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Research Article

Approximate Solution of Multi-Pantograph Equations With Variable Coefficients via Collocation Method Based on Hermite Polynomials

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Abstract. This research article presents an approximate solution of the non-homogenous Multi-Pantograph equation comprising of variable coefficients by utilizing a collocation method based on Hermite polynomials. These orthogonal polynomials along with its collocation points transform the equation and the initial conditions into matrix equation comprising of a system of linear algebraic equations. Subsequently, by solving this system, the unknown Hermite coefficients are calculated. To reveal the accuracy and efficiency of the method applied, the approximate results obtained by this technique have been compared with exact solutions. Moreover, some numerical illustrations in the form of examples are given to exhibit the applicability of the proposed technique.

Keywords. Multi-Pantograph equation; Collocation method; Hermite polynomials; Matrix equation; Approximate results; Accuracy

MSC. 65N30; 65M70; 33C45

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1. Introduction

In this research article, the following non-homogenous multi-pantograph equation with variable coefficients is considered:

$$z'(t) = \lambda z(t) + \sum_{p=1}^{l} \beta_p(t) z(q_p(t)) + f(t), \quad t \geq 0, \tag{1.1}$$

with initial condition

$$z(0) = c, \tag{1.2}$$

where λ, c and q_p are constants with $0 < p < l$, $\beta_p(t)$ and $f(t)$ are analytical functions. Functional differential equations with proportional delays are generally known as pantograph equations. The name pantograph started from the work of Ockendon and Tayler [16]. These equations emerge in numerous applications, for example, cell growth models, biology, economy, control, number theory, electrodynamics, astrophysics and many more [3,10,21]. A massive work has already been done by several authors in order to get the numerical solution of pantograph equations by using many different approaches [1,4,5,12,17,20,22,23]

Recently, the multi-pantograph equations were studied by numerous researchers in order to obtain its numerical as well as exact solution. For example, Muroya et al. [15] solved the multi-pantograph delay equation numerically by utilizing the collocation method. Few properties of the exact as well as approximate solution of the multi-pantograph equations are showed by Liu and Li [14]. The Runge-Kutta methods have been applied by Li and Liu [13] to the multi-pantograph delay equation. The numerical solution of non-homogenous multi-pantograph equation with variable coefficients are computed via the Taylor method [18]. Application of the *homotopy analysis method* (HAM) for solving the multi-pantograph equation is found in [2]. To solve these equations the successive approximations method has been applied by Jafari [11]. Multi-pantograph delay equations with variable coefficients are solved by the *homotopy perturbation method* (HPM) in [7].

The chief goal of this research work is to apply the Hermite Collocation method to the equation (1.1)-(1.2). In this methodology solution of the unknown function is expressed in the form of a linear combination of some basis functions involving unknown coefficients. These basis functions can be favored as orthogonal polynomials as per their specific properties, which make the problem under consideration much easier to solve.

This article is organized as follows: Section 2 depicts a few properties of Hermite polynomials. Application of the Hermite Collocation Method on Multi-Pantograph Delay Equations is described in Section 3. Subsequently, in Section 4, some examples are provided. Finally, conclusion is presented at the end.

2. Hermite Polynomials

Hermite polynomials are represented by H_n.

$$e^{t^2}(e^{-t^2}(H_n(t))')' + \lambda_n H_n(t) = 0, \quad \forall\ t \in A, \tag{2.1}$$

$$H_n''(t) - 2tH_n'(t) + 2nH_n(t), \quad n = 0,1,2,\dots, \tag{2.2}$$

where H_n are the eigenfunctions of the Sturm-Liouville problem (2.1) and are the solution of the second order ordinary differential equation (2.2). The Hermite polynomials $H_n(t)$ are set of orthogonal polynomials over the domain $(-\infty,\infty)$ with weighting function e^{-t^2}.

$$H_{n+1}(t) = 2tH_n(t) - 2nH_{n-1}(t), \quad n \ge 1. \tag{2.3}$$

In practice, the Hermite polynomials can be calculated by utilizing the recurrence relation (2.3).

$$H_n'(t) = 2nH_{n-1}(t), \quad n \ge 1,\ t \in A. \tag{2.4}$$

The derivative relation (2.4) is an important property of these polynomials [8, 19]. Some of the Initial Hermite polynomials are:

$$H_0(t) = 1, \qquad H_1(t) = 2t,$$
$$H_2(t) = 4t^2 - 2, \qquad H_3(t) = 8t^3 - 12t,$$
$$H_4(t) = 16t^4 - 48t^2 + 12, \qquad H_5(t) = 32t^5 - 160t^3 + 120t,$$
$$H_6(t) = 64t^6 - 480t^4 + 720t^2 - 120, \qquad H_7(t) = 128t^7 - 1344t^5 + 3360t^3 - 1680t,$$
$$H_8(t) = 256t^8 - 3584t^6 + 13440t^4 - 13440t^2 + 1680.$$

Any square integrable function $f(x) \in L^2_{w(x)}(-\infty,\infty)$ can be represented in terms of Hermite polynomials [9] as below:

$$y(t) = \sum_{j=0}^{\infty} a_j H_j(t).$$

The solution is assumed to be expressed in the form of truncated Hermite series as:

$$y(t) = \sum_{j=0}^{N} a_j H_j(t). \tag{2.5}$$

The Hermite polynomials are expressed in the vector form as:

$$\mathbf{H(t)} = \begin{bmatrix} H_0(t) & H_1(t) & H_2(t) & \dots & H_j(t) \end{bmatrix}.$$

Thus the finite series (2.5) can be put in a given below matrix form:

$$[y(t)] = \mathbf{H(t)A}, \tag{2.6}$$

where

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_j \end{bmatrix}^T.$$

By using the derivation relation (2.4) of Hermite polynomials the following matrix relation between the matrices $H^k(t)$ and $H(t)$ is obtained:

$$\begin{bmatrix} H_0'(t) \\ H_1'(t) \\ H_2'(t) \\ \vdots \\ H_{j-1}'(t) \\ H_j'(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2.1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2.2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2(j-1) & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 2j & 0 \end{bmatrix} \begin{bmatrix} H_0(t) \\ H_1(t) \\ H_2(t) \\ \vdots \\ H_{j-1}(t) \\ H_j(t) \end{bmatrix}$$

$$[\mathbf{H}'(\mathbf{t})]^T = M[\mathbf{H}(\mathbf{t})]^T \quad \Longrightarrow \quad \mathbf{H}'(\mathbf{t}) = \mathbf{H}(\mathbf{t})M^T \tag{2.7}$$

where M is referred to as the Hermite operational matrix of derivative. By using (2.7), we get

$$\mathbf{H}''(\mathbf{t}) = \mathbf{H}'(\mathbf{t})M^T = \mathbf{H}(\mathbf{t})(M^T)^2, \ \ \mathbf{H}'''(\mathbf{t}) = \mathbf{H}'(\mathbf{t})(M^T)^2 = \mathbf{H}(\mathbf{t})(M^T)^3, \ \ \mathbf{H}^{(k)}(\mathbf{t}) = \mathbf{H}(\mathbf{t})(M^T)^k.$$

3. Applying the Hermite Collocation Method on Multi-Pantograph Delay Equations

In this section, we apply the Hermite Collocation method on Multi-Pantograph Delay Equations. For this, it is assumed that solution of (1.1) can be expanded in the first $(N+1)$ terms Hermite polynomials as:

$$z(t) = \sum_{j=0}^{N} a_j H_j(t) = \mathbf{H}(\mathbf{t})\mathbf{A}. \tag{3.1}$$

With the help of (2.7) and (3.1) one may say that:

$$z'(t) = \mathbf{H}(\mathbf{t})M^T\mathbf{A}. \tag{3.2}$$

Therefore, (2.7), (3.1) and (3.2) implies (1.1) can be re-scripted as follows:

$$\mathbf{H}(\mathbf{t})M^T\mathbf{A} = \lambda\mathbf{H}(\mathbf{t})\mathbf{A} + \sum_{p=1}^{l} \beta_p(t)\mathbf{H}(q_p(t))\mathbf{A} + f(t), \quad t \geq 0$$

$$\Longrightarrow \quad \left[\mathbf{H}(\mathbf{t})M^T - \lambda\mathbf{H}(\mathbf{t}) - \sum_{p=1}^{l} \beta_p(t)\mathbf{H}\Big(q_p(t)\Big)\right]\mathbf{A} = f(t).$$

In order to obtain the unknown Hermite coefficient, substituting the collocation points $t_i = i/N$, $i = 0,1,2,\dots,N$ thus yields the system of the matrix equations which is as follows:

$$\left\{\mathbf{H}(t_i)M^T - \lambda\mathbf{H}(t_i) - \sum_{p=1}^{l} \beta_p(t_i)\mathbf{H}\Big(q_p(t_i)\Big)\right\}\mathbf{A} = f(t_i) = F. \tag{3.3}$$

Thus, (1.1) is converted into matrix equation comprising of a system of $(N+1)$ linear algebraic equations involving unknown Hermite coefficients. Here,

$$F = \begin{pmatrix} f(t_0) \\ f(t_1) \\ \vdots \\ f(t_N) \end{pmatrix}, \quad \lambda = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix}, \quad \beta_p(t_i) = \begin{pmatrix} \beta_p(t_0) & 0 & \cdots & 0 \\ 0 & \beta_p(t_1) & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_p(t_N) \end{pmatrix}$$

and

$$H(t_i) = \begin{pmatrix} H_0(t_0) & H_1(t_0) & \cdots & H_N(t_0) \\ H_0(t_1) & H_1(t_1) & \cdots & H_N(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(t_N) & H_1(t_N) & \cdots & H_N(t_N) \end{pmatrix}, \quad q_p = \begin{pmatrix} q_p & 0 & \cdots & 0 \\ 0 & q_p & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_p \end{pmatrix}.$$

Moreover, the system of matrix equation (3.3) corresponding to (1.1) can be written in a simplified way as follows:

$$WA = F \quad \text{or} \quad [W;F],$$

where

$$W=\left\{\mathbf{H}(t_i)M^T-\lambda\mathbf{H}(t_i)-\sum_{p=1}^{l}\beta_p(t_i)\mathbf{H}\Big(q_p(t_i)\Big)\right\}. \tag{3.4}$$

This can be elaborated in an augmented matrix formation as follows:

$$[W;F]=\begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & f(t_0)\\ w_{10} & w_{11} & \cdots & w_{1N} & ; & f(t_1)\\ \vdots & \vdots & \ddots & \vdots & & \vdots\\ w_{N0} & w_{N1} & \cdots & w_{NN} & ; & g(t_N)\end{pmatrix}.$$

With the help of (3.1), the initial condition presented in (1.2) can be presented in matrix form as:

$$\begin{aligned} & z(0)=z_0=\mathbf{H(0)A}=c\\ \Longrightarrow\quad & z_0\mathbf{A}=c \text{ or } [z_0;c]. \end{aligned} \tag{3.5}$$

Lastly, replacing the last row of the augmented matrix given above by the row matrix (3.5), the equation (1.1) under conditions (1.2) is reduced to the following linear system of algebraic equations:

$$\widetilde{W}\mathbf{A}=\widetilde{G}, \tag{3.6}$$

where

$$[\widetilde{W};\widetilde{G}]=\begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & f(t_0)\\ w_{10} & w_{11} & \cdots & w_{1N} & ; & f(t_1)\\ \vdots & \vdots & \ddots & \vdots & & \vdots\\ w_{(N-1)0} & w_{(N-1)1} & \cdots & w_{(N-1)N} & ; & f(t_{N-1})\\ z_{00} & z_{01} & \cdots & z_{0N} & ; & c\end{pmatrix}.$$

It is important to note that, If rank $\widetilde{W}=\text{rank}[\widetilde{W};\widetilde{G}]=N+1$ the linear system (3.6) has a unique solution and the matrix of Hermite coefficients A is determined by $A=\widetilde{W}^{-1}\widetilde{G}$. In another case, if determinant i.e. $|\widetilde{W}|=0$ and rank $\widetilde{W}=\text{rank}[\widetilde{W};\widetilde{G}]<N+1$, then we may obtain the particular solutions. But, no solution exists if rank $\widetilde{W}\neq\text{rank}[\widetilde{W};\widetilde{G}]$.

4. Numerical Examples

Example 4.1. Consider the following multi-pantograph delay equations [6]:

$$z'(t)=\frac{1}{2}z(t)+\frac{1}{2}e^{t/2}z\left(\frac{t}{2}\right),\quad 0\le t\le 3 \tag{4.1}$$

with initial conditions

$$z(0)=1. \tag{4.2}$$

The exact solution of this equation is $z(t)=e^t$. In order to obtain its approximate solution for $N=7$ we apply the presented methodology by expressing the solution in the form of truncated Hermite series:

$$z_7(t)=\sum_{j=0}^{7}a_jH_j(t)=H(t)A.$$

Using the collocation points for $N = 7$, which are calculated as:

$$\left\{0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, 1\right\}.$$

The matrix equation of this example is:

$$[H(t)M^T - \lambda H(t) - \mu_1 H(q_1 t)]A = G = 0$$

or

$$WA = G = 0,$$

where

$$W = H(t)M^T - \lambda H(t) - \mu_1 H\left(\frac{t}{2}\right).$$

Here $\lambda = \frac{1}{2}$, $\mu_1 = \frac{1}{2}e^{t/2}$, $q_1 = \frac{1}{2}$, $g(t) = 0$.

$$\lambda = \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix},$$

$$M^T = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mu_1 = \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 602/1121 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 631/1094 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1138/1837 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1523/2289 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 611/855 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 799/1041 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1225/1486 \end{bmatrix},$$

$$H(t) = \begin{bmatrix} 1 & 0 & -2 & 0 & 12 & 0 & -120 & 0 \\ 1 & 2/7 & -94/49 & -580/343 & 6517/591 & 4303/258 & -9601/91 & -18653/81 \\ 1 & 4/7 & -82/49 & -1112/343 & 4741/579 & 4531/148 & -6632/103 & -14146/35 \\ 1 & 6/7 & -62/49 & -1548/343 & 1858/499 & 7820/199 & -1989/560 & -44138/93 \\ 1 & 8/7 & -34/49 & -1840/343 & -1393/708 & 68361/1681 & 2183/33 & -136917/332 \\ 1 & 10/7 & 2/49 & -1940/343 & -4587/551 & 3569/107 & 11650/89 & -43719/205 \\ 1 & 12/7 & 46/49 & -1800/343 & -3745/256 & 4243/251 & 9815/56 & 8687/89 \\ 1 & 2 & 2 & -4 & -20 & -8 & 184 & 464 \end{bmatrix},$$

$$H\left(\frac{t}{2}\right)=\begin{bmatrix} 1 & 0 & -2 & 0 & 12 & 0 & -120 & 0 \\ 1 & 1/7 & -97/49 & -293/343 & 6924/589 & 5491/645 & -6864/59 & -57251/482 \\ 1 & 2/7 & -94/49 & -580/343 & 6517/591 & 4303/258 & -9601/91 & -18653/81 \\ 1 & 3/7 & -89/49 & -855/343 & 3116/317 & 3285/136 & -22250/253 & -15067/46 \\ 1 & 4/7 & -82/49 & -1112/343 & 4741/579 & 4531/148 & -6632/103 & -14146/35 \\ 1 & 5/7 & -73/49 & -1345/343 & 2983/486 & 10047/281 & -4695/131 & -22278/49 \\ 1 & 6/7 & -62/49 & -1548/343 & 1858/499 & 7820/199 & -1989/560 & -44138/93 \\ 1 & 1 & -1 & -5 & 1 & 41 & 31 & -461 \end{bmatrix}.$$

Thus, we obtain W as follows:

$$W=\begin{bmatrix} -1 & 2 & 2 & -12 & -12 & 120 & 120 & -1680 \\ -2325/2242 & 2092/1175 & 2051/648 & -3072/301 & -1217/48 & 2434/25 & 5992/19 & -35050/27 \\ -589/547 & 2129/1374 & 3362/795 & -13080/1757 & -1492/41 & 6379/112 & 28086/61 & -20395/36 \\ -787/703 & 2399/1837 & 3895/751 & -2123/560 & -4714/107 & 1078/411 & 22696/43 & 13667/35 \\ -5335/4578 & 867/827 & 1327/220 & 578/855 & -6112/129 & -2234/37 & 27875/56 & 35031/25 \\ -781/643 & 928/1197 & 2183/323 & 6398/1089 & -8003/176 & -13677/109 & 11894/33 & 56603/25 \\ -488/385 & 355/732 & 3260/443 & 2684/229 & -5066/135 & -7581/41 & 17102/145 & 88615/32 \\ -984/743 & 261/1486 & 2762/353 & 2827/156 & -8057/353 & -37687/164 & -63853/299 & 89893/33 \end{bmatrix}.$$

The augmented matrix for the problem is:

$$[W;G]=\begin{bmatrix} -1 & 2 & 2 & -12 & -12 & 120 & 120 & -1680 & ; & 0 \\ -2325/2242 & 2092/1175 & 2051/648 & -3072/301 & -1217/48 & 2434/25 & 5992/19 & -35050/27 & ; & 0 \\ -589/547 & 2129/1374 & 3362/795 & -13080/1757 & -1492/41 & 6379/112 & 28086/61 & -20395/36 & ; & 0 \\ -787/703 & 2399/1837 & 3895/751 & -2123/560 & -4714/107 & 1078/411 & 22696/43 & 13667/35 & ; & 0 \\ -5335/4578 & 867/827 & 1327/220 & 578/855 & -6112/129 & -2234/37 & 27875/56 & 35031/25 & ; & 0 \\ -781/643 & 928/1197 & 2183/323 & 6398/1089 & -8003/176 & -13677/109 & 11894/33 & 56603/25 & ; & 0 \\ -488/385 & 355/732 & 3260/443 & 2684/229 & -5066/135 & -7581/41 & 17102/145 & 88615/32 & ; & 0 \\ -984/743 & 261/1486 & 2762/353 & 2827/156 & -8057/353 & -37687/164 & -63853/299 & 89893/33 & ; & 0 \end{bmatrix}.$$

The matrix form for initial condition is:

$$[z_0;c_0]=\begin{bmatrix} 1 & 0 & -2 & 0 & 12 & 0 & -120 & 0 \end{bmatrix}.$$

By using the initial condition the new augmented matrix can be obtained as follows:

$$[W;G]=\begin{bmatrix} -1 & 2 & 2 & -12 & -12 & 120 & 120 & -1680 & ; & 0 \\ -2325/2242 & 2092/1175 & 2051/648 & -3072/301 & -1217/48 & 20835/214 & 5992/19 & -35050/27 & ; & 0 \\ -589/547 & 2129/1374 & 3713/878 & -13080/1757 & -1492/41 & 6379/112 & 28086/61 & -20395/36 & ; & 0 \\ -787/703 & 2399/1837 & 3895/751 & -2123/560 & -4714/107 & 1537/586 & 22696/43 & 13667/35 & ; & 0 \\ -5335/4578 & 867/827 & 1327/220 & 578/855 & -6112/129 & -2234/37 & 27875/56 & 35031/25 & ; & 0 \\ -781/643 & 928/1197 & 2183/323 & 6492/1105 & -8003/176 & -13677/109 & 11894/33 & 56603/25 & ; & 0 \\ -488/385 & 1178/2429 & 3260/443 & 2684/229 & -5066/135 & -7581/41 & 17102/145 & 88615/32 & ; & 0 \\ 1 & 0 & -2 & 0 & 12 & 0 & -120 & 0 & ; & 1 \end{bmatrix}.$$

Solving this system, the unknown Hermite coefficients vector is found as:

$$A=\left[\frac{1103}{863} \quad \frac{2317}{3547} \quad \frac{717}{4714} \quad \frac{55}{1752} \quad \frac{133}{66278} \quad \frac{137}{194211} \quad \frac{-14}{831489} \quad \frac{1}{116718}\right]^T.$$

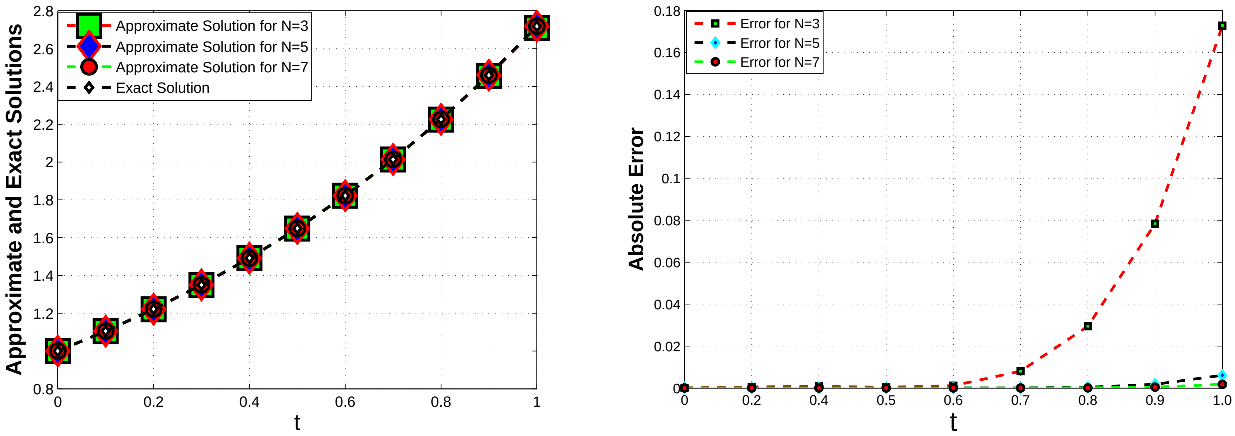
With the help of these coefficients and Hermite polynomial for $N=7$ as the approximate solution can be computed as.

$$[1,2t,4t^2-2,8t^3-12t,16t^4-48t^2+12,32t^5-160t^3+120t,$$
$$64t^6-480t^4+720t^2-120,128t^7-1344t^5+3360t^3-1680t].$$

By comparing the Exact and Approximate solutions the absolute error obtained for $N=3,5$ and 7 is presented in Table 1 and is graphically described in Figure 1.

Table 1. Absolute Error for Example 4.1

t	$\lvert Z(t)-z(t)\rvert$ $N=3$	$N=5$	$N=7$
0.00	9.99201E-16	8.88178E-16	4.10783E-14
0.20	5.33161E-04	1.77992E-06	2.37037E-07
0.40	8.05588E-04	1.42060E-06	3.21991E-07
0.60	3.59186E-04	2.51928E-06	3.81246E-07
0.80	1.13874E-03	8.63957E-07	6.54700E-07
1.00	8.02391E-03	3.69735E-05	3.09254E-06



(a) (b)

Figure 1. (a) represents the approximate and exact solutions at different value of N while (b) represents the absolute error at different value of N.

Example 4.2. Consider the multi-pantograph delay equation [24].

$$z'(t) = -\frac{5}{6}z(t) + 4z\left(\frac{t}{2}\right) + 9z\left(\frac{t}{3}\right) + t^2 - 1, \quad t \geq 0 \tag{4.3}$$

with initial conditions

$$z(0) = 1. \tag{4.4}$$

The exact solution of this equation is $z(t) = 1 + \frac{67}{6}t + \frac{1675}{72}t^2 + \frac{12157}{1296}t^3$. Equation (4.3) implies $\lambda = -\frac{5}{6}$, $\mu_1 = 4$, $\mu_2 = 9$, $q_1 = \frac{1}{2}$, $q_2 = \frac{1}{3}$ and $g(t) = t^2 - 1$. Using the collocation points $t = \frac{i}{N}$, the matrix equation of this example is:

$$[H(t)M^T - \lambda H(t) - \mu_1 H(q_1 t) - \mu_2 H(q_2 t)]A = G = t^2 - 1$$

or

$$WA = G = t^2 - 1,$$

where

$$W = H(t)M^T - \lambda H(t) - \mu_1 H(q_1 t) - \mu_2 H(q_2 t).$$

For $N = 5$, we have

$$\lambda = \begin{bmatrix} -5/6 & 0 & 0 & 0 & 0 & 0 \\ 0 & -5/6 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5/6 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5/6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5/6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5/6 \end{bmatrix},$$

$$M^T = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mu_1 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \quad \mu_2 = \begin{bmatrix} 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix},$$

$$H(t) = \begin{bmatrix} 1 & 0 & -2 & 0 & 12 & 0 \\ 1 & 2/5 & -46/25 & -292/125 & 1627/161 & 4887/215 \\ 1 & 4/5 & -34/25 & -536/125 & 2956/625 & 16073/422 \\ 1 & 6/5 & -14/25 & -684/125 & -2004/625 & 11140/279 \\ 1 & 8/5 & 14/25 & -688/125 & -7604/625 & 11767/479 \\ 1 & 2 & 2 & -4 & -20 & -8 \end{bmatrix},$$

$$H\left(\frac{t}{2}\right) = \begin{bmatrix} 1 & 0 & -2 & 0 & 12 & 0 \\ 1 & 1/5 & -49/25 & -149/125 & 3468/301 & 5932/501 \\ 1 & 2/5 & -46/25 & -292/125 & 1627/161 & 4887/215 \\ 1 & 3/5 & -41/25 & -423/125 & 4881/625 & 5113/161 \\ 1 & 4/5 & -34/25 & -536/125 & 2956/625 & 16073/422 \\ 1 & 1 & -1 & -5 & 1 & 41 \end{bmatrix},$$

$$H\left(\frac{t}{3}\right) = \begin{bmatrix} 1 & 0 & -2 & 0 & 12 & 0 \\ 1 & 2/15 & -446/225 & -875/1097 & 1992/169 & 1511/190 \\ 1 & 4/15 & -434/225 & -1951/1234 & 1617/145 & 2687/172 \\ 1 & 2/5 & -46/25 & -292/125 & 1627/161 & 4887/215 \\ 1 & 8/15 & -386/225 & -1073/352 & 3181/367 & 3191/110 \\ 1 & 2/3 & -14/9 & -100/27 & 556/81 & 8312/243 \end{bmatrix}.$$

Thus, we obtain W as follows:

$$W = \begin{bmatrix} -73/6 & 2 & 73/3 & -12 & -146 & 120 \\ -73/6 & 1/3 & 1931/75 & -17109/16451 & -6335/39 & 1997/1879 \\ -73/6 & -4/3 & 2009/75 & 296/25 & -15917/93 & -28667/188 \\ -73/6 & -3 & 2059/75 & 666/25 & -45026/267 & -31057/94 \\ -73/6 & -14/3 & 2081/75 & 1084/25 & -10879/72 & -4117/8 \\ -73/6 & -19/3 & 83/3 & 62 & -1030/9 & -18320/27 \end{bmatrix}.$$

The augmented matrix for the problem is:

$$[W;G] = \begin{bmatrix} -73/6 & 2 & 73/3 & -12 & -146 & 120 & ; & -1 \\ -73/6 & 1/3 & 1931/75 & -17109/16451 & -6335/39 & 1997/1879 & ; & -24/25 \\ -73/6 & -4/3 & 2009/75 & 296/25 & -15917/93 & -28667/188 & ; & -21/25 \\ -73/6 & -3 & 2059/75 & 666/25 & -45026/267 & -31057/94 & ; & -16/25 \\ -73/6 & -14/3 & 2081/75 & 1084/25 & -10879/72 & -4117/8 & ; & -9/25 \\ -73/6 & -19/3 & 83/3 & 62 & -1030/9 & -18320/27 & ; & 0 \end{bmatrix}.$$

The matrix form for initial condition is:

$$[z_0; c_0] = \begin{bmatrix} 1 & 0 & -2 & 0 & 12 & 0 \end{bmatrix}.$$

By using the initial condition the new augmented matrix can be obtained as follows:

$$[W;G]=\begin{bmatrix} -73/6 & 2 & 73/3 & -12 & -146 & 120 & ; & -1 \\ -73/6 & 1/3 & 1931/75 & -17109/16451 & -6335/39 & 1997/1879 & ; & -24/25 \\ -73/6 & -4/3 & 2009/75 & 296/25 & -15917/93 & -28667/188 & ; & -21/25 \\ -73/6 & -3 & 2059/75 & 666/25 & -45026/267 & -31057/94 & ; & -16/25 \\ -73/6 & -14/3 & 2081/75 & 1084/25 & -10879/72 & -4117/8 & ; & -9/25 \\ 1 & 0 & -2 & 0 & 12 & 0 & ; & 1 \end{bmatrix}.$$

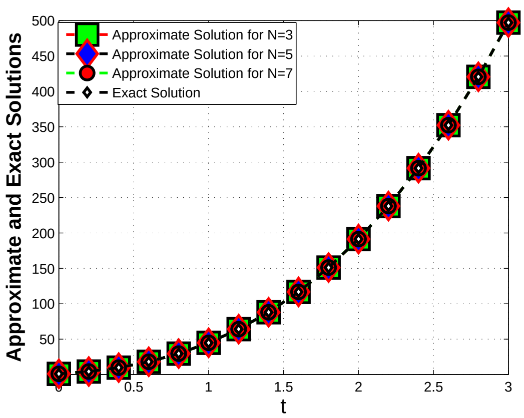
Solving this system, the unknown Hermite coefficients vector is found as:

$$A=\begin{bmatrix} 2539/201 \\ 2549/202 \\ 1832/315 \\ 890/759 \\ -4/531023 \\ 1/573568 \end{bmatrix}.$$

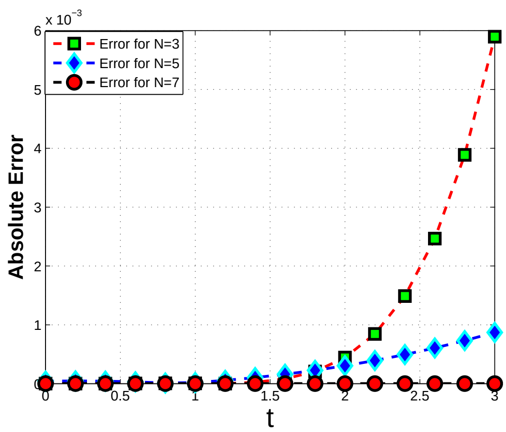
By applying the Hermite collocation method, for $N = 3$, 4 and 7, the approximate solution is also calculated. Later, it is compared with exact solution in order to obtain the absolute error. The absolute errors obtained for $N = 3,4$ and 7 are tabulated in Table 2 and is graphically described in Figure 2.

Table 2. Absolute Error for Example 4.2

t	$\lvert Z(t)-z(t)\rvert$		
	$N=3$	$N=5$	$N=7$
0.00	3.99680E-15	4.26040E-05	3.99680E-14
0.50	1.84323E-06	3.67841E-05	2.45031E-10
1.00	5.37369E-06	1.84480E-05	2.46523E-10
1.50	4.51151E-05	1.29901E-04	4.35122E-10
2.00	4.42980E-04	3.04382E-04	1.48663E-09
2.50	1.92844E-03	5.48701E-04	1.37845E-09
3.00	5.89781E-03	8.69664E-04	2.62413E-08



(a)



(b)

Figure 2. (a) represents the approximate and exact solutions at different value of N while (b) represents the absolute error at different value of N.

Example 4.3. Consider the following multi-pantograph equation [15]:

$$z'(t) = -z(t) + 0.25z(0.5t) - 0.25e^{(-0.5t)}, \quad t \geq 0 \tag{4.5}$$

with initial conditions

$$z(0) = 1. \tag{4.6}$$

The exact solution of this equation is $z(t) = e^{-t}$. Here $\lambda = -1$, $\mu_1 = \frac{1}{4}$, $q_1 = \frac{1}{2}$ and $g(t) = -\frac{1}{4}e^{\frac{-t}{2}}$. Using the collocation points $t = \frac{i}{N}$ the matrix equation of this example is:

$$[H(t)M^T + H(t) - \mu_1 H(q_1 t)]A = G = -\frac{1}{4}e^{\frac{-t}{2}}$$

or

$$WA = G = -\frac{1}{4}e^{\frac{-t}{2}},$$

where

$$W = H(t)M^T + H(t) - \mu_1 H(q_1 t).$$

For $N = 3$, we have

$$\lambda = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad M^T = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mu_1 = \begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix},$$

$$H(t) = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 2/3 & -14/9 & -100/27 \\ 1 & 4/3 & -2/9 & -152/27 \\ 1 & 2 & 2 & -4 \end{bmatrix}, \quad H\left(\frac{t}{2}\right) = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 1/3 & -17/9 & -53/27 \\ 1 & 2/3 & -14/9 & -100/27 \\ 1 & 1 & -1 & -5 \end{bmatrix}.$$

Thus we obtain W as follows:

$$W = \begin{bmatrix} 3/4 & 2 & -3/2 & -12 \\ 3/4 & 31/12 & 19/12 & -1355/108 \\ 3/4 & 19/6 & 11/2 & -163/27 \\ 3/4 & 15/4 & 41/4 & 37/4 \end{bmatrix}.$$

The augmented matrix for the problem is:

$$[W;G] = \begin{bmatrix} 3/4 & 2 & -3/2 & -12 & ; & -1/4 \\ 3/4 & 31/12 & 19/12 & -1355/108 & ; & -397/1876 \\ 3/4 & 19/6 & 11/2 & -163/27 & ; & -376/2099 \\ 3/4 & 15/4 & 41/4 & 37/4 & ; & -274/1807 \end{bmatrix}.$$

The matrix form for initial condition is:

$$[z_0; c_0] = \begin{bmatrix} 1 & 0 & -2 & 0 \end{bmatrix}.$$

By using the initial condition the new augmented matrix can be obtained as follows:

$$[W;G] = \begin{bmatrix} 3/4 & 2 & -3/2 & -12 & ; & -1/4 \\ 3/4 & 31/12 & 19/12 & -1355/108 & ; & -397/1876 \\ 3/4 & 19/6 & 11/2 & -163/27 & ; & -376/2099 \\ 1 & 0 & -2 & 0 & ; & 1 \end{bmatrix}.$$

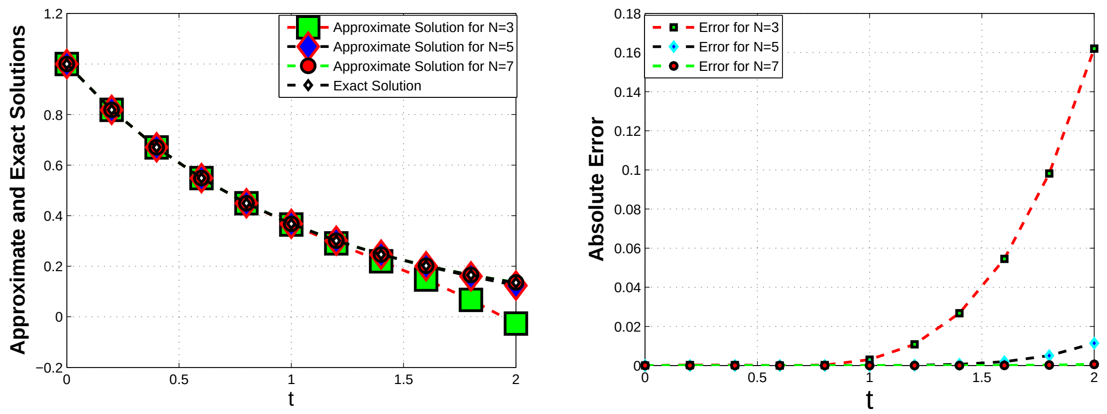
Solving this system, the unknown Hermite coefficients vector is found as:

$$A = \begin{bmatrix} 956/769 \\ -1093/1849 \\ 236/1941 \\ -131/8625 \end{bmatrix}.$$

By applying the Hermite collocation method for $N = 5$ and 7 the approximate solution is also calculated. Later, it is compared with exact solution in order to obtain the absolute error. The absolute errors obtained for $N = 3$, 5 and 7 are tabulated in Table 3 and is graphically described in Figure 3.

Table 3. Absolute Error for Example 4.3

t	$\lvert Z(t) - z(t)\rvert$		
	$N = 3$	$N = 5$	$N = 7$
0.00	9.99201E-16	2.88658E-15	1.99840E-14
0.50	1.32295E-04	4.86271E-07	1.63040E-08
1.00	3.03959E-03	1.34588E-05	2.54904E-07
1.50	3.89361E-02	1.11731E-03	4.83368E-05
2.00	1.62005E-01	1.14402E-02	6.42735E-04



(a) (b)

Figure 3. (a) represents the approximate and exact solutions at different value of N while (b) represents the absolute error at different value of N.

5. Conclusion

The Hermite collocation method is presented for solving the Multi-Pantograph Equations. The results obtained by the present methodology reveals that this technique is very accurate and effective. The numerical results demonstrates that the accuracy improves by increasing the number of collocation points. It is observed that, by increasing the value of N, the errors start decreasing more rapidly. Therefore, for better results,using large number N is suggested.

Another significant advantage of the technique is that Hermite coefficients of the solution are discovered very easily using the computer based programs.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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