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Research Article

Balanced Shifts

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Abstract. The notion of balanced generator of the Dyke system has been extended to a general subshift called *balanced shift* and it has been shown that they are essentially mixing. All balanced shifts are half synchronized and full shifts are the only balanced and synchronized subshifts. Also, a formula for the Gurevic entropy of a balanced shift has been given.

Keywords. Synchronized systems; Half synchronized; Dyke systems; Balanced block; Entropy

MSC. 37B10; 54H20; 37B40

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1. Introduction

One of the most studied dynamical systems is a *subshift of finite type* (SFT). An SFT is a system whose set of forbidden blocks is finite [5]. Equivalently, an SFT X is a subshift whose any block of length greater than a certain number M is synchronizing; that is, if m is *any* block with $|m| \geq M$ and if v_1m and mv_2 are both blocks of X, then v_1mv_2 is a block of X. If an irreducible system has at least one synchronizing block, then it is called a *synchronized system* and examples are *sofics*: factors of SFT's. Synchronized systems has attracted much attention [1] and extension of them has been of interest as well; notably *half synchronized systems* which are systems having *half synchronizing* blocks. In fact, if for a left transitive ray (not just a block as above) such as $rm \in X^- = \{\dots x_{-1}x_0 : x = \dots x_{-1}x_0x_1x_2\dots \in X\}$ and mv any block in X, one has again $rmv \in X$, then m is called *half synchronizing* [2]. Clearly, any synchronized system is half synchronized. Dyke subshifts and certain β-shifts are non-synchronized but half synchronized systems [2].

Balanced blocks was first introduced for Dyke systems [6]. We will extend them to some certain subshifts and we will show that for these systems, each block is a half synchronizing block and any subshift with a balanced generator, that is a generator consisting of balanced blocks, is mixing. If subshift X has a balanced generator, then X is a synchronized system if and only if X is a full shift.

2. Background and definitions

This section is devoted to the required basic definitions. The notations has been taken from [5] and [2] for the relevant concepts.

First we present some elementary concept from [5]. Let $\mathscr{A}$ be an alphabet, that is a non-empty finite set of symbols. The full *$\mathscr{A}$-shift* denoted by $\mathscr{A}^{\mathbb{Z}}$, is the collection of all *bi-infinite* sequences of symbols in $\mathscr{A}$. Equip $\mathscr{A}$ with discrete topology and $\mathscr{A}^{\mathbb{Z}}$ with product topology. A *block* over $\mathscr{A}$ is a finite sequence of symbols from $\mathscr{A}$. If x is a point in $\mathscr{A}^{\mathbb{Z}}$ and $i \le j$, then we will denote a block of length $j-i+1$ by $x_{[i,j]} = x_i x_{i+1} \dots x_j$. If $n \ge 1$, then u^n denotes the concatenation of n copies of u, and put $u^0 = \varepsilon$ where ε is the *empty block*. The *shift map* σ on the full shift $\mathscr{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose i-th coordinate is $y_i = x_{i+1}$. By our topology, σ is a homeomorphism. For a full shift $\mathscr{A}^{\mathbb{Z}}$, define $X_{\mathscr{F}}$ to be the subset of sequences in $\mathscr{A}^{\mathbb{Z}}$ not containing any block from a set of blocks $\mathscr{F}$. A *shift space* or a *subshift* is a subset X of a full shift $\mathscr{A}^{\mathbb{Z}}$ such that $X = X_{\mathscr{F}}$ for some collection $\mathscr{F}$ called the *forbidden blocks*. The complement of $\mathscr{F}$ is the set of *admissible blocks* or just blocks in X. A shift space X is called a *shift of finite type* (SFT) if for some finite $\mathscr{F}$, $X = X_{\mathscr{F}}$. A SFT is M-step if it can be described by a collection of forbidden blocks all of which have length $M+1$. A shift of *sofic* is the image of an SFT by a factor code (an onto sliding block code). Every SFT is sofic [5, Theorem 3.1.5], but the converse is not true.

Let $W_n(X)$ denote the set of all admissible n-blocks. The *language* of X is the collection $W(X) = \cup_n W_n(X)$. A shift space X is *irreducible* if for every ordered pair of blocks $u, v \in W(X)$ there is a block $w \in W(X)$ so that $uwv \in W(X)$. It is *mixing* if for every ordered pair $u, v \in W(X)$, there is an $N \in \mathbb{N}$ such that for each $n \ge N$ there is a block $w \in W_n(X)$ such that $uwv \in W(X)$.

Let G be a graph with edge set $\mathscr{E} = \mathscr{E}(G)$ and the set of vertices $\mathscr{V} = \mathscr{V}(G)$. The *edge shift* X_G is the shift space over the alphabet $\mathscr{A} = \mathscr{E}$ defined by

$$X_G = \left\{\xi = (\xi_i)_{i\in\mathbb{Z}} \in \mathscr{E}^{\mathbb{Z}} : t(\xi_i) = i(\xi_{i+1})\right\}.$$

Each edge e initiates at a vertex denoted by $i(e)$ and terminates at a vertex $t(e)$.

A labeled graph is a pair $\mathscr{G} = (G, \mathscr{L})$, where G is a graph with edge set $\mathscr{E}$, and the labeling $\mathscr{L} : \mathscr{E}(G) \to \mathscr{A}$ assigns to each edge e of G a label $\mathscr{L}(e)$ from the finite alphabet $\mathscr{A}$. For a path $\pi = \pi_0 \dots \pi_k$, $\mathscr{L}(\pi) = \mathscr{L}(\pi_0) \dots \mathscr{L}(\pi_k)$ is the label of π. By π_u we mean a path labeled u.

Let $\mathscr{L}_\infty(\xi)$ be the sequence of bi-infinite labels of a bi-infinite path ξ in G and set

$$X_{\mathscr{G}} := \{\mathscr{L}_\infty(\xi) : \xi \in X_G\} = \mathscr{L}_\infty(X_G).$$

We say $\mathscr{G}$ is a *presentation* or *cover* of $X = \overline{X_{\mathscr{G}}}$. In particular, X is sofic if and only if $X = X_{\mathscr{G}}$ for a finite graph G [5, Proposition 3.2.10].

In this part we bring some concepts from [2]. Let X be a subshift and $x \in X$. Then, $x_+ = (x_i)_{i \in Z^+}$ (resp. $x_- = (x_i)_{i \leq 0}$) is called right (resp. left) infinite X-ray. Let $X^+ = \{x_+ : x \in X\}$. For a left infinite X-ray, say x_-, its follower set is $w_+(x_-) = \{x_+ \in X^+ : x_- x_+ \in X\}$ and for $m \in W(X)$ its follower set is $w_+(m) = \{x_+ \in X^+ : m x_+ \in X^+\}$. Analogously, we define predecessor sets $w_-(x_+) = \{x_- \in X^- : x_- x_+ \in X\}$ and $w_-(m) = \{x_- \in X^- : x_- m \in X^-\}$. Consider the collection of all follower sets $w_+(x_-)$ as the set of vertices of a graph. There is an edge from I_1 to I_2 labeled a if and only if there is an X-ray x_- such that $x_- a$ is an X-ray and $I_1 = w_+(x_-), I_2 = w_+(x_- a)$. This labeled graph is called the *Krieger graph* for X. A block $m \in W(X)$ is *synchronizing* if whenever um and mv are in $W(X)$, we have $umv \in W(X)$. An irreducible shift space X is *synchronized system* if it has a synchronizing block. A block $m \in W(X)$ is *half synchronizing* if there is a left transitive point $x \in X$ such that $x_{[-|m|+1,0]} = m$ and $w_+(x_{(-\infty,0]}) = w_+(m)$ which is called the *magic vertex* in the Krieger graph. If X is a half synchronized system and m any half synchronizing block, the irreducible component of the Krieger graph containing the vertex $w_+(m)$ is denoted by X_0^+ and is called the *right Fischer cover* of X. The left Fischer cover is defined similarly. We only use right Fischer cover and we call it just Fischer cover.

Let X be a shift space. The *entropy* of X is defined by

$$h(X) = \lim_{n \to \infty} \frac{1}{n} \log |W_n(X)|.$$

A shift space that is the closure of the set of sequences obtained by freely concatenating the blocks in a list of countable blocks, called the set of generators, is a *coded system* [5].

3. Balanced Generator

Dyke system S_{2n} is a well-known non-synchronized but half synchronized system. Its alphabet $\mathscr{A}$ consists of $2n$ brackets; for instance $\mathscr{A} = \{(,),[,]\}$ for S_4. A point $x \in \mathscr{A}^{\mathbb{Z}}$ is a point in S_{2n} if and only if any subblock of x, obeys the standard bracket rules [2]. In a Dyke system, a block is called balanced if the delimiters appear in a "balanced" form which we will elaborate this notion by extending the balanced blocks to other subshifts.

Definition 3.1. Let X be a shift space. A block $u \in W(X)$ is called *balanced* whenever

(i) For all $x \in X$ and $i \in \mathbb{Z}$, $x_{(-\infty,i)} u x_{[i,+\infty)} \in X$.

(ii) If $x \in X$ and $u = x_{[i,j]}$, then $x_{(-\infty,i)} x_{(j,+\infty)} \in X$.

Let X be a coded system generated by a set G whose all elements are balanced. Then, G is called a *balanced generator* and X is called a *balanced shift*.

The set of all balanced blocks for X, including the empty block ε, will be denoted by $\mathrm{BW}(X)$.

Example 3.2. (i) In the full shift $\mathscr{A}^{\mathbb{Z}}$, $G = \mathscr{A}$ is a balanced generator for $\mathscr{A}^{\mathbb{Z}}$.

(ii) The set of all balanced blocks in the Dyke system S_4 is a balanced generator for S_4. In fact, if X is a subshift of S_4 such that (), [ ] are two balanced blocks for X, then $X = S_4$.

Let $u := u_1u_2 \dots u_l$, $v := v_1v_2 \dots v_k$ be two balanced blocks for the subshift X. We define $u \bowtie v$ to be

$$u \bowtie v = \{v_1 \dots v_i u v_{i+1} \dots v_k : 1 \le i < k\} \cup \{uv, vu\},$$

and we say that u is tied with v. Use the convention that if $|v| = 1$, then $u \bowtie v = \{uv, vu\}$. For instance in S_4 let $u = ()$ and $v = [\,]$. Then, $u \bowtie v = \{()[\,], [()], [\,]()\}$.

Let X be a subshift over $\mathscr{A}$. Suppose $\mathscr{C}_0 \subseteq \mathrm{BW}(X)$ and set

$$\mathscr{C}_1 := \bigcup_{u,v \in \mathscr{C}_0} u \bowtie v, \mathscr{C}_2 := \bigcup_{u,v \in \mathscr{C}_0 \cup \mathscr{C}_1} u \bowtie v, \dots, \mathscr{C}_n := \bigcup_{u,v \in \mathscr{C}_0 \cup \dots \cup \mathscr{C}_{n-1}} u \bowtie v, \dots. \tag{1}$$

It is easy to see that all elements of $\mathscr{C} := \cup_{i \in \mathbb{N} \cup \{0\}} \mathscr{C}_i$ are balanced blocks for the subshift $Z := \overline{\langle \mathscr{C} \rangle_{\mathscr{C}_0}}$ where $\langle \mathscr{C} \rangle_{\mathscr{C}_0}$ presents the set of all concatenations of the elements of $\mathscr{C}$. We call $\mathscr{C}_0$ a *balanced constructor* for Z.

Example 3.3. (i) $\mathscr{C}_0^{S_4} = \{(), [\,]\}$ is a balanced constructor for S_4.

(ii) $\mathscr{C}_0$ may have infinitely many elements. For instance, let Z be a balanced shift over $\{(,), [,], \{, \}, 1\}$ with

$$\mathscr{C}_0 := \{(), [\,]\} \cup \{v_n = a1^n b : a = \{, b = \}, n \in \mathbb{N}\}$$

as its balanced constructor. This Z is not conjugate to any S_{2n}. This is because, it has 7 fixed points while any S_{2n} has $2n$ fixed points.

(iii) Any $\mathscr{C}_n$ in (1) is a constructor for a balanced subsystem of the associated balanced shift.

Definition 3.4. Let X be a shift space. A block $u \in W(X)$ is called *right balanced* (resp. *left balanced*) whenever for each $x \in X$, $ux_+ \in X^+$ (resp. $x_-u \in X^-$) where $x_+ = x_{(0,+\infty)}$ (resp. $x_- = x_{(-\infty,0]}$).

So u is right balanced (resp. left balanced) if and only if $w_+(u) = X^+$ (resp. $w_-(u) = X^-$).

Let X be a coded system generated by G. If for all $u \in G$, u is a right (resp. *left*) balanced block, then G is called the *right balanced* (resp. *left balanced*) generator and X is called a *right balanced shift* (resp. *left balanced shift*).

The balanced system in the part (ii) of Example 3.3, unlike Dyke systems, has characters which are neither right nor left balanced. Here, 1 is not right nor left balanced; for 1] and [1 are not allowed.

Proposition 3.5. *A balanced block is both right and left balanced.*

The converse of the above proposition is not necessarily true. For instance, for the even shift a sofic shift on $\mathscr{A} = \{0, 1\}$ with $\mathscr{F} = \{10^{2n+1}1 : n \in \mathbb{N} \cup \{0\}\}$, the block 0^n is right and left balanced but 0^{2n+1} is not balanced. Also, no other right balanced block exist which means that even shift does not have right balanced generator.

Lemma 3.6. *Let X be a subshift. If $u = a_0 \cdots a_{k-1}$ is a right balanced block for X, then all blocks in*

$$\{a_i \cdots a_{k-1} : 0 \le i < k\} \cup \{u^n : n \ge 1\}$$

are right balanced blocks. In particular, if X has a right balanced block, then for any $N \in \mathbb{N}$ there is a right balanced block v with $|v| = N$.

Proposition 3.7. *Any edge shift with a right balanced block is a full shift.*

Proof. Let u be a right balanced block for an edge shift X_G. Then, there is only one path π_u in G labeled u. But $uu \in W(X_G)$ and so π_u must be a cycle. Set $I := i(\pi_u) = t(\pi_u)$ and let J be any other vertex of G. Let π_v be a path with $i(\pi_v) = J$ and $t(\pi_v) = I$. Since X_G is an edge shift, π_v is unique, that is, other paths between these two vertices must have different labels. Since u is right balanced, $uv \in W(X_G)$ and so $i(\pi_v) = I$ or $\mathscr{V}(G) = \{I\}$ which means X_G represents a full shift. □

This proposition can be used to show that having a right balanced block is not a property preserved by conjugacy. For instance, the golden shift, that is the shift $X_{\{11\}}$, has 0^n as its right balanced block for $n \in \mathbb{N}$. Also, the golden shift as an SFT is conjugate to an edge shift say X_H and $h(X_{\{11\}}) = h(X_H) = \log \frac{1+\sqrt{5}}{2}$. So the golden shift as well as its edge shift are not conjugate to any full shift. Now, Proposition 3.7 shows that X_H (conjugate to $X_{\{11\}}$) does not have any right balanced block.

The point in Proposition 3.7 is that u is both right balanced and synchronizing. In fact, more restrictions arise if u is balanced and synchronizing (see Proposition 3.11). Here, we give an example of a synchronized system which is not a full shift and has a block which is both synchronized and right balanced.

Example 3.8. Add a new symbol $*$ to the set of four delimiters of S_4. Let X be the subshift consisting of all bi-infinite sequences of these five symbols such that any finite subblock which does not contain a $*$ is a block in S_4 [2]. Then, $*$ is both right balanced and synchronizing.

In full shifts any block is synchronizing. A similar terminology will be used for half synchronizing.

Definition 3.9. The shift space X is called *full half* whenever each block in $W(X)$ is a half synchronizing block.

Dyke system and β-shifts are full half [2, Example 0.10]. (See [4] for the definition of β-shifts).

Proposition 3.10. *Every shift space with a right balanced generator G is a full half.*

Proof. Pick $a \in W(X)$ and let $G^* = \{v_1, v_2, \dots\}$ be the set of all finite concatenation of G. Set $x_- := \cdots v_l v_{l-1} \dots v_2 v_1 a$. Note that x_- is left transitive and consider $ay_+ \in X^+$. Since each v_i is right balanced, so $v_1 a y_+ \in X^+$ and consequently $v_2 v_1 a y_+ \in X^+$. Hence $y_+ \in w_+(x_-)$ and so $w_+(x_-) = w_+(a)$. This means a is a half synchronizing block for X and as a result X is a full half system. □

Note that any edge shift X_G is full half and if X_G is not a full shift, by Proposition 3.7 X_G cannot have a right balanced block. This shows that the converse of the above proposition is not necessarily true.

Proposition 3.11. *Every synchronized system X with a balanced synchronizing block u, is a full shift.*

Proof. Let $a,b \in W(X)$. Then, $au, ub \in W(X)$ and so $aub \in W(X)$. Since u is a balanced block, so $ab \in W(X)$. Thus X is a full shift. □

Corollary 3.12. *Let G be a balanced generator for X. Then, X is a synchronized system if and only if X is the full shift.*

Proof. Let m be a synchronizing block for X. Choose $\{u_1,\dots,u_n\} \subseteq G$, such that $m \subseteq u := u_1 \dots u_n$. This u is a balanced synchronizing block. Thus by Proposition 3.11, X is a full shift. □

By applying Corollary 3.12, one can give examples of half synchronized but not synchronized systems. Clearly, generator of such examples must have infinitely many elements; because, for any system with finite generator is SFT and in fact by Proposition 3.11, must be a full shift.

The conclusion of Corollary 3.12 is not true when one is dealing with right balanced generators.

Example 3.13. Let X be a subshift of S_4 generated by

$$G = \{u \in W(S_4) : u \text{ is a right balanced block for } S_4 \textit{ and } [\not\subseteq u\}.$$

Then,] is a synchronizing block for X and G is a right balanced generator for X. This shows that the hypothesis of Corollary 3.12 cannot be weakened to right balanced generator.

Proposition 3.14. *Any subshift with a balanced generator is mixing.*

Proof. Let X be a subshift with a balanced generator G and let $u, v \in W(X)$. Then, a finite concatenation of elements of G such as u' and v' exist where $u \subseteq u'$ and $v \subseteq v'$. Write u' and v' as u_1uu_2 and v_1vv_2, respectively. Let $|u_2v_1| = N$. Now, let $n \geq N$ and $w \in W(X)$ such that $|w| = n - N$. Since $uu_2wv_1v \in W(X)$ and $|u_2wv_1| = n$, so X is mixing. □

3.1 Components of the Krieger graphs of balanced shifts

Whenever the Fischer cover exists, it is useful for visualizing some dynamics of a subshift and always exists for a half synchronized system [2]. For depicting the Fischer cover of a balanced shift, it is very helpful to start from $w_+(u^\infty)$ where u is a right balanced block. Then, for any right balanced block v, $w_+(u^\infty) = w_+(u^\infty v) = X^+$ and in particular, $w_+(u^\infty)$ will be a magic vertex and so a vertex in the Fischer cover. Below we will give an example of a balanced shift, starting at some $w_+(x_-)$ and not of the above form which ends up to a cover of the system which is not Fischer.

Due to the popularity of Dyke systems, we have shown the Fischer cover of S_4 in Figure 1 and such covers for other S_{2n}'s has the same pattern. For S_4, we start at $w_+(u^\infty)$ where $u =)$ is a right balanced for S_4.

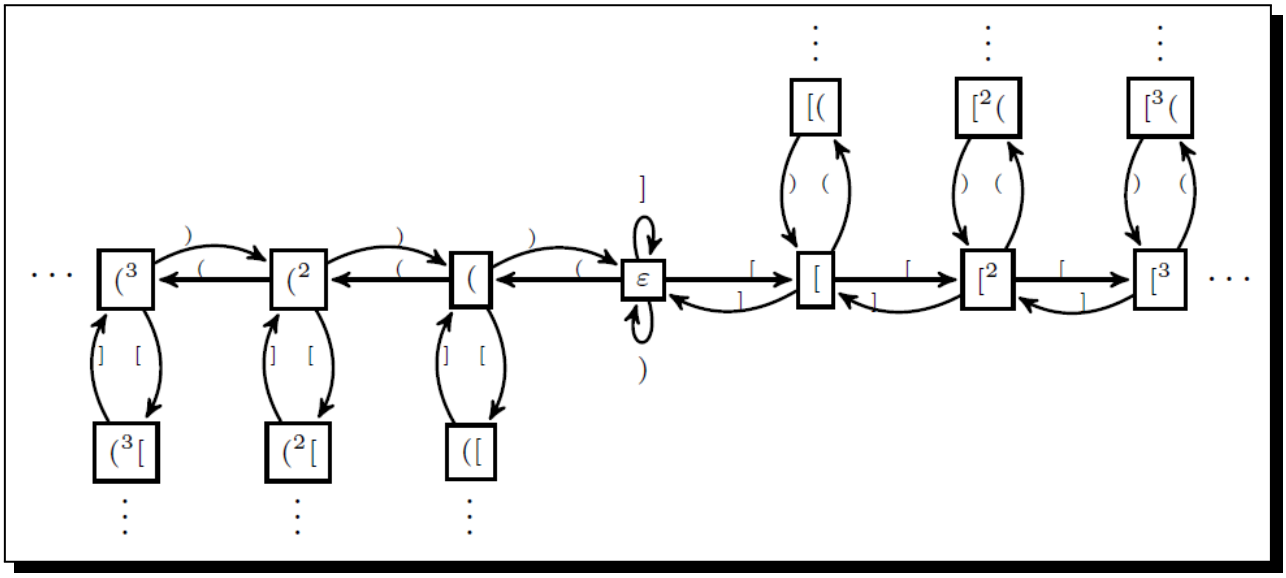

Figure 1. The Fischer cover $(S_4)_0^+$. If $J = \boxed{v} \neq \boxed{\varepsilon} = w_+()^\infty)$, then v is left balanced and J is the initial and terminal of 3 cycles $\{\pi_{[]}, \pi_{()}, \pi_{][}\}$ or $\{\pi_{[]}, \pi_{()}, \pi_{)(}\}$.

Let X be a subshift. The subgraph H of the Krieger graph X is called an *irreducible component* of the Krieger graph X, whenever H is an irreducible and if H' is any irreducible subgraph of the Krieger graph X such that H is a subgraph of H', then $H = H'$ [2]. An irreducible component of the Krieger is called a *component cover* if it is a cover for the system.

The next example shows that a half synchronized system may have infinitely many component covers.

Example 3.15. Let S_4 be the Dyke system and H_0 the irreducible component cover in the Krieger graph of S_4 containing the vertex $I_0 := w_+([^\infty)$ (Figure 2). This is in fact a cover, for it has all the balanced blocks as labels of some paths. Also, note that $u = [$ is not right balanced and so one cannot guarantee that this cover is Fischer. Indeed it is not, for it is easy to see that there is no path in the Krieger graph of S_4 from I_0 to $I := w_+()^\infty) \in \mathscr{V}((S_4)_0^+)$ and so $H_0 \neq (S_4)_0^+$.

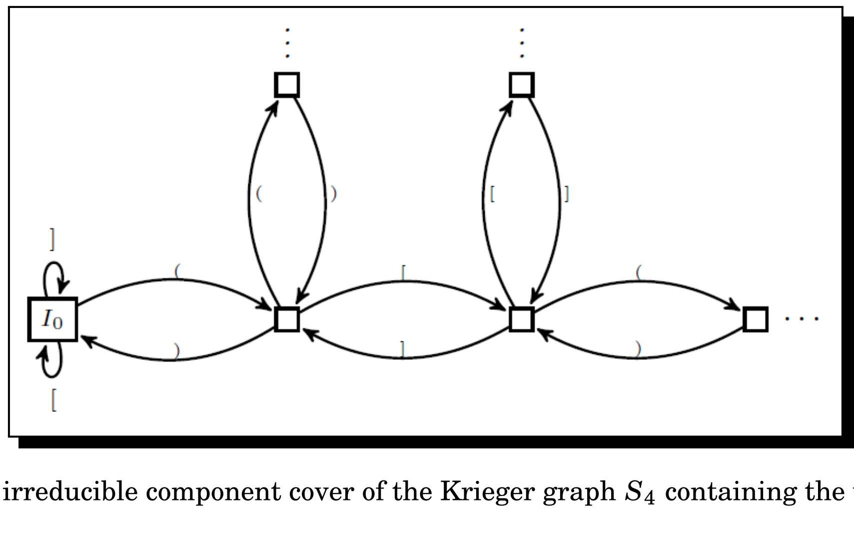

Figure 2. An irreducible component cover of the Krieger graph S_4 containing the vertex $w_+([^\infty)$.

One can do the same routine and find infinitely many such irreducible component covers. In fact, let H_i be the irreducible component cover in the Krieger graph S_4 containing the vertex $I_i := w_+\left(\cdots([^i([^i\right)$. Then, it is not hard to see that if $i,j \geq 1$ and $i \neq j$, then $H_i \neq H_j$.

Next proposition shows that for synchronized systems there is a unique irreducible component cover in the Krieger graph.

Proposition 3.16. *Let X be a synchronized system and let H be a cover for X which is an irreducible component in the Krieger graph of X. Then, $H = X_0^+$.*

Proof. Let m be a synchronizing block of X. Since H is a cover for X and $m \in W(X)$, so there is a finite path π_m in H labeled m. Thus there is $y \in X$ such that $w_+(y_-) = i(\pi_m) \in \mathscr{V}(H)$ and so $w_+(y_-m) = t(\pi_m) \in \mathscr{V}(H)$. Also, m is a synchronizing block of X, so there is $x \in X$ such that $x_{[-|m|+1,0]} = m$ and $w_+(x_-) = w_+(m)$.

But $w_+(x_-) = w_+(y_-m)$. Hence H is an irreducible component of the Krieger graph X containing $w_+(m)$ and so $H = X_0^+$ [2, p. 146]. □

3.2 Gurevic entropy

Let $H = (\mathscr{V}, \mathscr{E})$ be a connected graph. For each pair of vertices $I, J \in \mathscr{V}$, let $r_n(I,J)$ denote the number of paths of length n starting at I and terminating at J. Then,

$$h(H) = \limsup_{n\to\infty} \frac{1}{n} \log r_n(I,J)$$

is independent of I, J, and it is called the *Gurevic entropy* of H [7].

For any synchronized system X, the *synchronized entropy* h_{syn} of X is defined as

$$h_{\text{syn}}(X) = \limsup_{n} \frac{1}{n} \log(\text{cardinal}\{a \in W_n(X) : mam \in W(X)\}), \tag{2}$$

where $m \in W(X)$ is an arbitrary synchronizing block [8].

Let m be a half synchronizing block for X. Fix m and x provided by the definition of half synchronizing. Notice that x_- terminates at m and set

$$h(m,X) := \limsup_{n\to\infty} \frac{1}{n} \log|\{a \in W_n(X) : w_+(x_-am) = w_+(m)\}|. \tag{3}$$

Proposition 3.17. *Let X be a right balanced shift and fix m and x as in* (3). *Then,*

$$h(m,X) = \limsup_{n\to\infty} \frac{1}{n} \log|\{v \in W_n(X) : v \textit{ is a right balanced block}\}|. \tag{4}$$

Proof. Let G be a right balanced generator for X and choose $\{u_1, \ldots, u_n\} \subseteq G$ such that $m \subseteq u_1 \cdots u_n$. So there are $a, b \in W(X)$ such that $amb = u_1 \cdots u_n$. Then, by Lemma 3.6, mb is a right balanced block. Since m is half synchronizing, mb is half synchronizing as well. Set

$$t := \limsup_{n\to\infty} \frac{1}{n} \log|\{v \in W_n(X) : v \text{ is a right balanced}\}|$$

and let

$$v_0 \in \{a \in W_n(X) : w_+(x_-am) = w_+(m)\}.$$

This implies $mv_0m = \overline{x}_{[-|mv_0m|+1,0]}$ and $w_+(\overline{x}_-) = w_+(m)$ where $\overline{x}_- = x_-v_0m$. By definition of a right balanced block, for any $y_+ \in X^+$, $mby_+ \in X^+$. Hence $\overline{x}_-by_+ \in X$ and so $v_0mby_+ \in X^+$. This means that $v_0mb \in W_{n+|m|+|b|}(X)$ is right balanced block. Thus

$$v_0mb \in \{v \in W_{n+|m|+|b|}(X) : v \text{ is a right balanced block}\}$$

and so

$$h(m,X) \le \limsup \frac{1}{n} \log \left|\{v \in W_{n+|m|+|b|}(X) : v \text{ is a right balanced block}\}\right| = t.$$

Conversely, let $v' \in W_n(X)$ be right balanced. Similar to the proof of Proposition 3.10, $z_- = \cdots v_2v_1mbv'm$ is left transitive and $w_+(z_-) = w_+(m)$. Also, $w_+(x_-bv'm) = w_+(m)$. Hence

$$bv' \in \{v \in W_{n+|b|}(X) :\ w_+(x_-am) = w_+(m)\}.$$

Thus $t \le h(m,X)$ and we are done. □

The right statement in (4) is independent of m and so is the left and comparing (2), it is plausible to denote it by $h_{\text{hsyn}}(X)$. Recall that for the synchronized systems, we have $h(X_0^+) = h_{\text{syn}}(X)$ [3]. Similarly, we have:

Corollary 3.18. *Let X be a right balanced shift. Then, $h(X_0^+) = h_{hsyn}(X)$.*

Proof. Let u be a right balanced block and note that then $w_+(u^\infty) \in \mathscr{V}(X_0^+)$. Let $(RB)_n(X)$ denote the set of right balanced blocks of length n. Set

$$\mathscr{C}_n := \{\mathscr{L}(C) : C \text{ is a cycle in } X_0^+ \text{ starting at } w_+(u^\infty), |C| = n\}.$$

By Proposition 3.17, to show that $h(X_0^+) = h_{\text{hsyn}}(X)$, it is enough to prove that $(RB)_n(X) = \mathscr{C}_n$.

But $v \in W_n(X)$ is a right balanced if and only if $w_+(u^\infty) = w_+(u^\infty v)$ and this equality is satisfied if and only if there is a cycle in X_0^+ starting at $w_+(u^\infty)$ labeled v and $|C| = n$. □

A rough estimate for h_{hsyn} is:

Example 3.19. $h_{\text{hsyn}}(S_4) \ge \frac{3\log 2}{2}$.

Proof. Let RW_{2n} and BW_{2n} be the set of right balanced and balanced blocks of length $2n$, respectively. Then, $|RW_{2n}| > |BW_{2n}|$ and by [6, Lemma 3.6],

$$|BW_{2n}| = \frac{\binom{2n}{n}2^n}{n+1}.$$

Thus

$$h_{\text{hsyn}}(S_4) \ge \limsup \frac{1}{2n} \log \frac{\binom{2n}{n}2^n}{n+1} = \frac{\log 8}{2}.$$ □

4. Conclusion

The well known Dyke system is just a typical example of a subshift which is half synchronized but not synchronized. Here, by some appropriate conventions, it was shown that a Dyke system is actually a member of a large family of subshifts sharing some interesting common properties; in particular, giving a big class of half synchronized but not synchronized systems. All results achieved are due to the fact that how freely admissible words sit inside the points of subshift.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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