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Research Article

SO$(3,\mathbb{C})$ Representation and Action on A Homogeneous Space in $\mathbb{C}^3$

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Abstract. A homogeneous manifold $\mathscr{H}$ embedded in $\mathbb{C}^3$ composed of complex vectors with constraints, potentially representing space of complex velocities, is considered. The imposed constraints include orthogonality between the real and the imaginary parts of vectors which together with the non-conjugate scalar product provide real vector magnitudes. The corresponding representation of the group SO$(3,\mathbb{C})$ acting on $\mathscr{H}$ which is in agreement to the polar decomposition of complex orthogonal matrices is also discussed.

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1. Introduction

The complex space $\mathbb{C}^3$ offers several advantages over the commonly used 4-dimensional real space regarding relativistic applications. It allows the use of the customary vector product which is basic in describing of many physical laws. The greater number of vector components in $\mathbb{C}^3$ permits involvement of additional physical quantities. Also, the complex vectors offer a possibility to make distinction between physical concepts by employing their real/imaginary separation. These observations explain why the complex rotation group SO$(3,\mathbb{C})$ which naturally acts on $\mathbb{C}^3$ is occasionally used as an alternative tool to represent elements of physical theories commonly expressed in terms of its isomorphic group $\mathrm{SO}(1,3)^+$.

In this paper, a space $\mathscr{H}$ of vectors with constraints, potentially representing relativistic velocities in $\mathbb{C}^3$, is introduced. A complex bundle is constructed and it is parameterized by seven parameters a_1,a_2,a_3, b_1,b_2,b_3, t, such that the first three parameterize the real part, the second three parameterize the imaginary part of the bundle vectors and t parameterizes the base. The space $\mathscr{H}$ is composed of constrained vectors in the tangent spaces of the complex bundle with structure group $\mathrm{SO}(3,\mathbb{C})$ acting on $\mathscr{H}$.

The common representation of $\mathrm{SO}(3,\mathbb{C})$ acting on vectors in $\mathbb{C}^3$ is occasionally used to examine elements of physical theories, for example see [1, 3, 7, 9], among others. The representation of the group $\mathrm{SO}(3,\mathbb{C})$ is adapted according to the constraints imposed on the complex vectors in $\mathscr{H}$. This $\mathrm{SO}(3,\mathbb{C})$ representation is discussed through the polar decomposition of the $\mathrm{SO}(3,\mathbb{C})$ matrices on (real orthogonal)/(positive definite Hermitian), i.e. rotation/boost matrices. It is proved that boosts act transitively on $\mathscr{H}$. The boost representation matrix is introduced as a special case of $\mathrm{SO}(3,\mathbb{C})$ boost link between vectors in $\mathscr{H}$. Finally, an alternative boost representation suitable for applications in physics is given.

2. Representations of Vectors in $\mathscr{H}$

It is assumed that the time is represented by a real, affine one-dimensional space $\mathscr{T}$, and space-time by a complex (3+3)-dimensional manifold $\mathscr{U}$ homeomorphic to $\mathbb{C}^3$, fibered over $\mathscr{T}$ by a map $\pi:\mathscr{U}\to\mathscr{T}$. So, the fibers are complex vector spaces with vectors $(a_1,a_2,a_3)+i(b_1,b_2,b_3)$ where the coordinates (a_1,a_2,a_3) could represent a real 3-dimensional time, while the coordinates (b_1,b_2,b_3) could represent an imaginary 3-dimensional space. The use of 3-dimensional time in physics is not so unusual (see for example [2, 8]). Now, the vectors in the tangent spaces on $\mathscr{U}$ will be constrained in the following way.

For $\frac{d}{dt}(a_1,a_2,a_3)=\vec{x}$ and $\frac{d}{dt}(b_1,b_2,b_3)=\vec{y}$,

(i) $\vec{x}\,\vec{y}=0$ (orthogonality constraint),

(ii) $\vec{x}^{\,2}-\vec{y}^{\,2}=c^2$, $c\in\mathbb{R}$, $c>0$ (hyperbolic norm constraint).

Only the subsets of tangent vectors which satisfy the above two constraints are considered. As it is customary, one can pullback the constrained tangent vectors to $\mathscr{U}$ and work directly in $\mathscr{U}$, i.e. in $\mathbb{C}^3$. Thus, one can consider a subset $\mathscr{H}$ of complex vectors $v\in\mathbb{C}^3$ of the form

$$v=\vec{x}+i\vec{y},\quad \vec{x},\vec{y}\in\mathbb{R}^3,\quad \vec{x}\,\vec{y}=0,$$

with magnitude $v\cdot v=v^2=c^2$ for a constant $c\in\mathbb{R}$, where $\cdot$ is a scalar product given by ordinary (non-conjugate) multiplication. Of course, the scalar product $\cdot$ is a complex valued scalar product in $\mathbb{C}^3$ with weakened conditions. Notice that the orthogonality constraint $\vec{x}\,\vec{y}=0$ provides real vector magnitudes.

Definition 2.1. The space $\mathscr{H} \subset \mathbb{C}^3$ defined by $\mathscr{H} = \{v = \vec{x} + i\vec{y} \mid v^2 = c^2,\ \vec{x}\vec{y} = 0\}$ is a space of h-vectors in $\mathbb{C}^3$.

Of course, the space $\mathscr{H}$ is not closed regarding the inherited vector addition and multiplication with scalar from $\mathbb{C}^3$, so it is not a vector subspace of $\mathbb{C}^3$. However, it is useful to notice that when $v \in \mathscr{H}$, then also $-v \in \mathscr{H}$, $\overline{v} \in \mathscr{H}$ and $-\overline{v} \in \mathscr{H}$. The dimension of the space $\mathscr{H}$ is four (real dimensions) since one dimension is lost by the orthogonality constraint $\vec{x}\vec{y} = 0$ and another dimension is lost by the fixed vector magnitudes. Compare this with the space of 4-vectors of velocity in Minkowski space where one component depends on the remaining three and therefore, the corresponding homogeneous space is 3-dimensional.

3. Transformations in $\mathscr{H}$

Now, it is of interest to examine a complex representation G of SO(3,ℂ) adapted to $\mathscr{H}$. The standard polar decomposition of a complex orthogonal matrix on (real orthogonal)/(positive definite Hermitian) [4] should correspond to the rotation/boost representation of transformations in G. So, every matrix in G can be represented as a product of SO(3,ℝ) matrix representing a rotation and a positive definite Hermitian matrix representing a boost.

The rotations in G that act on $\mathscr{H}$ are simply reduced to the standard 3-dimensional rotations from SO(3,ℝ). Their common representation is given by the Rodrigues' rotation formula $Rot_{\vec{s},\theta} = \cos\theta \cdot I + \sin\theta[\vec{s}\,]_\times + (1-\cos\theta)\vec{s} \otimes \vec{s}$, where $\vec{s}$ is the rotation axis (unit vector), θ is the rotation angle, I stands for the identity matrix and $[\vec{s}\,]_\times$ is the cross product matrix.

Boosts deserve special attention since they change the magnitudes of the real and the imaginary parts of h-vectors. How can one define the boost representation acting on $\mathscr{H}$? One way is to consider the G-link problem, i.e. finding transformation(s) in G that carries a given h-vector u to a given h-vector v.

Firstly,one can show that G acts on $\mathscr{H}$ transitively, i.e. for each pair of h-vectors $u, v \in \mathscr{H}$ there is an element $T \in G$ such that $Tu = v$.

Proposition 1. *$\mathscr{H}$ is a homogeneous space.*

Proof. Let u and v be two h-vectors in $\mathscr{H}$. Since the scalar product is Euclidean, the standard boost G-link between u and v is analogous to the corresponding link in Minkowski space, e.g. [5] given by

$$T_{u\to v} = I - \frac{2}{(u+v)^2}(u+v)\bigotimes(u+v) + \frac{2}{v^2}v\bigotimes u, \quad \text{when } (u+v)^2 \neq 0, \text{ and}$$
$$T_{u\to v} = I + \frac{2}{v^2}v\bigotimes u, \quad \text{when } (u+v)^2 = 0. \tag{3.1}$$

Since $(u+v)\cdot u = c^2 + u\cdot v$ and $(u+v)^2 = 2(c^2 + u\cdot v)$, one can obtain orthogonality and $T_{u\to v}u = v$ by straightforward calculations. □

Since the polar decomposition of the $SO(3,\mathbb{C})$ matrices corresponds to the polar decomposition of $SO(1,3)^+$ matrices, the condition $T_{u\to u} = I$, can be used to show that $T_{u\to v}$ is a boost. Actually, there is a continuum of boosts that links u and v [6, pp. 13 – 14].

However, it is not a conventional approach to represent the boost through a boost G-link $T_{u\to v}$, since such representation is in a basis containing the h-vector u, while in the standard boost representation the boost matrix should refer only to a h-vector v determined in a basis containing corresponding "zero"h-vector. So, for $v = \vec{x} + i\vec{y} \in \mathscr{H}$, $\vec{y} \neq \vec{0}$, the corresponding "zero"h-vector will be $u = \alpha\vec{x}$ for the scalar $\alpha \in \mathbb{R}$, $0 \ll 1$ (notice that the vector $x = \vec{x} \notin \mathscr{H}$, since $\|\vec{x}\| > c$). Now, to obtain a standard boost representation one can simply put $u = \alpha\vec{x}$ in (3.1) and obtain

$$\begin{aligned} B_{\alpha\vec{x}\to v} &= I - \frac{c^2(1+\alpha)^2 - 2\alpha\left(c^2+\alpha\vec{x}^2\right)}{c^2\left(c^2+\alpha\vec{x}^2\right)}\vec{x}\bigotimes\vec{x} + \frac{c^2}{c^2\left(c^2+\alpha\vec{x}^2\right)}\vec{y}\bigotimes\vec{y} \\ &\quad + i\left(\frac{c^2(1+\alpha) - 2\alpha\left(c^2+\alpha\vec{x}^2\right)}{c^2\left(c^2+\alpha\vec{x}^2\right)}\vec{y}\bigotimes\vec{x} - \frac{c^2(1+\alpha)}{c^2\left(c^2+\alpha\vec{x}^2\right)}\vec{x}\bigotimes\vec{y}\right) \\ &= I - \frac{\alpha(\alpha-1)}{c^2}\vec{x}\bigotimes\vec{x} + \frac{\alpha}{c^2(1+\alpha)}\vec{y}\bigotimes\vec{y} + i\frac{\alpha}{c^2}\left(\vec{y}\bigotimes\vec{x} - \vec{x}\bigotimes\vec{y}\right). \end{aligned} \tag{3.2}$$

Now, since $\alpha^2\vec{x}^2 = c^2$ and $\vec{x}^2 - \vec{y}^2 = c^2$ it follows that $\alpha = 1/\sqrt{1+\vec{y}^2/c^2}$. Thus, the boost B_v on $\mathscr{H}$ is a boost G-link between a "zero"h-vector which corresponds to v, and the vector v. It seems more convenient to put $k = \frac{1}{\alpha} = \sqrt{1+\vec{y}^2/c^2}$, $k > 1$ and obtain a boost matrix as in the following proposition.

Proposition 2. *The matrix B_v, parameterized by a h-vector $v = \vec{x} + i\vec{y}$ and given by*

$$B_v = I + \frac{k-1}{c^2k^2}\vec{x}\bigotimes\vec{x} + \frac{1}{c^2(1+k)}\vec{y}\bigotimes\vec{y} + i\frac{1}{c^2k}\left(\vec{y}\bigotimes\vec{x} - \vec{x}\bigotimes\vec{y}\right), \tag{3.3}$$

where $k = \sqrt{1+\vec{y}^2/c^2}$, is a representation of a boost in G.

Now, one can show that the matrix B_v is in agreement with the polar decomposition of complex orthogonal matrices.

Proposition 3. *The matrix B_v given by* (3.3) *is an orthogonal and positive definite Hermitian matrix with respect to the scalar product* $\cdot$.

Proof. Since $B_v = B_{k\vec{x}\to v}$ it immediately follows that B_v is an orthogonal matrix, i.e. $B_v^{-1} = B_v^T = B_{\overline{v}}$. It is obvious that $\overline{B_v} = B_v^T$, and so $\overline{B_v}^T = B_v$, and $B_v\overline{B_v} = I$ which means that B_v is a Hermitian coninvolutory matrix. Let $u = \vec{a} + i\vec{b}$ and $v = \vec{x} + i\vec{y}$. From

$$\begin{aligned} B_v u &= \left[\vec{a} + \left(\frac{k-1}{c^2k^2}\vec{a}\,\vec{x} + \frac{1}{c^2k}\vec{b}\,\vec{y}\right)\vec{x} + \left(\frac{1}{c^2(1+k)}\vec{a}\,\vec{y} - \frac{1}{c^2k}\vec{b}\,\vec{x}\right)\vec{y}\right] \\ &\quad + i\left[\vec{b} + \left(\frac{k-1}{c^2k^2}\vec{b}\,\vec{x} - \frac{1}{c^2k}\vec{a}\,\vec{y}\right)\vec{x} + \left(\frac{1}{c^2(1+k)}\vec{b}\,\vec{y} - \frac{1}{c^2k}\vec{a}\,\vec{x}\right)\vec{y}\right], \end{aligned}$$

using simple algebraic operations,one can obtain $\overline{u}(B_v u) > 0$. □

It is interesting to compare the boost representation (3.3) acting on $\mathscr{H}$ with the standard boost representation in SO(3, $\mathbb{C}$) acting on $\mathbb{C}^3$ which in the present notation takes the form

$$B_{\vec{y}} = kI - \frac{1}{c^2(1+k)}\vec{y}\otimes\vec{y} + i\frac{1}{ck}[\vec{y}]_\times$$

where $\vec{y}$ is a real 3-vector representing velocity (see for example [7]). Compare it with the boost representation

$$B_v = I + \frac{k-1}{c^2k^2}\vec{x}\otimes\vec{x} + \frac{1}{c^2(1+k)}\vec{y}\otimes\vec{y} + i\frac{1}{c^2k}[\vec{x}\times\vec{y}]_\times \tag{3.4}$$

(rewritten (3.3) using cross product matrix) adapted to $\mathscr{H}$. Obviously, the existence of $\vec{x}$ in (3.4) reflects the properties of the space $\mathscr{H}$ more accurately.

It can be easily checked that, in general, two boosts do not commute even when they are generated from vectors with a common real part, i.e. $B_uB_v \neq B_vB_u$ for $u = \vec{x}+i\vec{z}$ and $v = \vec{x}+i\vec{y}$. From the fact that, generally, a product of two Hermitian matrices is not a Hermitian matrix, it follows that the product of boosts is not a boost. The boost representation (3.3) corresponds to the observer-dependent (basis-free) Lorentz boost in Minkowski space [6, p. 25].

4. Discussion

In practice, if one wants to represent velocity as an h-vector, the following form is more relevant,

$$v = \gamma_v(c\vec{n} + i\vec{v}), \tag{4.1}$$

where $\vec{n}$ is an unit vector orthogonal to $\vec{v}$ (a standard velocity vector in $\mathbb{R}^3$), i.e. $\|\vec{n}\| = 1$ and $\vec{n}\vec{v} = 0$ and $\gamma_v = 1/\sqrt{1-\vec{v}^2/c^2}$ is Lorentz factor. The vector $\vec{n}$ can be called orthogonal component of the vector v.

Corollary. *The representations of the h-vector v, given by Definition* 2.1 *and* (4.1) *are equivalent.*

Proof. Let $v = \vec{x} + i\vec{y}$ be vector in $\mathbb{C}^3$ with constraints $\vec{x}\vec{y} = 0$ and $v^2 = c^2$. Then, the representation $v = \vec{x} + i\vec{y} \Leftrightarrow v = p\vec{n} + i\vec{y}$ where $\|\vec{n}\| = 1$ and $\vec{n}\vec{y} = 0$. From the condition $p^2 - \vec{y}^2 = c^2$ it follows that $p^2 \geq c^2$ and so, $p = qc$ for some $q \geq 1$. Now, for $\vec{y} = q\vec{v}$, the hyperbolic norm constraint gives $q^2c^2 - q^2\vec{v}^2 = c^2$, i.e. $q^2 = \frac{1}{1-\vec{v}^2/c^2} = \gamma_v{}^2$. Conversely, if $v = \gamma_v(c\vec{n} + i\vec{v})$ then obviously $\vec{x} = \gamma_v c\vec{n}$ and $\vec{y} = \gamma_v\vec{v}$. □

The space $\mathscr{H}$ can be interpreted as a space of relativistic velocities represented by the complex h-vectors of the form $v = \vec{x} + i\vec{y}$. The imaginary part $\vec{y}$ of v represents relativistic 3-velocity. The real part of v, given by $\vec{x}$ could be associated with an observer representing a velocity of the observer's time. The transformations presented in the previous section vary the velocity given in the imaginary part of vectors, but also they change the velocity of the observer's time given by the real part of vectors.

Using vector representation (4.1), the boost (3.3) takes the form

$$B_v = I + (\gamma_v - 1)\vec{n}\otimes\vec{n} + \frac{\gamma_v{}^2}{(1+\gamma_v)c^2}\vec{v}\otimes\vec{v} + i\frac{\gamma_v}{c}(\vec{v}\otimes\vec{n} - \vec{n}\otimes\vec{v})$$

which is more convenient representation for physical applications.

5. Conclusion

The intention of this work is to provide a representation of $\mathrm{SO}(3,\mathbb{C})$ transformations acting on space $\mathscr{H}$ of vectors in $\mathbb{C}^3$ with two constraints. The orthogonality constraint, together with the non-conjugate scalar product provides real vector magnitudes, while the scalar product remains complex valued. The transformations on $\mathscr{H}$ by representation G of the group $\mathrm{SO}(3,\mathbb{C})$ acting on $\mathscr{H}$ are explicitly presented.The representation G of the group $\mathrm{SO}(3,\mathbb{C})$ is analyzed through the polar decomposition of the $\mathrm{SO}(3,\mathbb{C})$ matrices on (real orthogonal)/(positive definite Hermitian), i.e. rotation/boost matrices. It is confirmed that boosts act transitively on $\mathscr{H}$. The boost representation matrix is introduced as a special case of $\mathrm{SO}(3,\mathbb{C})$ boost link between vectors in $\mathscr{H}$. Finally, an alternative boost representation suitable for applications in physics is given.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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