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Research Article

Split Jacobsthal and Jacobsthal-Lucas Quaternions

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Abstract. In this paper, we introduce split Jacobsthal and split Jacobsthal-Lucas quaternions. We obtain generating functions and Binet's formulas for these quaternions. We also investigate some properties of them.

Keywords. Jacobsthal numbers; Jacobsthal-Lucas numbers; Split quaternions; Recurrence relations

MSC. 11B39; 11B37; 11R52

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1. Introduction

The quaternion numbers have been introduced by William Rowan Hamilton in the mid nineteenth century. Quaternions are four-dimensional hyper-complex numbers.

A quaternion is defined by

$$p = p_0 + p_1 e_1 + p_2 e_2 + p_3 e_3,$$

where p_0, p_1, p_2 and p_3 are real numbers, and the units e_1, e_2, e_3 satisfy the rules

$$\begin{aligned} &e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1, \\ &e_1 e_2 = e_3 = -e_2 e_1, \quad e_2 e_3 = e_1 = -e_3 e_2, \quad e_3 e_1 = e_2 = -e_1 e_3. \end{aligned} \tag{1}$$

For more details on quaternions, one can see, for example [5, 17].

The split quaternions, in other words coquaternions, have been introduced by James Cockle in 1849. Split quaternions form a four-dimensional non-commutative associative algebra over the real numbers with basis $\{1, e_1, e_2, e_3\}$.

A split quaternion q is of the form

$$q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 = (q_0, q_1, q_2, q_3),$$

where q_0, q_1, q_2 and q_3 are real numbers, and the units e_1, e_2, e_3 satisfy the rules

$$e_1^2 = -1, \quad e_2^2 = e_3^2 = e_1 e_2 e_3 = 1,$$
$$e_1 e_2 = e_3 = -e_2 e_1, \quad e_2 e_3 = -e_1 = -e_3 e_2, \quad e_3 e_1 = e_2 = -e_1 e_3. \tag{2}$$

The conjugate of split quaternion q denoted by $\overline{q}$ is

$$\overline{q} = q_0 + q_1 e_1 - q_2 e_2 - q_3 e_3,$$

and the norm of q is

$$N(q) = q\overline{q} = q_0^2 + q_1^2 - q_2^2 - q_3^2.$$

The Fibonacci sequence is defined recursively by the relation $F_n = F_{n-1} + F_{n-2}$ with initial conditions $F_0 = 0$ and $F_1 = 1$. Similarly, the Lucas sequence is defined as $L_n = L_{n-1} + L_{n-2}$, where $L_0 = 2$ and $L_1 = 1$.

The Jacobsthal sequence is defined by the recurrence relation $J_n = J_{n-1} + 2J_{n-2}$ with initial conditions $J_0 = 0$ and $J_1 = 1$. Also, the Jacobsthal-Lucas sequence is defined recursively by the relation $j_n = j_{n-1} + 2j_{n-2}$, where $j_0 = 2$ and $j_1 = 1$.

The generating functions of the Jacobsthal and Jacobsthal-Lucas sequences are given by

$$G(t) = \frac{t}{1 - t - 2t^2}$$

and

$$g(t) = \frac{2 - t}{1 - t - 2t^2},$$

respectively. Moreover, the Binet's formulas for these sequences are defined as

$$J_n = \frac{2^n - (-1)^n}{3} \tag{3}$$

and

$$j_n = 2^n + (-1)^n, \tag{4}$$

respectively. There have been many studies on the Jacobsthal and Jacobsthal-Lucas sequences (see, for example [3, 7, 9, 16]).

Horadam [6] defined the Fibonacci and Lucas quaternions as

$$Q_n = F_n + F_{n+1} e_1 + F_{n+2} e_2 + F_{n+3} e_3$$

and

$$K_n = L_n + L_{n+1} e_1 + L_{n+2} e_2 + L_{n+3} e_3,$$

respectively, where F_n is the nth Fibonacci number, L_n is the nth Lucas number, and e_1, e_2, e_3 satisfy the rules (1).

Iyer [8] investigated the relations between the Fibonacci and Lucas quaternions. Moreover, Halici [4] obtained some properties of the Fibonacci quaternions. In [11], Ramirez defined k-Fibonacci and k-Lucas quaternions. Furthermore, Tan et al. [13, 14] introduced the bi-periodic Fibonacci and Lucas quaternions.

Akyigit et al. [1] defined the split Fibonacci and Lucas quaternions as

$$Q_n = F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3$$

and

$$T_n = L_n + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3,$$

respectively, where F_n is the nth Fibonacci number, L_n is the nth Lucas number, and e_1, e_2, e_3 satisfy the rules (2).

Polatli et al. [10] studied the split k-Fibonacci and k-Lucas quaternions, and in [15], Tokeser et al. introduced the split Pell and Pell-Lucas quaternions.

The Jacobsthal and Jacobsthal-Lucas quaternions are defined by Szynal-Liana and Włoch [12] as

$$JQ_n = J_n + J_{n+1}e_1 + J_{n+2}e_2 + J_{n+3}e_3$$

and

$$JLQ_n = j_n + j_{n+1}e_1 + j_{n+2}e_2 + j_{n+3}e_3,$$

respectively, where J_n is the nth Jacobsthal number, j_n is the nth Jacobsthal-Lucas number, and e_1, e_2, e_3 satisfy the rules (1).

Aydin and Yuce [2] investigated some properties of the Jacobsthal and Jacobsthal-Lucas quaternions.

The main objective of this paper is to introduce split Jacobsthal quaternions and split Jacobsthal-Lucas quaternions. We also aim to obtain some properties of these quaternions including generating functions, Binet's formulas, determinantal representations, matrix representations, Cassini's identities, Catalan's identities, and d'Ocagne's identities.

2. Split Jacobsthal and Split Jacobsthal-Lucas Quaternions

The nth split Jacobsthal quaternion and nth split Jacobsthal-Lucas quaternion are defined, for $n \geq 0$, by

$$SJQ_n = J_n + J_{n+1}e_1 + J_{n+2}e_2 + J_{n+3}e_3$$

and

$$SJLQ_n = j_n + j_{n+1}e_1 + j_{n+2}e_2 + j_{n+3}e_3,$$

respectively, where J_n is the nth Jacobsthal number, j_n is the nth Jacobsthal-Lucas number, and e_1, e_2, e_3 are split quaternionic units which satisfy the rules (2).

It is easy to see that

$$SJQ_n = SJQ_{n-1} + 2SJQ_{n-2} \tag{5}$$

and

$$SJLQ_n = SJLQ_{n-1} + 2SJLQ_{n-2}. \tag{6}$$

The generating functions for the split Jacobsthal and Jacobsthal-Lucas quaternions are given in the following theorem.

Theorem 1. *The generating functions of the split Jacobsthal and split Jacobsthal-Lucas quaternions are*

$$J(t) = \frac{SJQ_0(1-t) + SJQ_1 t}{1-t-2t^2} \tag{7}$$

and

$$JL(t) = \frac{SJLQ_0(1-t) + SJLQ_1 t}{1-t-2t^2}, \tag{8}$$

respectively.

Proof. Let us write

$$J(t) = \sum_{n=0}^{\infty} SJQ_n t^n = SJQ_0 + SJQ_1 t + SJQ_2 t^2 + SJQ_3 t^3 + \ldots + SJQ_n t^n + \ldots.$$

Then, we have

$$tJ(t) = SJQ_0 t + SJQ_1 t^2 + SJQ_2 t^3 + \ldots + SJQ_{n-1} t^n + \ldots$$

and

$$2t^2 J(t) = 2SJQ_0 t^2 + 2SJQ_1 t^3 + \ldots + 2SJQ_{n-2} t^n + \ldots .$$

Thus, we obtain

$$\begin{aligned}(1-t-2t^2)J(t) &= SJQ_0 + (SJQ_1 - SJQ_0)t + \sum_{n=2}^{\infty}(SJQ_n - SJQ_{n-1} - 2SJQ_{n-2})t^n \\ &= SJQ_0 + (SJQ_1 - SJQ_0)t\end{aligned}$$

which completes the proof of eq. (7).
Eq. (8) can be proved similarly. □

The following theorem gives Binet's formulas for the split Jacobsthal and Jacobsthal-Lucas quaternions.

Theorem 2. *The nth term of the split Jacobsthal quaternion and the nth term of the split Jacobsthal-Lucas quaternion are*

$$SJQ_n = \frac{\alpha^* 2^n - \beta^*(-1)^n}{3} \tag{9}$$

and

$$SJLQ_n = \alpha^* 2^n + \beta^*(-1)^n, \tag{10}$$

respectively, where $\alpha^* = (1,2,4,8)$ *and* $\beta^* = (1,-1,1,-1)$.

Proof. The characteristic equation of the recurrence relations (5) and (6) is $t^2 - t - 2 = 0$, and the roots of this equation are 2 and -1. From the recurrence relation and initial values $SJQ_0 = (0,1,1,3)$, $SJQ_1 = (1,1,3,5)$, Binet's formula for SJQ_n is obtained as

$$SJQ_n = c_1 2^n + c_2(-1)^n = \frac{1}{3}[(1,2,4,8)2^n - (1,-1,1,-1)(-1)^n],$$

where $c_1 = \frac{SJQ_0+SJQ_1}{3} = \frac{\alpha^*}{3}$ and $c_2 = \frac{2SJQ_0-SJQ_1}{3} = \frac{-\beta^*}{3}$.

Thus, we get

$$SJQ_n = \frac{\alpha^* 2^n - \beta^*(-1)^n}{3}.$$

Similarly, from the recurrence relation and initial values $SJLQ_0 = (2,1,5,7)$, $SJLQ_1 = (1,5,7,17)$, Binet's formula for $SJLQ_n$ is obtained as

$$SJLQ_n = (1,2,4,8)2^n + (1,-1,1,-1)(-1)^n = \alpha^* 2^n + \beta^*(-1)^n.$$

□

Theorem 3. *For $n \geq 1$, let $\mathbf{P_n}$ be $n \times n$ tridiagonal matrix defined by*

$$\mathbf{P_n} = \begin{pmatrix} P_{11} & P_{12} & 0 & 0 & \cdots & 0 \\ -2 & 1 & 2 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 2 & \ddots & 0 \\ 0 & 0 & -1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 2 \\ 0 & \ldots\ldots & & 0 & -1 & 1 \end{pmatrix}$$

and for $P_{11} = SJQ_1$ and $P_{12} = SJQ_0$, let $\mathbf{P_0} = SJQ_0$, and for $P_{11} = SJLQ_1$ and $P_{12} = SJLQ_0$, let $\mathbf{P_0} = SJLQ_0$. Then

$$\det \mathbf{P_n} = SJQ_n,$$

where $P_{11} = SJQ_1$ and $P_{12} = SJQ_0$, and

$$\det \mathbf{P_n} = SJLQ_n,$$

where $P_{11} = SJLQ_1$ and $P_{12} = SJLQ_0$.

Proof. We prove the theorem for $P_{11} = SJQ_1$ and $P_{12} = SJQ_0$. The other condition can be done similarly.

We use mathematical induction on n. For $n = 1$ and $n = 2$, we have

$$\det \mathbf{P_1} = P_{11} = SJQ_1 \quad \text{and} \quad \det \mathbf{P_2} = P_{11} + 2P_{12} = SJQ_2.$$

Let us assume that the equality holds for $n-1$ and $n-2$, that is,

$$\det \mathbf{P_{n-1}} = SJQ_{n-1} \quad \text{and} \quad \det \mathbf{P_{n-2}} = SJQ_{n-2}.$$

Finally, for n, we get

$$\det \mathbf{P_n} = \det \mathbf{P_{n-1}} + 2\det \mathbf{P_{n-2}} = SJQ_{n-1} + 2SJQ_{n-2} = SJQ_n.$$

□

Theorem 4. *Let n be positive integer. Then*

$$\begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}^{n-1}\begin{pmatrix}SJQ_2 & SJQ_1\\ SJQ_1 & SJQ_0\end{pmatrix}=\begin{pmatrix}SJQ_{n+1} & SJQ_n\\ SJQ_n & SJQ_{n-1}\end{pmatrix}$$

and

$$\begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}^{n-1}\begin{pmatrix}SJLQ_2 & SJLQ_1\\ SJLQ_1 & SJLQ_0\end{pmatrix}=\begin{pmatrix}SJLQ_{n+1} & SJLQ_n\\ SJLQ_n & SJLQ_{n-1}\end{pmatrix}.$$

This theorem can be proved easily by using mathematical induction on n. Moreover, the consequence of this theorem, which gives the Cassini's identities involving the split Jacobsthal and Jacobsthal-Lucas quaternions, is the following theorem.

Theorem 5. *For positive integer n, we have*

$$SJQ_{n+1}SJQ_{n-1}-SJQ_n^2=(-2)^{n-1}\lambda$$

and

$$SJLQ_{n+1}SJLQ_{n-1}-SJLQ_n^2=(-1)^n2^{n-1}9\lambda,$$

where $\lambda=(1,-5,-3,-9)$.

Proof. By taking determinants of the matrices defined in Theorem 4, the proof can be done easily. □

Now we give the Catalan's identities involving the split Jacobsthal and Jacobsthal-Lucas quaternions in the following theorem.

Theorem 6. *For $r\le n$, let n and r be positive integers. Then*

$$SJQ_{n+r}SJQ_{n-r}-SJQ_n^2=(-2)^{n-r}\frac{1}{3}(\mu_1 2^r-\mu_2(-1)^r)J_r$$

and

$$SJLQ_{n+r}SJLQ_{n-r}-SJLQ_n^2=(-1)^{n-r+1}2^{n-r}(\mu_2+\mu_1 4^r-(\mu_1+\mu_2)(-2)^r),$$

where $\mu_1=(1,-13,1,-13)$ and $\mu_2=(1,11,-11,-1)$.

Proof. By using the Binet's formula (9), we have

$$\begin{aligned}
&SJQ_{n+r}SJQ_{n-r}-SJQ_n^2\\
&=\frac{\alpha^*2^{n+r}-\beta^*(-1)^{n+r}}{3}\,\frac{\alpha^*2^{n-r}-\beta^*(-1)^{n-r}}{3}-\frac{\alpha^*2^{n}-\beta^*(-1)^{n}}{3}\,\frac{\alpha^*2^{n}-\beta^*(-1)^{n}}{3}\\
&=\frac{1}{9}[\alpha^*\beta^*(-2)^n+\beta^*\alpha^*(-2)^n-\alpha^*\beta^*(-1)^{n-r}2^{n+r}-\beta^*\alpha^*(-1)^{n+r}2^{n-r}]\\
&=\frac{1}{9}[\beta^*\alpha^*(-1)^r(-2)^{n-r}(2^r-(-1)^r)-\alpha^*\beta^*(-1)^{n-r}2^n(2^r-(-1)^r)]\\
&=\frac{2^r-(-1)^r}{3}(-2)^{n-r}\frac{1}{3}[\beta^*\alpha^*(-1)^r-\alpha^*\beta^*2^r].
\end{aligned}$$

Since $\alpha^*=(1,2,4,8)$ and $\beta^*=(1,-1,1,-1)$, and also by considering eq. (3), we obtain the desired result.

The other identity can be proved similarly by using the Binet's formula (10). □

Note that if we set $r = 1$ in Theorem 6, the Cassini's identities involving the split Jacobsthal and Jacobsthal-Lucas quaternions, which are given in Theorem 5 can be obtained again.

The following theorem gives the d'Ocagne's identities involving the split Jacobsthal and Jacobsthal-Lucas quaternions.

Theorem 7. *Let m and n be two positive integers. Then*

$$SJQ_m SJQ_{n+1} - SJQ_n SJQ_{m+1} = (-1)^{n+1} 2^n \rho J_{m-n}$$

and

$$SJLQ_m SJLQ_{n+1} - SJLQ_n SJLQ_{m+1} = (-2)^n 3\rho(2^{m-n} - (-1)^{m-n}),$$

where $\rho = (1,3,-7,-5)$.

Proof. By using the Binet's formula (10), we have

$$\begin{aligned} &SJLQ_m SJLQ_{n+1} - SJLQ_n SJLQ_{m+1} \\ &\quad = (\alpha^* 2^m + \beta^*(-1)^m)(\alpha^* 2^{n+1} + \beta^*(-1)^{n+1}) - (\alpha^* 2^n + \beta^*(-1)^n)(\alpha^* 2^{m+1} + \beta^*(-1)^{m+1}) \\ &\quad = \alpha^*\beta^*(-1)^{n+1}2^m + \beta^*\alpha^*(-1)^m 2^{n+1} - \alpha^*\beta^*(-1)^{m+1}2^n - \beta^*\alpha^*(-1)^n 2^{m+1} \\ &\quad = (-2)^n(-\alpha^*\beta^* - 2\beta^*\alpha^*)(2^{m-n} - (-1)^{m-n}). \end{aligned}$$

Since $\alpha^* = (1,2,4,8)$ and $\beta^* = (1,-1,1,-1)$, we obtain

$$SJLQ_m SJLQ_{n+1} - SJLQ_n SJLQ_{m+1} = (-2)^n 3\rho(2^{m-n} - (-1)^{m-n}).$$

In a similar way, the first identity can be proved. □

3. Results

In this section, we derive some identities of the split Jacobsthal quaternions and split Jacobsthal-Lucas quaternions.

Theorem 8. *Let m, n and r be positive integers. Then*

$$2SJQ_{n-1} + SJQ_{n+1} = SJQ_n, \tag{11}$$

$$9SJQ_n^2 - SJLQ_n^2 = (-2)^{n+2}(1,-1,-5,-7), \tag{12}$$

$$SJQ_{m+n} + (-2)^n SJQ_{m-n} = j_n SJQ_m, \tag{13}$$

$$SJLQ_{m+n} + (-2)^n SJLQ_{m-n} = j_n SJLQ_m, \tag{14}$$

$$SJQ_{m+n} = J_{n+1} SJQ_m + 2J_n SJQ_{m-1}, \tag{15}$$

$$SJLQ_{m+n} = \frac{1}{3}(j_{n+1} SJLQ_m + 2j_n SJLQ_{m-1}), \tag{16}$$

$$SJQ_{2n} = J_{n+1} SJQ_n + 2J_n SJQ_{n-1}, \tag{17}$$

$$SJQ_{2n+1} = J_{n+1} SJQ_{n+1} + 2J_n SJQ_n, \tag{18}$$

$$SJQ_{m+n} SJLQ_{m+r} - SJQ_{m+r} SJLQ_{m+n} = (-1)^{m+n} 2^{m+n+1}(1,-1,-5,-7) J_{r-n}. \tag{19}$$

Proof. Throughout the proof, we consider $\alpha^* = (1,2,4,8)$ and $\beta^* = (1,-1,1,-1)$.

(11): By using the Binet's formula (9), we have

$$\begin{aligned} 2SJQ_{n-1} + SJQ_{n+1} &= 2\frac{\alpha^* 2^{n-1} - \beta^*(-1)^{n-1}}{3} + \frac{\alpha^* 2^{n+1} - \beta^*(-1)^{n+1}}{3} \\ &= \frac{1}{3}(3\alpha^* 2^n + 3\beta^*(-1)^n) \\ &= \alpha^* 2^n + \beta^*(-1)^n. \end{aligned}$$

From the Binet's formula (10), the proof of the identity (11) is completed.

(12): From the Binet's formulas (9) and (10), we have

$$\begin{aligned} 9SJQ_n^2 - SJLQ_n^2 &= 9\frac{\alpha^* 2^n - \beta^*(-1)^n}{3}\frac{\alpha^* 2^n - \beta^*(-1)^n}{3} - (\alpha^* 2^n + \beta^*(-1)^n)(\alpha^* 2^n + \beta^*(-1)^n) \\ &= (-2)^{n+1}(\alpha^*\beta^* + \beta^*\alpha^*) \\ &= (-2)^{n+2}(1,-1,-5,-7). \end{aligned}$$

(13): By using the Binet's formula (9), we have

$$\begin{aligned} SJQ_{m+n} + (-2)^n SJQ_{m-n} &= \frac{\alpha^* 2^{m+n} - \beta^*(-1)^{m+n}}{3} + (-2)^n \frac{\alpha^* 2^{m-n} - \beta^*(-1)^{m-n}}{3} \\ &= \frac{1}{3}(2^n + (-1)^n)(\alpha^* 2^m - \beta^*(-1)^m). \end{aligned}$$

From the eqs. (4) and (9), we obtain the desired result.

The proof of the identity (14) can be done similarly by using the Binet's formula (10).

(15): From the definition of the split Jacobsthal quaternion and the identity $J_{m+n} = J_m J_{n+1} + 2J_{m-1}J_n$ (see [9]), we have

$$\begin{aligned} SJQ_{m+n} &= J_{m+n} + J_{m+n+1}e_1 + J_{m+n+2}e_2 + J_{m+n+3}e_3 \\ &= J_{n+1}(J_m + J_{m+1}e_1 + J_{m+2}e_2 + J_{m+3}e_3) + 2J_n(J_{m-1} + J_m e_1 + J_{m+1}e_2 + J_{m+2}e_3) \\ &= J_{n+1}SJQ_m + 2J_n SJQ_{m-1}. \end{aligned}$$

The identity (16) can be proved similarly by using the identity $j_{m+n} = j_m j_{n+1} + 2j_{m-1}j_n$. The identities (17) and (18) can be proved by taking, respectively, $m = n$ and $m = n+1$ into eq. (15).

(19): By using the Binet's formulas (9) and (10), we have

$$\begin{aligned} &SJQ_{m+n}SJLQ_{m+r} - SJQ_{m+r}SJLQ_{m+n} \\ &= \frac{\alpha^* 2^{m+n} - \beta^*(-1)^{m+n}}{3}(\alpha^* 2^{m+r} + \beta^*(-1)^{m+r}) - \frac{\alpha^* 2^{m+r} - \beta^*(-1)^{m+r}}{3}(\alpha^* 2^{m+n} + \beta^*(-1)^{m+n}) \\ &= \frac{1}{3}[\alpha^*\beta^*(-1)^{m+r}2^{m+n} - \beta^*\alpha^*(-1)^{m+n}2^{m+r} - \alpha^*\beta^*(-1)^{m+n}2^{m+r} \\ &\quad - \beta^*\alpha^*(-1)^{m+r}2^{m+n}] \\ &= \frac{2^{r-n} - (-1)^{r-n}}{3}(-1)^{m+n+1}2^{m+n}(\alpha^*\beta^* + \beta^*\alpha^*). \end{aligned}$$

By considering α^*, β^*, and the Binet's formula (3), we get

$$SJQ_{m+n}SJLQ_{m+r} - SJQ_{m+r}SJLQ_{m+n} = (-1)^{m+n}2^{m+n+1}(1,-1,-5,-7)J_{r-n}. \qquad \square$$

4. Conclusion

In this study, the split Jacobsthal and Jacobsthal-Lucas quaternions were introduced. Some results including Binet's formulas, generating functions and determinantal representations for these quaternions were given. Moreover, some well-known identities, such as Catalan's, Cassini's and d'Ocagne's identities, involving the split Jacobsthal and Jacobsthal-Lucas quaternions were obtained.

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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