Communications in Mathematics and Applications
Vol. 9, No. 4, pp. 661–669, 2018
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v9i4.879

Research Article

A Note on the Generalized Solutions of the Third-order Cauchy-Euler Equations

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Abstract. In this paper, we propose the generalized solutions of the third order Cauchy-Euler equations

$$at^3y'''(t)+bt^2y''(t)+cty'(t)+dy(t)=0,$$

where a,b,c and d are real constants with $a\neq 0$ and $t\in\mathbb{R}$ using Laplace transform technique. We find that the types of solutions depend on the conditions of the values of a,b,c and d. Precisely, we obtain a distributional solution if $(k^3+3k^2+2k)a-(k^2+k)b+kc-d=0$, for all $k\in\mathbb{N}$ and a weak solution if $(k^3-3k^2+2k)a+(k^2-k)b+kc+d=0$, for all $k\in\mathbb{N}\cup\{0\}$. Our work improves the result of A. Kananthai [Distribution solutions of the third order Euler equation, *Southeast Asian Bull. Math.* **23** (1999), 627–631].

Keywords. Generalized solutions; Distributional solutions; Weak solutions; Dirac delta function; Cauchy-Euler equation; Laplace transform

MSC. 34A37; 44A10; 46F10; 46F12

Received: April 3, 2018 **Accepted:** July 25, 2018

1. Introduction

A linear ordinary differential operator L order n defined by

$$Ly=\left(a_n(t)\frac{d^n}{dt^n}+a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}}+\cdots+a_1(t)\frac{d}{dt}+a_0(t)\right)y=\sum_{m=0}^{n}a_m(t)\frac{d^m(y)}{dt^m}, \tag{1.1}$$

where the coefficients $a_m(t)$ are infinitely differentiable functions and y is a distribution. Consider the solution of the ordinary differential equation

$$Ly = \sum_{m=0}^{n} a_m(t)\frac{d^m(y)}{dt^m} = \tau, \tag{1.2}$$

where τ is an arbitrary known distribution. The fundamental solution is the solution for $\tau = \delta(t)$, where $\delta(t)$ is the Dirac delta function. A distribution y is a solution of (1.2) if for every test function φ, we have

$$\langle Ly, \varphi\rangle = \langle \tau, \varphi\rangle. \tag{1.3}$$

On searching for a solution y of (1.2), we may encounter the following situations (see Kanwal [10]):

(i) The solution y is a sufficiently smooth functions, so that the operator in (1.2) can be performed in the usual sense and the resulting equation is an identity. Then y is the *classical solution*.

(ii) The solution y is not sufficiently smooth, so that the operator in (1.2) cannot be performed in the usual sense, but it satisfies (1.3) in the sense of distribution. Then y is a *weak solution*.

(iii) The solution y is a singular distribution and satisfies (1.3). Then y is a *distributional solution*.

All of these solutions are called *generalized solutions*.

The Cauchy-Euler differential equation is one of the first, and simplest, forms of a higher order non-constant coefficient ordinary differential equation that is encountered in an undergraduate differential equations course. Let us consider Cauchy-Euler differential equation has the form

$$a_n t^n y^{(n)}(t) + a_{n-1}t^{n-1}y^{(n-1)}(t) + \cdots + a_0 y(t) = 0, \tag{1.4}$$

where $a_0, a_1, \dots, a_n$ are real constants and $a_n \neq 0$. For finding a solution y of differential equation (1.4), the classical solution is typically determined by either using the method of variation of parameters or transforming the equation to a constant-coefficient equation and applying the method of undetermined coefficients, see [1–3, 6, 7] for more details.

In 2016, H. Kim [11] checked the method to find a basis of (1.4) by transforms. The most common form is the second order Cauchy-Euler equation of the form

$$t^2 y''(t) + aty'(t) + by(t) = 0. \tag{1.5}$$

This equation has been used in several areas of physics and engineering applications. It appears in solving Laplace equation in a polar coordinates, describing time-harmonic vibrations of a thin elastic rod, boundary value problem in spherical coordinates and so on. Kim [12] studied the solution of (1.5) expressed by the differential operator using Laplace transform. Moreover, Ghil and Kim [5] studied the classical solutions of Cauchy-Euler equation using Laplace transform. They verified the solutions of (1.5) and the third order Cauchy-Euler equation of the form

$$t^3 y'''(t) + at^2 y''(t) + bty'(t) + cy(t) = 0 \tag{1.6}$$

by using Laplace transform technique. In 2001, Kananthai [9] studied the distributional solutions of ordinary differential equation with polynomial coefficients. Next, Nonlaopon *et al*. [14] studied generalized solutions of a certain n order differential equations with polynomial coefficients

$$ty^{(n)}(t) + my^{(n-1)}(t) + ty(t) = 0, \tag{1.7}$$

where m and n are any integers with $n \geq 2$ and $t \in \mathbb{R}$. They found that the types of solutions of (1.7), which is either the distributional solutions or weak solutions, depend on the values of m and n. Liangprom and Nonlaopon [13] studied on the generalized solutions of a certain fourth order Cauchy-Euler equations of the form

$$t^4y^{(4)}(t) + t^3y'''(t) + t^2y''(t) + ty'(t) + my(t) = 0, \tag{1.8}$$

where m is some integers and $t \in \mathbb{R}$. They found that the types of solutions of (1.8) depend on the value of m. Moreover, Kananthai [8] studied the generalized solutions of a certain third order Euler differential equation of the form

$$t^3y'''(t) + t^2y''(t) + ty'(t) + my(t) = 0, \tag{1.9}$$

where m is some integers and $t \in \mathbb{R}$. He found that the types of solutions of (1.9) depend on the values of m.

Now, we consider the third order Cauchy-Euler equation of the form

$$at^3y'''(t) + bt^2y''(t) + cty'(t) + dy(t) = 0, \tag{1.10}$$

where a, b, c and d are real constants and $t \in \mathbb{R}$ using Laplace transform. The purpose of our work is to find the solutions of (1.10) in the space of distributions and use Laplace transform of distribution to solve the equation. For more details on the applications of the theory of distributions to differential equations (see [4, 15, 17]).

2. Preliminaries

Before reaching the main result, the following definitions and concepts are required.

Definition 2.1 ([8])**.** Let $T \in \mathbb{R}$ and $f(t)$ be a locally integrable function which satisfies the following conditions:

(i) $f(t) = 0$ for all $t < T$;

(ii) there exists a real number c such that $e^{-ct}f(t)$ is absolutely integrable over $\mathbb{R}$.

The Laplace transform of $f(t)$ is defined by

$$F(s) = \mathscr{L}\{f(t)\} = \int_T^\infty f(t)e^{-st}dt, \tag{2.1}$$

where s is a complex variable.

Furthermore, it is also know that if f is continuous, then $F(s)$ is an analytic function on the half-plane $\mathscr{R}e(s) > \sigma_a$, where σ_a is an abscissa of absolute convergence for $\mathscr{L}\{f(t)\}$.

Definition 2.2 ([8])**.** Let $f(t)$ be a function satisfying the same conditions as in Definition 2.1 and $\mathscr{L}\{f(t)\} = F(s)$. The inverse Laplace transform of $F(s)$ is defined by

$$f(t) = \mathscr{L}^{-}\{F(s)\} = \frac{1}{2\pi i}\lim_{\omega\to\infty}\int_{c-i\omega}^{c+i\omega} F(s)e^{st}dt, \tag{2.2}$$

where $\mathscr{R}e(s) > \sigma_a$.

Recall that the Laplace transform $G(s)$ of a locally integrable function $g(t)$ that satisfies the conditions of Definition 2.1, that is

$$G(s) = \mathscr{L}\{g(t)\} = \int_T^{\infty} g(t)e^{-st}dt, \tag{2.3}$$

where $\mathscr{R}e(s) > \sigma_a$, may be written in the form $G(s) = \langle g(t), e^{st}\rangle$.

Definition 2.3 ([8])**.** A distribution T is a continuous linear functional on the space $\mathscr{D}$ of the complex-valued functions with infinitely differentiable and bounded support. The space of all such distributions is denoted by $\mathscr{D}'$.

For every $T \in \mathscr{D}'$ and $\varphi \in \mathscr{D}$, the value that T acts on φ is denoted by $\langle T, \varphi\rangle$. Note that $\langle T, \varphi\rangle \in \mathbb{C}$. Now φ is called a *test function* in $\mathscr{D}$.

Example 2.1 ([8])**.** (i) The locally integrable function f is a distribution generated by the locally integrable function f. Then we define $\langle T, \varphi\rangle = \int_\Omega f(t)\varphi(t)dt$, where Ω is a support of φ and $\varphi \in \mathscr{D}$.

(ii) The Dirac delta function is a distribution defined by $\langle \delta, \varphi\rangle = \varphi(0)$ and the support of δ is $\{0\}$.

A distribution T generated by a locally integrable function is called a *regular distribution*; otherwise, it is called a *singular distribution*.

Definition 2.4 ([8], The Differentiation of Distribution)**.** the k-order derivative of a distribution T, denoted by $T^{(k)}$, is defined by $\langle T^{(k)}, \varphi\rangle = (-1)^k\langle T, \varphi^{(k)}\rangle$ for all $\varphi \in \mathscr{D}$.

Example 2.2 ([8])**.** (i) $\langle \delta', \varphi\rangle = -\langle \delta, \varphi'\rangle = -\varphi'(0)$.

(ii) $\langle \delta^{(k)}, \varphi\rangle = (-1)^k\langle \delta, \varphi^{(k)}\rangle = (-1)^k\varphi^{(k)}(0)$.

Definition 2.5 ([8])**.** Let $f(t)$ be a distribution satisfying the following properties:

(i) f is a right-sided distribution, that is, $f \in \mathscr{D}'_R$;

(ii) there exists a real number c such that $e^{-ct}f(t)$ is a tempered distribution.

The Laplace transform of a right-sided distribution $f(t)$ satisfying (ii) is defined by

$$F(s) = \mathscr{L}\{f(t)\} = \langle e^{-ct}f(t), X(t)e^{-(s-c)t}\rangle, \tag{2.4}$$

where $X(t)$ is an infinitely differentiable function with support bounded on the left, which equals 1 over the neighbourhood of the support of $f(t)$.

For $\mathscr{R}e(s) > c$, the function $X(t)e^{-(s-c)t}$ is a test function in the space S of testing functions of rapid descent and that $e^{-ct}f(t)$ is in the space S' of tempered distributions. Equation (2.4) can be deduced to

$$F(s) = \mathscr{L}\{f(t)\} = \langle f(t), e^{-st}\rangle, \tag{2.5}$$

which possesses the sense given by the right-hand side of (2.4). Now, $F(s)$ is a function of s defined over the right half-plane $\mathscr{R}e(s) > c$. Zemanian [17] proved that $F(s)$ is an analytic function in the region of convergence $\mathscr{R}e(s) > \sigma_1$, where σ_1 is the abscissa of convergence where $e^{-ct}f(t) \in S'$ for some real number $c > \sigma_1$.

Example 2.3 ([8]). Let $\delta(t)$ be the Dirac delta function and $f(t)$ be a Laplace-transformable distribution in $\mathscr{D}'_R$ such that $\mathscr{L}\{f(t)\} = F(s)$ for $\mathscr{R}e(s) > \sigma_1$. For all positive integer k, we have the following properties:

(i) $\mathscr{L}\left\{\dfrac{t^k H(t)}{k!}\right\} = \dfrac{1}{s^{k+1}}$, $\mathscr{R}e(s) > \sigma_1$;

(ii) $\mathscr{L}\{\delta(t)\} = 1$, $-\infty < \mathscr{R}e(s) < \infty$;

(iii) $\mathscr{L}\left\{\delta^{(k)}(t)\right\} = s^k$, $-\infty < \mathscr{R}e(s) < \infty$;

(iv) $\mathscr{L}\left\{t^k f(t)\right\} = (-1)^k F^{(k)}(s)$, $\mathscr{R}e(s) > \sigma_1$;

(v) $\mathscr{L}\left\{f^{(k)}(t)\right\} = s^k F(s)$, $\mathscr{R}e(s) > \sigma_1$.

The proof of the following Lemma is given in [10].

Lemma 2.1 ([10]). *Let $\psi(t)$ be an infinitely differentiable function. Then*

$$\begin{aligned}\psi(t)\delta^{(m)}(t) &= (-1)^m \psi^{(m)}(0)\delta(t) + (-1)^{m-1} m\psi^{(m-1)}(0)\delta'(t) \\ &\quad + (-1)^{m-2}\frac{m(m-1)}{2!}\psi^{(m-1)}(0)\delta''(t) + \cdots + \psi(0)\delta^{(m)}(t).\end{aligned} \tag{2.6}$$

A useful formula that follows from (2.6) for any monomial $\psi(t) = t^n$ is

$$t^n\delta^{(m)}(t) = \begin{cases} 0, & \text{for } m < n; \\ (-1)^n \frac{m!}{(m-n)!}\delta^{(m-n)}(t), & \text{for } m \geq n. \end{cases} \tag{2.7}$$

Lemma 2.2 ([16]). *If the equation*

$$\sum_{i=0}^{n} t^i a_i(t) y^{(i)}(t) = 0 \tag{2.8}$$

with infinitely differentiable coefficients $a_i(t)$ and $a_n(0) \neq 0$ has a solution

$$y(t) = \sum_{i=0}^{k} a_i \delta^{(i)}(t), \quad a_k \neq 0, \tag{2.9}$$

of order (of the distribution) k, then

$$\sum_{i=0}^{n} (-1)^i a_i(0)(k+i)! = 0. \tag{2.10}$$

Conversely, if k is the smallest nonnegative integer root of (2.10)*, then there exists an order-k solution of* (2.9) *at $t = 0$.*

The proof of this Lemma is given in [16].

3. Main Results

Our main results and their proofs will be revealed in this section.

Theorem 3.1. *The types of solutions of the third-order Cauchy-Euler equation*

$$at^3y'''(t)+bt^2y''(t)+cty'(t)+dy(t)=0, \tag{3.1}$$

which depend on the conditions of the value of a,b,c *and* d *are given by the following cases:*

(i) *If there exists* $k\in\mathbb{N}$ *such that*

$$(k^3+3k^2+2k)a-(k^2+k)b+kc-d=0, \tag{3.2}$$

then the solutions of (3.1) *are distributional solutions in the form* $\delta^{(k-1)}(t)$.

(ii) *If there exists* $k\in\mathbb{N}\cup\{0\}$ *such that*

$$(k^3-3k^2+2k)a+(k^2-k)b+kc+d=0, \tag{3.3}$$

then the solution of (3.1) *are weak solutions in the form* $H(t)\dfrac{t^k}{k!}$.

Proof. Taking the Laplace transform on the both sides of (3.1), we have

$$\mathscr{L}\{at^3y'''(t)\}+\mathscr{L}\{bt^2y''(t)\}+\mathscr{L}\{cty'(t)\}+\mathscr{L}\{dy(t)\}=0.$$

Using Example 2.3(iv) and (v), we get

$$-a\frac{d^3}{ds^3}[s^3Y(s)]+b\frac{d^2}{ds^2}[s^2Y(s)]-c\frac{d}{ds}[sY(s)]+dY(s)=0.$$

By Leibniz's rule for derivative, we obtain

$$e_3s^3Y'''(s)+e_2s^2Y''(s)+e_1sY'(s)+e_0Y(s)=0, \tag{3.4}$$

where

$$\left.\begin{aligned} e_0&=6a-2b+c-d, & e_1&=18a-4b+c,\\ e_2&=9a-b, & e_3&=a. \end{aligned}\right\} \tag{3.5}$$

Suppose that $Y(s)=s^r$ is a solution of (3.4) with a given constant r. Then (3.4) becomes

$$r(r-1)(r-2)e_3s^r+r(r-1)e_2s^r+re_1s^r+e_0s^r=0.$$

Cancelling a common factor s^r, we have

$$r(r-1)(r-2)e_3+r(r-1)e_2+re_1+e_0=0$$

or equivalently,

$$(r^3-3r^2+2r)e_3+(r^2-r)e_2+re_1+e_0=0. \tag{3.6}$$

If r is a non-negative integer, then substituting $r=k-1$ yields

$$\left[(k-1)^3-3(k-1)^2+2(k-1)\right]e_3+\left[(k-1)^2-(k-1)\right]e_2+(k-1)e_1+e_0=0. \tag{3.7}$$

By substituting (3.5) into (3.7), we obtain (3.2).

Therefore, if the condition (3.2) holds, then the solution of (3.1) is $Y(s)=s^{k-1}$ for all $k\in\mathbb{N}$. Now $Y(s)$ are analytic functions over the entire s-plane. By taking the inverse Laplace transform of $Y(s)$ and using Example 2.3, we find that the solutions of (3.1) are the distributional solutions of the form

$$y(t) = \delta^{(k-1)}(t), \tag{3.8}$$

which satisfies (3.2).

On the other hand, if r is a negative integer, then $r = -(k+1)$ for all $k \in \mathbb{N} \cup \{0\}$. Then (3.6) becomes

$$\left[-(k+1)^3 - 3(k+1)^2 - 2(k+1)\right] e_3 + \left[(k+1)^2 + (k+1)\right] e_2 - (k+1)e_1 + e_0 = 0. \tag{3.9}$$

By substituting (3.5) into (3.9), we get (3.3).

Hence, if the condition (3.3) holds, then the solution of (3.1) is $Y(s) = \dfrac{1}{s^{k+1}}$ for all $k \in \mathbb{N} \cup \{0\}$. Now $Y(s)$ are analytic functions over the entire s-plane. By taking the inverse Laplace transform to $Y(s)$ and using Example 2.3, we have the solutions of (3.1) are the weak solutions of the form

$$y(t) = H(t)\frac{t^k}{k!}, \tag{3.10}$$

which satisfies (3.3). This completes the proof of Theorem 3.1. □

Theorem 3.2. *The distributional solution of the third-order Cauchy-Euler equation*

$$at^3 y'''(t) + bt^2 y''(t) + cty'(t) + dy(t) = 0, \tag{3.11}$$

where a, b, c and d are constants and $t \in \mathbb{R}$, depends on the relationship of a, b, c and d in such a way that

$$(k^3 + 3k^2 + 2k)a - (k^2 + k)b + kc - d = 0, \tag{3.12}$$

where $k \in \mathbb{N} \cup \{0\}$ is the order of distribution.

Proof. By Lemma 2.2, we substitute $n = 3, a_3(0) = a, a_2(0) = b, a_1(0) = c$ and $a_0(0) = d$ into (2.10), we obtain

$$a(k+2)! - b(k+1)! + c(k)! - d(k-1)! = 0. \tag{3.13}$$

Therefore, we get (3.12) as required. □

Remark 3.3 If $a = b = c = 1$ and $d = m$, then (3.1) becomes

$$t^3 y'''(t) + t^2 y''(t) + ty'(t) + my(t) = 0 \tag{3.14}$$

and (3.2) reduces to

$$m = k^3 + 2k^2 + 2k, \tag{3.15}$$

which appeared in A. Kananthai [8].

Example 3.1. (i) For $a = 2, b = 7, c = 1$ and $d = 8$, then (3.1) becomes

$$2t^3 y'''(t) + 7t^2 y''(t) + ty'(t) + 8y(t) = 0. \tag{3.16}$$

It follows from (3.2) that its distributional solution is $y(t) = \delta'(t)$. By applying (2.7), it is easy to verify that $y(t) = \delta'(t)$ satisfies (3.16).

(ii) For $a = \frac{1}{36}, b = \frac{5}{18}, c = \frac{1}{6}$ and $d = -\frac{2}{9}$, then (3.1) becomes

$$\frac{1}{36}t^3y'''(t) + \frac{5}{18}t^2y''(t) + \frac{1}{6}ty'(t) - \frac{2}{9}y(t) = 0. \tag{3.17}$$

It follows from (3.2) that its distributional solution is $y(t) = \delta(t)$. By applying (2.7), it is easy to verify that $y(t) = \delta(t)$ satisfies (3.17).

Example 3.2. (i) For $a = 5, b = -2, c = -7$ and $d = 0$, then (3.1) becomes

$$5t^3y'''(t) - 2t^2y''(t) - 7ty'(t) = 0. \tag{3.18}$$

It follows from (3.3) that its weak solution is $y(t) = H(t)$. It is easy to verify directly that $y(t) = H(t)$ satisfies (3.18).

(ii) For $a = \frac{1}{10}, b = -\frac{3}{8}, c = \frac{1}{2}$ and $d = -\frac{1}{4}$, then (3.1) becomes

$$\frac{1}{10}t^3y'''(t) - \frac{3}{8}t^2y''(t) + \frac{1}{2}ty'(t) - \frac{1}{4}y(t) = 0. \tag{3.19}$$

It follows from (3.3) that its weak solution is $y(t) = H(t)\frac{t^2}{2!}$. It is easy to verify directly that $y(t) = H(t)\frac{t^2}{2!}$ satisfies (3.19).

4. Conclusions

We find the generalized solutions in the space of distributions of the third-order Cauchy-Euler equation

$$at^3y'''(t) + bt^2y''(t) + cty'(t) + dy(t) = 0$$

by using Laplace transform. We find that if the condition $(k^3 + 3k^2 + 2k)a - (k^2 + k)b + kc - d = 0$ holds for all $k \in \mathbb{N}$, then there exists the distributional solutions of such equation, and if the condition $(k^3 - 3k^2 + 2k)a + (k^2 - k)b + kc + d = 0$ holds, then there exists the weak solutions of such equation for all $k \in \mathbb{N} \cup \{0\}$.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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