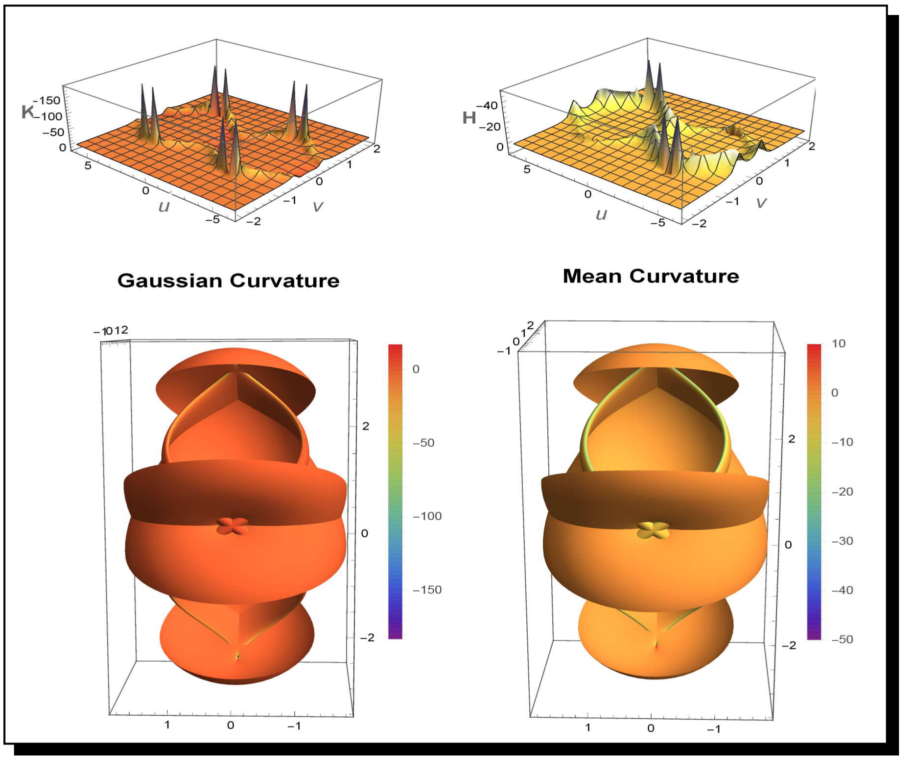

Figure 8. Gaussian and mean curvatures functions' graphics and the variations of Gaussian and mean curvatures on embankmentlike surface (36)

5. Tubembankmentlike Surfaces in E^3

In this section, we define tubembankmentlike surfaces and by obtaining the Gaussian, mean and second Gaussian curvatures of tubembankmentlike surfaces, we give some characterizations about Weingarten tubembankmentlike surfaces. Also, we create some visualizations for tubembankentlike surfaces according to different directrix.

Definition 3. Let Γ_{el} be the embankmentlike surface parametrized by (28). If $\Psi(u,v) = c$ =constant on Γ_{el}, then we'll call that*tubembankmentlike surface*.

So, we can give the following Corollary from the definition of tubembankmentlike surface and (28):

Corollary 2. *Let* $\alpha : I \subseteq \mathbb{R} \to E^3$ *be a regular space curve with non-zero curvature. Then, the parametrization of a tubembankmentlike surface* $\Gamma_{tel}(u,v) = (\Gamma_{tel_1}(u,v), \Gamma_{tel_2}(u,v), \Gamma_{tel_3}(u,v))$ *can be given by*

$$\Gamma_{tel}(u,v) = \alpha(u) \mp c_1\{\cos v.N(u) + \sin v.B(u)\}, \tag{37}$$

where $c_1 = \sqrt{m^2+1}.c$, $m \in \mathbb{R}$, $m > 0$ *with* $|m\dot{\alpha}_3| < \sqrt{\dot{\alpha}_1^2 + \dot{\alpha}_2^2}$.

Now, we'll give a characterization for tubembankmentlike surfaces using quaternionic approach.

Corollary 3. *Let* $\alpha : I \subseteq \mathbb{R} \to E^3$ *be a unit speed directrix curve with non-zero curvature of tubembankmentlike surface* Γ_{tel} *and* $q(u,v) = \cos v + \sin v.T(u)$ *be a unit quaternion in* $S^3 \subset \mathbb{R}^4$*. Then, the tubembankmentlike surface* Γ_{tel} *can be given by*

$$\Gamma_{tel}(u,v) = \alpha(u) \mp c_1.q(u,v) \times N(u).$$

The components of the first and second fundamental forms of tubembankmentlike surface (37) are obtained by

$$g_{11} = (1 \pm c_1\kappa\cos v)^2 + c_1^2\tau^2, \quad g_{12} = c_1^2\tau, \quad g_{22} = c_1^2$$

and

$$L_{11} = \pm(1 \pm c_1\kappa\cos v)\kappa\cos v + c_1\tau^2, \ L_{12} = c_1\tau, \ L_{22} = c_1,$$

respectively. So, from (4), (5) and (6), we obtain the Gaussian, mean and second Gaussian curvatures of tubembankmentlike surface Γ_{tel} as

$$K = \frac{\pm\kappa\cos v}{(1 \pm c_1\kappa\cos v).c_1}, \tag{38}$$

$$H = \frac{1 \pm 2c_1\kappa\cos v}{2(1 \pm c_1\kappa\cos v).c_1} \tag{39}$$

and

$$K_{II} = \frac{1 + \cos^2 v(1 \pm 6c_1\kappa\cos v + 4c_1^2\kappa^2\cos^2 v)}{4c_1(1 \pm c_1\kappa\cos v)^2.\cos^2 v}, \tag{40}$$

respectively.

Hence we have,

Theorem 3. *If* $\alpha : I \subseteq \mathbb{R} \to E^3$ *is a regular space curve with non-zero curvature, then the tubembankmentlike surface* Γ_{tel}*, with directrix* α *and non-degenerate second fundamental form, is a* (H,K)*-Weingarten surface.*

Proof. From (38) and (39), we have

$$K_u = \frac{\pm\kappa'\cos v}{(1 \pm c_1\kappa\cos v)^2.c_1}, \quad K_v = \frac{\mp\kappa\sin v}{(1 \pm c_1\kappa\cos v)^2.c_1} \tag{41}$$

and

$$H_u = \frac{\pm\kappa'\cos v}{2(1 \pm c_1\kappa\cos v)^2}, \quad H_v = \frac{\mp\kappa\sin v}{2(1 \pm c_1\kappa\cos v)^2}. \tag{42}$$

Thus, from (41) and (42), we get $\Phi(H,K) = H_uK_v - H_vK_u = 0$. So, the proof completes. □

We know that, a surface is said that linear Weingarten surface, if it satisfies $aH + bK = c$, where a,b,c are constants [17]. Thus;

Corollary 4. *If* $\alpha : I \subseteq \mathbb{R} \to E^3$ *is a regular space curve with non-zero curvature, then the tubembankmentlike surface* Γ_{tel}*, with directrix* α *and non-degenerate second fundamental form, is a linear Weingarten surface.*

Proof. From (38) and (39), we have $2H - c_1K = \frac{1}{c_1}$, which completes the proof. □

Theorem 4. *Let $\alpha : I \subseteq \mathbb{R} \to E^3$ be a regular space curve with non-zero curvature. If the tubembankmentlike surface Γ_{tel}, with directrix α and non-degenerate second fundamental form, is a (H,K_{II}) or (K,K_{II})-Weingarten surface, then the curvature of α is a non-zero constant.*

Proof. Taking derivatives of (40) with respect to u and v, we have

$$K_{II_u} = \frac{\pm 2\kappa' \cos^2 v + c_1\kappa\kappa' \cos^3 v \mp \kappa'}{2(1 \pm c_1\kappa\cos v)^3.\cos v} \quad \text{and} \tag{43}$$

$$K_{II_v} = \frac{\mp 2c_1\kappa \sin v \cos^3 v \pm 2c_1\kappa \sin v \cos v - c_1^2\kappa^2 \sin v \cos^4 v - \sin v}{2c_1(1 \pm c_1\kappa\cos v)^3.\cos^3 v} \tag{44}$$

respectively. If Γ_{tel} is a (H,K_{II}) or (K,K_{II})-Weingarten surface, then from (41)-(44) and the definitions of (H,K_{II}) or (K,K_{II})-Weingarten surface, we get $\kappa' \sin v = 0$ and this completes the proof. □

5.1 Visualization for Tubembankmentlike Surfaces in E^3

Finally, let us give some visualizations for tubembankmentlike surfaces:

Example 5. Let us take directrix as helix (32) and let us suppose that $\Omega(u,v) = u + d_1$, d_1 =constant and $m = 1$. Then, we obtain the tubembankmentlike surface (37) under these assumptions as

$$\Gamma_{tel}(u,v) = \Big(2\cos(u) - 4\sqrt{2}\cos(u)\cos(v) + 4.\sqrt{\tfrac{2}{5}}\sin(u)\sin(v),$$
$$2\sin(u) - 4\sqrt{2}\sin(u)\cos(v) - 4.\sqrt{\tfrac{2}{5}}\cos(u)\sin(v), u + 8.\sqrt{\tfrac{2}{5}}\sin(v)\Big), \tag{45}$$

where we take $d_1 = 4$. In Figure 9, one can see the directrix (32) and tubembankmentlike surface (45).

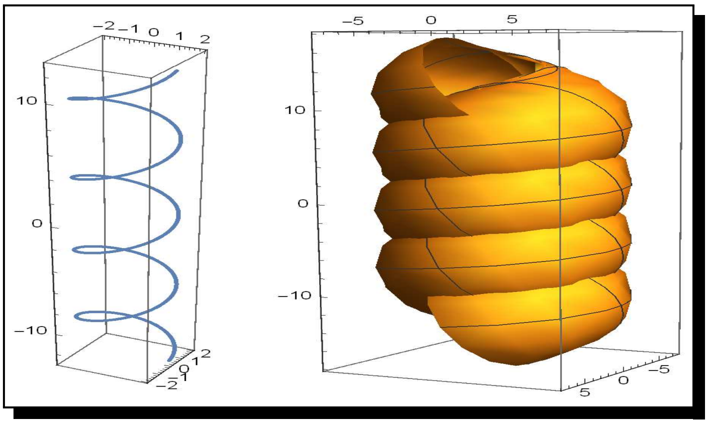

Figure 9. The directrix (32) and tubembankmentlike surface (45)

From (38), (39) and (40), we obtain the Gaussian, mean and second Gaussian curvatures of tubembankmentlike surface (45) as

$$K = \frac{-\cos v}{10\sqrt{2} - 32\cos v},$$

$$H = \frac{5 - 16\sqrt{2}\cos v}{40\sqrt{2} - 128\cos v}$$

and

$$K_{II} = \frac{1 + \cos^2 v\left(1 - \frac{48\sqrt{2}}{5}\cos v + \frac{512}{25}\cos^2 v\right)}{16\sqrt{2}\left(1 - \frac{8\sqrt{2}}{5}\cos v\right)^2 \cos^2 v},$$

respectively. In Figure 10, one can see the Gaussian, mean and second Gaussian curvatures functions' graphics above and the variations of Gaussian, mean and second Gaussian curvatures on tubembankmentlike surface (45) below.

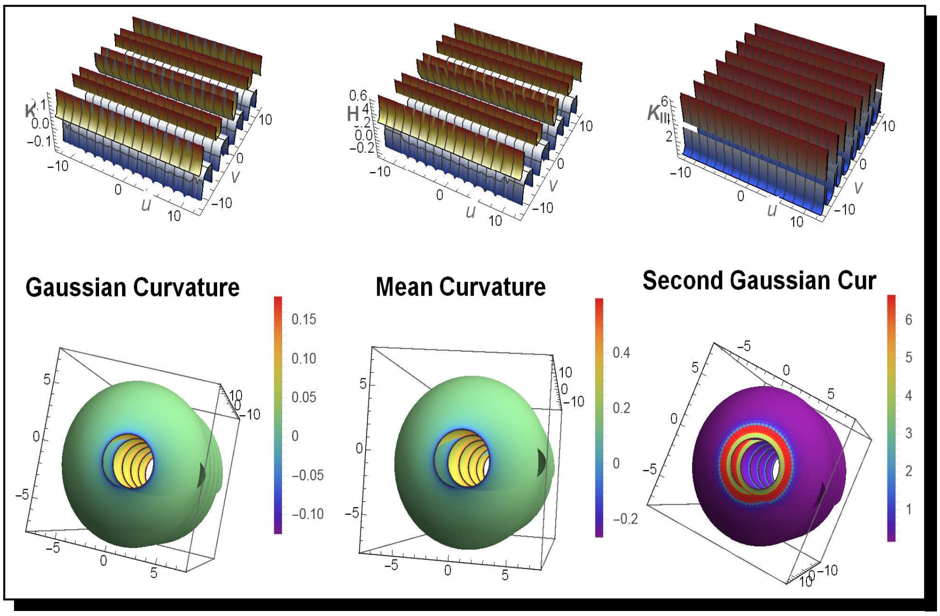

Figure 10. Gaussian, mean and second Gaussian curvatures functions' graphics and the variations of Gaussian, mean and second Gaussian curvatures on tubembankmentlike surface (45)

Example 6. Let us take directrix as Viviani curve (34) and let us suppose that $\Omega(u,v) = 2\sin(\frac{u}{2}) + d_2$, d_2 =constant and $m = 0,5$. Then, we obtain the tubembankmentlike surfaces (37) under these assumptions as

$$\begin{aligned}\Gamma_{tel}(u,v) = \Bigg(&1 + \cos(u) + \frac{3}{4\sqrt{5}}\cos(v)N_1(u) + \frac{3}{4\sqrt{5}}\sin(v)B_1(u),\\ &\sin u + \frac{3}{4\sqrt{5}}\cos(v)N_2(u) + \frac{3}{4\sqrt{5}}\sin(v)B_2(u),\\ &2\sin\left(\frac{u}{2}\right) + \frac{3}{4\sqrt{5}}\cos(v)N_3(u) + \frac{3}{4\sqrt{5}}\sin(v)B_3(u)\Bigg),\end{aligned} \tag{46}$$

where we take $d_2 = 0,3$. In Figure 11, one can see the directrix (34) and tubembankmentlike surface (46).

From (38), (39) and (40), we obtain the Gaussian, mean and second Gaussian curvatures of tubembankmentlike surface (46) as

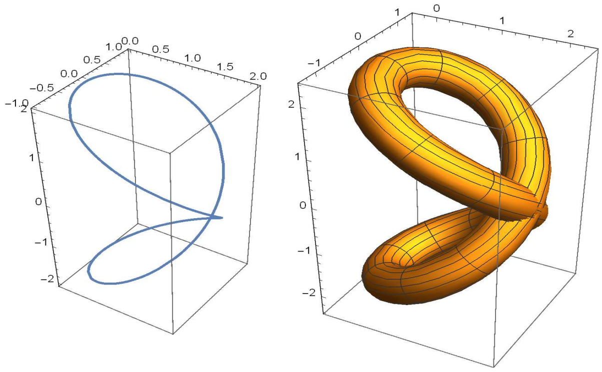

Figure 11. The directrix (34) and tubembankmentlike surface (46)

$$K=\frac{-80\sqrt{5}.\sqrt{13+3\cos u}.\cos v}{60.(3+\cos u)^{\frac{3}{2}}-9\sqrt{5}.\sqrt{13+3\cos u}.\cos v},\quad H=\frac{40\sqrt{5}.(3+\cos u)^{\frac{3}{2}}-60.\sqrt{13+3\cos u}.\cos v}{60.(3+\cos u)^{\frac{3}{2}}-9\sqrt{5}.\sqrt{13+3\cos u}.\cos v}$$

and

$$K_{II}=\frac{1+\cos^2 v\left(1-\frac{9}{2\sqrt{5}}\frac{\sqrt{13+3\cos u}}{(3+\cos u)^{\frac{3}{2}}}\cos v+\frac{9}{20}\frac{13+3\cos u}{(3+\cos u)^3}\cos^2 v\right)}{\frac{3}{\sqrt{5}}\left(1-\frac{3}{4\sqrt{5}}\frac{\sqrt{13+3\cos u}}{(3+\cos u)^{\frac{3}{2}}}\cos v\right)^2\cos^2 v},$$

respectively. In Figure 12, one can see the Gaussian, mean and second Gaussian curvatures functions' graphics above and the variations of Gaussian, mean and second Gaussian curvatures on tubembankmentlike surface (46) below.

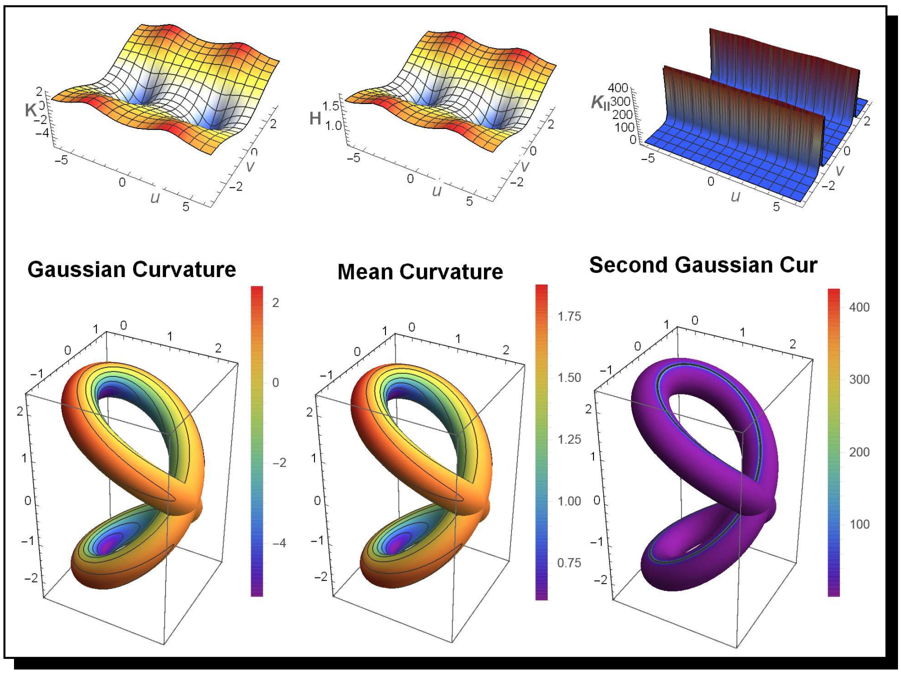

Figure 12. Gaussian, mean and second Gaussian curvatures functions' graphics and the variations of Gaussian, mean and second Gaussian curvatures on tubembankmentlike surface (46)

Example 7. Let us take directrix as

$$\alpha(u) = (e^u, u+3, 2u), \tag{47}$$

which is an arbitrary curve in E^3 and let us suppose that $\Omega(u,v) = 2u + d_3$, d_3 =constant and $m = 0,1$. Here, one can easily calculate Frenet-Serret apparatus of the curve (47) as

$$\left.\begin{aligned} T(u) &= \frac{1}{\sqrt{e^{2u}+5}}(e^u, 1, 2) \\ N(u) &= \frac{1}{\sqrt{e^{2u}+5}}\left(\sqrt{5}, -\frac{e^u}{\sqrt{5}}, -\frac{2e^u}{\sqrt{5}}\right) \\ B(u) &= \frac{1}{\sqrt{5}}(0, 2, -1) \\ \kappa &= \frac{\sqrt{5}e^u}{(5+e^{2u})^{\frac{3}{2}}}, \quad \tau = 0. \end{aligned}\right\}. \tag{48}$$

Then, we obtain the tubembankmentlike surfaces (37) under these assumptions as

$$\begin{aligned} \Gamma(u,v) = \Bigg(& e^u + \frac{3\sqrt{5,05}}{\sqrt{e^{2u}+5}}\cos(v), u + 3 - \frac{3\sqrt{1,01}}{\sqrt{5}\sqrt{e^{2u}+5}}e^u\cos(v) + \frac{6\sqrt{1,01}}{\sqrt{5}}\sin(v), \\ & 2u - \frac{6\sqrt{1,01}}{\sqrt{5}\sqrt{e^{2u}+5}}e^u\cos(v) - \frac{3\sqrt{1,01}}{\sqrt{5}}\sin(v)\Bigg), \end{aligned} \tag{49}$$

where we take $d_3 = 3$. In Figure 13, one can see the directrix (47) and tubembankmentlike surface (49).

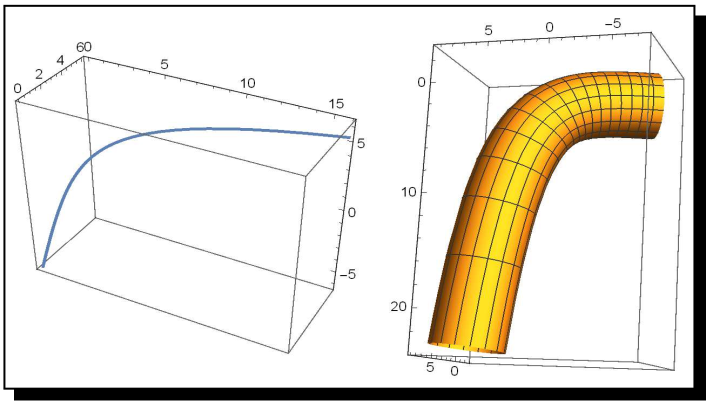

Figure 13. The directrix (47) and tubembankmentlike surface (49)

From (38), (39) and (40), we obtain the Gaussian, mean and second Gaussian curvatures of tubembankmentlike surface (49) as

$$K = -\frac{\sqrt{5}e^u\cos v}{3\sqrt{1,01}(5+e^{2u})^{\frac{3}{2}} - (9,09)\sqrt{5}e^u\cos v}, \quad H = \frac{(5+e^{2u})^{\frac{3}{2}} - 6\sqrt{5,05}e^u\cos v}{6\sqrt{1,01}((5+e^{2u})^{\frac{3}{2}} - 3\sqrt{5,05}e^u\cos v)}$$

and

$$K_{II} = \frac{1 + \cos^2 v\left(1 - 18\sqrt{5,05}\frac{e^u}{(e^{2u}+5)^{\frac{3}{2}}}\cos v + (181,8)\frac{e^{2u}}{(e^{2u}+5)^3}\cos^2 v\right)}{12\sqrt{1,01}\left(1 - 3\sqrt{5,05}\frac{e^u}{(e^{2u}+5)^{\frac{3}{2}}}\cos v\right)^2\cos^2 v},$$

respectively. In Figure 14, one can see the Gaussian, mean and second Gaussian curvatures functions' graphics above and the variations of Gaussian, mean and second Gaussian curvatures on tubembankmentlike surface (49) below.

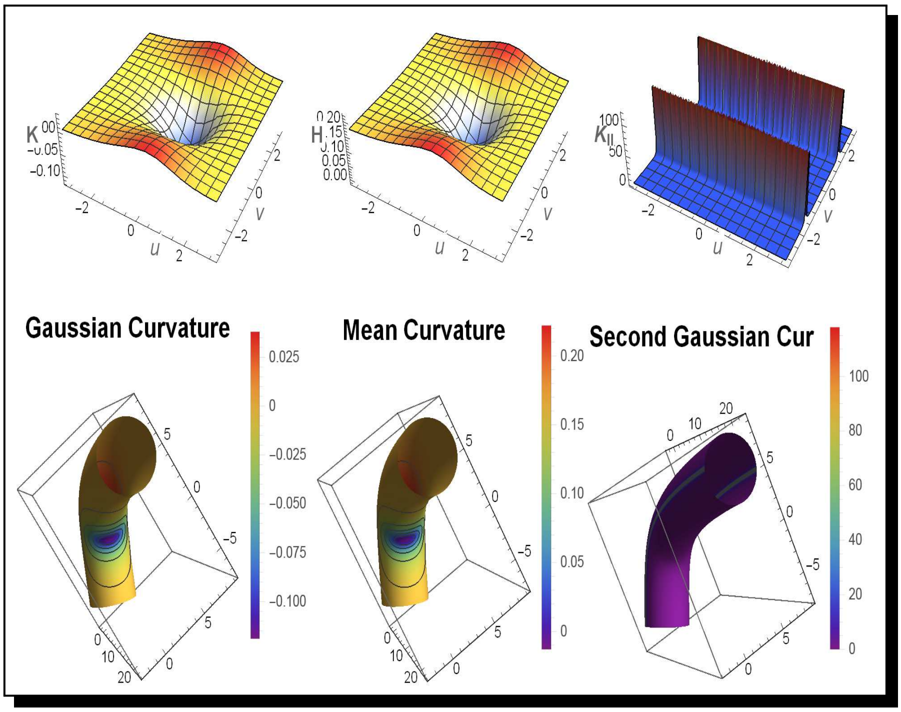

Figure 14. Gaussian, mean and second Gaussian curvatures functions' graphics and the variations of Gaussian, mean and second Gaussian curvatures on tubembankmentlike surface (49)

Remark 1. In the calculations of above examples, we've taken the "$\mp$" which are in (22), (28) and (37) as "$+$". Similarly, one can make these calculations by taking "$\mp$" as "$-$".

6. Conclusion and Future Work

We think that, embankment, embankmentlike and tubembankmentlike surfaces which are stated in the present paper will bring a new viewpoint to geometers and we hope, these surfaces will be useful for future structural engineers. Furthermore, in the future works we can investigate these surfaces in Minkowski 3-space, Galilean 3-space, pseudo Galilean 3-space and etc.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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