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Research Article

A Subdivision Approach to the Approximate Solution of 3rd Order Boundary Value Problem

S.A. Manan[1], A. Ghaffar[2,*], M. Rizwan[3], G. Rahman[2] and G. Kanwal[1]

[1] *Department of Mathematics, SBK Women University, Queeta, Pakistan*
[2] *Department of Mathematical Science, BUITEMS Quetta, Pakistan*
[3] *Department of General Studies, Jubail Industrial College, Jubail, Kingdom of Saudi Arabia*
***Corresponding author:** abdulghaffar.jaffar@gmail.com

Abstract. An algorithm to solve 3rd order boundary value problem is focused in this paper which is 8-point approximating scheme. It concludes the results with stability and convergence that is evaluated with the illustration of numerical example. This paper also contains the analysis of approximation properties for the mentioned collocation algorithm.

Keywords. Subdivision scheme; Boundary value problem; Convergence; Stability

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1. Introduction

Subdivision scheme being an algorithmic technique and generating smooth curves with surface too is to have through refinement of the curve by inserting new points at each level to the early existing control polygon where the sequences of inserting points may remain existed within the control polygon and the insertion of any number of points on an edge from one level to another is known as arity. The insertion of $2,3,\dots,n$ points are termed as binary, ternary up to n-ary, respectively.

Subdivision got an introduction to the field of research (de Rham [2]) producing c_0 continuous curves with corner cutting method. Dyn *et al*. [3] studied interpolating 4-point scheme incorporating tension parameter. In 2009, all the m-point binary schemes got a unified form with the general formula presented by Mustafa *et al*. [7]. Aslam *et al*. [1] contributed to the unification of all the odd point ternary schemes indexed with the mask of $(2n-1)$. An extensive form for the generalization of mask of 3-point a-ary ($a \geq 2$) approximating subdivision scheme is presented by Ghaffar *et al*. [5]. Ghaffar and Mustafa [6] introduced an algorithm to produce a family of even point ternary approximating subdivision scheme for curve design. The Chaikin's perturbation scheme presented by Dyn [4] is revised in its non-stationary form by Salam *et al*. [16] considering ω as a tension parameter that efficiently reproduces geometrical shapes such as circles, ellipses and hyperbolic shapes of the analytical curves.

An approximate solution to the b.v.p (*boundary value problem*) of order 2 is presented and reformulated by Qu and Agarwal [14] introducing interpolating scheme. Mustafa and Ejaz [8] formulated the results for the solution of 3rd order b.v.p using 8-point interpolating scheme producing smooth curves. Mustafa and Ejaz [10] numerically solved singularly perturbed b.v.p of order 3 and also discussed the convergence of the constructed collocation algorithm. Mustafa *et al*. [9] presented numerically approached schemes of subdivision for the solution of non linear fourth order b.v.p and explored its uses in the field of engineering and physics.

The plan of this article is as follows: Section 2 includes the general form of approximating subdivision scheme with some of it's important results. It is then proceeded by presenting 8-point binary approximating subdivision scheme along with the derivatives of its basis function in Section 4. A numerical approximating scheme is formulated with the proper adjustment of required boundary conditions in Section 3. Numerical example with the graphical representation of the solution is presented to evaluate the performance of the scheme in Section 5. Concluding remarks and future research insight is the part of last section.

2. Approximating Subdivision Scheme for Curve Design

This section is devoted to have an approximate solution of two point b.v.p of order three with the use of 8-point approximating subdivision scheme to reformulate the collocation algorithm. It has an approximation power of $O(h^2)$. Consider the following two point b.v.p of order three that is being dealt with by collocation algorithm:

$$\begin{aligned} &y'''(x) = a(x)y(x) + b(x), \qquad 0 \leq x \leq 1, \\ &y(0) = y_r, \quad y'(0) = 0, \quad y(1) = y_l, \end{aligned} \tag{2.1}$$

where $a(x)$ is positive and for $0 \leq x \leq 1$, $a(x)$ and $b(x)$ are continuous.

8-point approximating subdivision scheme is given by:

$$\begin{aligned} q_{2i}^{j+1} &= \frac{11}{2048}(q_{i-2}^j + q_{i+4}^j) + \frac{83}{1024}(q_{i-1}^j + q_{i+3}^j) + \frac{517}{2048}(q_i^j + q_{i+2}^j) + \frac{165}{512}(q_{i+1}^j), \\ q_{2i+1}^{j+1} &= \frac{1}{2048}(q_{i-3}^j + q_{i+4}^j) + \frac{55}{2048}(q_{i-2}^j + q_{i+3}^j) + \frac{341}{2048}(q_{i-1}^j + q_{i+2}^j) + \frac{627}{2048}(q_i^j + q_{i+1}^j). \end{aligned} \tag{2.2}$$

Remark 1. Let $\Phi(x)$ be the limit curve generated from the cardinal data $\{p_i = (i, \delta_0)^T\}$; that is, $\Phi(x)$ is the fundamental solution of the subdivision scheme (2.1); then

$$\Phi(i) = \begin{cases} 1, & i = 0 \\ 0, & i \neq 0. \end{cases} \tag{2.3}$$

Furthermore, $\Phi(x)$ satisfies the following two scale equation:

$$\begin{aligned}\Phi(x) &= \Phi_n(x) \\ &= \Phi(2x) + \sum_{j=-n}^{n} L_{n,|j|}\Phi(2x-2j+1), \quad x \in \mathbb{R}.\end{aligned} \tag{2.4}$$

Lemma 2.1 (Qu [11], Qu and Agarwal [12]). *The support of the fundamental solution $\Phi_n(x)$ to scheme* (2.2) *is finite. Explicitly, support $\Phi_n(x) = (-2n-1, 2n+1)$.*

Lemma 2.2 (Qu and Agarwal [13]). *Given a square matrix A of order n, let the normalized left and right* (*generalized*) *eigenvectors of A be denoted by $\{\eta_i, \xi_i\}$. Then for any vector $f \in R^n$, there exists following Fourier expansion:*

$$f = \sum_{i=1}^{n} (f^T \eta_i)\xi_i\,. \tag{2.5}$$

Lemma 2.3. *Suppose $F(t)$, $t \in \mathbb{R}$ is a regular and C^{2n+2} curve in R^m, $m \ge 2$. Let $p(t)$, $t \in \mathbb{R}$ be the limit curve generated by* (2.2) *from the initial data $p_i = F(ih)$, $i \in \mathbb{Z}$, $0 < h < 1$. Then on any finite interval $[a,b]$, the following estimates hold:*

$$\begin{aligned}&\|F(ht) - p(t)\|_\infty \le \frac{M_{2n+2}(F)}{(2n+2)!} h^{2n+2} = O(h^{2n+2}), \\ &\|h^j F^j(ht) - p^j(t)\|_\infty = O(h^{2n+2-j}), \quad j = 0,1,2,\dots,\frac{n+2}{2},\end{aligned} \tag{2.6}$$

where the number $M_{2n+2}(F)$ depends only on the derivatives of $F(t)$ and n.

2.1 The 8-Point Approximating Scheme

The local subdivision matrix of the considered scheme is given by

$$\begin{bmatrix}
K_{3,3} & K_{3,2} & K_{3,1} & K_{3,0} & K_{3,1} & K_{3,2} & K_{3,3} & 0 & 0 & 0 & 0 & 0 & 0 \\
L_{3,3} & L_{3,2} & L_{3,1} & L_{3,0} & L_{3,0} & L_{3,1} & L_{3,2} & L_{3,3} & 0 & 0 & 0 & 0 & 0 \\
0 & K_{3,3} & K_{3,2} & K_{3,1} & K_{3,0} & K_{3,1} & K_{3,2} & K_{3,3} & 0 & 0 & 0 & 0 & 0 \\
0 & L_{3,3} & L_{3,2} & L_{3,1} & L_{3,0} & L_{3,0} & L_{3,1} & L_{3,2} & L_{3,3} & 0 & 0 & 0 & 0 \\
0 & 0 & K_{3,3} & K_{3,2} & K_{3,1} & K_{3,0} & K_{3,1} & K_{3,2} & K_{3,3} & 0 & 0 & 0 & 0 \\
0 & 0 & L_{3,3} & L_{3,2} & L_{3,1} & L_{3,0} & L_{3,0} & L_{3,1} & L_{3,2} & L_{3,3} & 0 & 0 & 0 \\
0 & 0 & 0 & K_{3,3} & K_{3,2} & K_{3,1} & K_{3,0} & K_{3,1} & K_{3,2} & K_{3,3} & 0 & 0 & 0 \\
0 & 0 & 0 & L_{3,3} & L_{3,2} & L_{3,1} & L_{3,0} & L_{3,0} & L_{3,1} & L_{3,2} & L_{3,3} & 0 & 0 \\
0 & 0 & 0 & 0 & K_{3,3} & K_{3,2} & K_{3,1} & K_{3,0} & K_{3,1} & K_{3,2} & K_{3,3} & 0 & 0 \\
0 & 0 & 0 & 0 & L_{3,3} & L_{3,2} & L_{3,1} & L_{3,0} & L_{3,0} & L_{3,1} & L_{3,2} & L_{3,3} & 0 \\
0 & 0 & 0 & 0 & 0 & K_{3,3} & K_{3,2} & K_{3,1} & K_{3,0} & K_{3,1} & K_{3,2} & K_{3,3} & 0 \\
0 & 0 & 0 & 0 & 0 & L_{3,3} & L_{3,2} & L_{3,1} & L_{3,0} & L_{3,0} & L_{3,1} & L_{3,2} & L_{3,3} \\
0 & 0 & 0 & 0 & 0 & 0 & K_{3,3} & K_{3,2} & K_{3,1} & K_{3,0} & K_{3,1} & K_{3,2} & K_{3,3}
\end{bmatrix}, \tag{2.7}$$

where

$$\begin{aligned}&K_{3,3} = \frac{11}{2048}, \qquad K_{3,2} = \frac{83}{1024}, \qquad K_{3,1} = \frac{517}{2048}, \qquad K_{3,0} = \frac{165}{512}, \\ &L_{3,3} = \frac{1}{2048}, \qquad L_{3,2} = \frac{55}{2048}, \qquad L_{3,1} = \frac{341}{2048}, \qquad L_{3,0} = \frac{627}{2048}.\end{aligned}$$

Some of the eigenvalues of S are given by

$$\lambda = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512}, \frac{1}{1024}, \frac{1}{2048}, \frac{-1}{2048}. \tag{2.8}$$

The corresponding normalized left and right eigenvectors being presented by ξ and η and also holding the conditions that $S\xi = \lambda\xi$ and $\eta S^T = \eta\lambda$ are given by

$$\begin{aligned}
\xi_0 &= (1,1,1,1,1,1,1,1,1,1,1,1,1)^T, \\
\eta_0 &= \left(\frac{1}{119750400}\right)(1,2037,154674,2358161,12094239,27665850,35200476,27665850, \\
&\quad 12094239,2358161,154674,2037,1)^T, \qquad (2.9) \\
\xi_1 &= (6,5,4,3,2,1,0,-1,-2,-3,-4,-5,-6)^T, \\
\eta_1 &= \left(\frac{1}{-2917890}\right)(-1,-1013,-47840,-455191,-1309341,-1262514,0,1262514,1309341, \\
&\quad 455191,47840,1013,1)^T, \\
\xi_2 &= (103,70,43,22,7,-2,-5,-2,7,22,43,70,103)^T, \\
\eta_2 &= \left(\frac{1}{6531840}\right)(1,501,14106,73625,67455,-82062,-147252,-82062,67455,73625, \\
&\quad 14106,501,1)^T, \\
\xi_3 &= (-93,-50,-22,-6,1,2,0,-2,-1,6,22,50,93)^T, \\
\eta_3 &= \left(\frac{1}{362880}\right)(-1,-245,-3800,-7279,11571,15126,0,-15126,-11571,7279,3800,245,1)^T.
\end{aligned} \tag{2.10}$$

It is known that $\xi^T\eta_j = 1$ in case $i = j$ and is 0 for $i \neq j$ then the following consequences can be drawn with the use of theorem (2.1) and (2.2).

Lemma 2.4. *The fundamental solution* (*Cardinal basis*) $\phi(x)$ *is thrice continuously differentiable and supported on* $(-7,7)$ *and its derivatives at integers are given by*

$$\left.\begin{aligned}
\Phi'(i) &= 2\,\mathrm{sign}(i) e_{|i|}^T \eta_1, \\
\Phi''(i) &= 2^2 e_{|i|}^T \eta_2, \\
\Phi'''(i) &= 2^3\,\mathrm{sign}(i) e_{|i|}^T \eta_3
\end{aligned}\right\}, \quad -6 \le i \le 6, \tag{2.11}$$

where

$$\left.\begin{aligned}
&e_0 = (0,0,0,0,0,0,1,0,0,0,0,0,0)^T, && e_1 = (0,0,0,0,0,1,0,0,0,0,0,0,0)^T, \\
&e_2 = (0,0,0,0,1,0,0,0,0,0,0,0,0)^T, && e_3 = (0,0,0,1,0,0,0,0,0,0,0,0,0)^T, \\
&e_4 = (0,0,1,0,0,0,0,0,0,0,0,0,0)^T, && e_5 = (0,1,0,0,0,0,0,0,0,0,0,0,0)^T, \\
&e_6 = (1,0,0,0,0,0,0,0,0,0,0,0,0)^T,
\end{aligned}\right\} \tag{2.12}$$

$$\left.\begin{aligned}
&\Phi'(0) = 0, && \Phi'(\pm 1) = \pm\frac{1262514}{1458945}, && \Phi'(\pm 2) = \pm\frac{436447}{486315}, \\
&\Phi'(\pm 3) = \mp\frac{455191}{1458945}, && \Phi'(\pm 4) = \pm\frac{9568}{291789}, && \Phi'(\pm 5) = \pm\frac{1013}{1458945}, \\
&\Phi'(\pm 6) = \pm\frac{1}{1458945},
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&\Phi''(0)=-\frac{1753}{19440}, \quad \Phi''(\pm1)=-\frac{4559}{90720}, \quad \Phi''(\pm2)=\frac{1499}{36288},\\
&\Phi''(\pm3)=\frac{14725}{326592}, \quad \Phi''(\pm4)=\frac{2351}{272160}, \quad \Phi''(\pm5)=\frac{167}{544320},\\
&\Phi''(\pm6)=\frac{1}{1632960},
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&\Phi'''(0)=0, \quad \Phi'''(\pm1)=\pm\frac{2521}{7560}, \quad \Phi'''(\pm2)=\pm\frac{3857}{15120},\\
&\Phi'''(\pm3)=\mp\frac{7279}{45360}, \quad \Phi'''(\pm4)=\mp\frac{95}{1134}, \quad \Phi'''(\pm5)=\mp\frac{49}{9072},\\
&\Phi'''(\pm6)=\mp\frac{1}{45360}.
\end{aligned}\right\} \tag{2.13}$$

3. Numerical Approximating Collocation Algorithm

This section leads to the formulation of numerical approximation by using collocation algorithm. The algorithm will solve 2-point third order boundary value problems. This require an accurate adjustment of the boundary conditions to find unique solution.

3.1 The Collocation Algorithm

Let N be a positive integer ($N \ge 6$), $h = 1/N$, and $x_i = i/N = ih$, $i = 0,1,2,3,\dots,N$, and set $a_i = a(x_i)$, $b_i = b(x_i)$. Let

$$Z(x)=\sum_{i=-6}^{N+6} Z_i\phi\left(\frac{x-x_i}{h}\right), \quad 0\le x\le 1 \tag{3.1}$$

be the approximate solution to (2.1), where Z_i are the unknown to be determined by (2.1). The collocation algorithm, together with the boundary conditions to be discussed, is given by the setting

$$Z'''(x_j)=a(x_j)Z(x_j)+b(x_j), \quad j=0,1,2,3,\dots,N, \tag{3.2}$$

$$\left.\begin{aligned}
Z'(x_j)&=\frac{1}{h}\sum_{i=-6}^{N+6} Z_i\phi'\left(\frac{x_j-x_i}{h}\right),\\
Z''(x_j)&=\frac{1}{h^2}\sum_{i=-6}^{N+6} Z_i\phi''\left(\frac{x_j-x_i}{h}\right),\\
Z'''(x_j)&=\frac{1}{h^3}\sum_{i=-6}^{N+6} Z_i\phi'''\left(\frac{x_j-x_i}{h}\right).
\end{aligned}\right\} \tag{3.3}$$

Using (3.1) and (3.2) in (3.3), we get following $N+1$ system of equations:

$$\begin{aligned}
&\frac{1}{h^3}\sum_{i=-6}^{N+6} Z_i\phi'''\left(\frac{x_j-x_i}{h}\right)-a_j\sum_{i=-6}^{N+6} z_i\phi\left(\frac{x_j-x_i}{h}\right)=b_j,\\
\Rightarrow\ &\sum_{i=-6}^{N+1} z_i\Phi'''\left(\frac{x_j-x_i}{h}\right)-h^3a_j\sum_{i=-6}^{N+1} z_i\Phi\left(\frac{x_j-x_i}{h}\right)=h^3b_j, \quad j=0,1,2,\dots,N.
\end{aligned} \tag{3.4}$$

Now, we simplify the above system in the following theorems.

Theorem 3.1. *For $j=0$ (3.4) gives,*

$$\begin{aligned} &z_{-6}\Phi_6''' + z_{-5}\Phi_5''' + z_{-4}\Phi_4''' + z_{-3}\Phi_3''' + z_{-2}\Phi_2''' + z_{-1}\Phi_1''' + z_0 q_0 \\ &\quad + z_1\Phi_{-1}''' + z_2\Phi_{-2}''' + z_3\Phi_{-3}''' + z_4\Phi_{-4}''' + z_5\Phi_{-5}''' + z_6\Phi_{-6}''' = h^3 b_0, \end{aligned} \tag{3.5}$$

where $\Phi_j''' = \Phi'''(j)$ and $q_0 = \Phi_0''' - a_0h^3$.

Proof. Since

$$\begin{aligned} &\left\{z_{-6}\Phi'''\left(\frac{x_j - x_{-6}}{h}\right) + z_{-5}\Phi'''\left(\frac{x_j - x_{-5}}{h}\right) + z_{-4}\Phi'''\left(\frac{x_j - x_{-4}}{h}\right) + \dots\right. \\ &\quad \left. + z_{N+4}\Phi'''\left(\frac{x_j - x_{N+4}}{h}\right) + z_{N+5}\Phi'''\left(\frac{x_j - x_{N+5}}{h}\right) + z_{N+6}\Phi'''\left(\frac{x_j - x_{N+6}}{h}\right)\right\} \\ &\quad - a_0h^3\left\{z_{-6}\Phi\left(\frac{x_j - x_{-6}}{h}\right) + z_{-5}\Phi\left(\frac{x_j - x_{-5}}{h}\right) + z_{-4}\Phi'''\left(\frac{x_j - x_{-4}}{h}\right) + \dots\right. \\ &\quad \left. + z_{N+4}\Phi\left(\frac{x_j - x_{N+4}}{h}\right) + z_{N+5}\Phi\left(\frac{x_j - x_{N+5}}{h}\right) + z_{N+6}\Phi\left(\frac{x_j - x_{N+6}}{h}\right)\right\} = h^3 b_0. \end{aligned} \tag{3.6}$$

Substituting $j=0$ in (3.4), we get

$$\begin{aligned} &\left\{z_{-6}\Phi'''\left(\frac{x_0 - x_{-6}}{h}\right) + z_{-5}\Phi'''\left(\frac{x_0 - x_{-5}}{h}\right) + z_{-4}\Phi'''\left(\frac{x_0 - x_{-4}}{h}\right) + \dots\right. \\ &\quad \left. + z_{N+4}\Phi'''\left(\frac{x_0 - x_{N+4}}{h}\right) + z_{N+5}\Phi'''\left(\frac{x_0 - x_{N+5}}{h}\right) + z_{N+6}\Phi'''\left(\frac{x_0 - x_{N+6}}{h}\right)\right\} \\ &\quad - a_0h^3\left\{z_{-6}\Phi\left(\frac{x_0 - x_{-6}}{h}\right) + z_{-5}\Phi\left(\frac{x_0 - x_{-5}}{h}\right) + z_{-4}\Phi'''\left(\frac{x_0 - x_{-4}}{h}\right) + \dots\right. \\ &\quad \left. + z_{N+4}\Phi\left(\frac{x_0 - x_{N+4}}{h}\right) + z_{N+5}\Phi\left(\frac{x_0 - x_{N+5}}{h}\right) + z_{N+6}\Phi\left(\frac{x_0 - x_{N+6}}{h}\right)\right\} = h^3 b_0. \end{aligned} \tag{3.7}$$

For $x_j = jh$, $j = 0,1,2,3,\dots,N$, this implies

$$\begin{aligned} &z_{-6}\Phi'''(6) + z_{-5}\Phi'''(5) + z_{-4}\Phi'''(4) + \dots + z_{N+5}\Phi'''(-N-5) + z_{N+6}\Phi'''(-N-6) \\ &\quad + a_0h^3[z_{-6}\Phi(6) + z_{-5}\Phi(5) + z_{-4}\Phi(4) + \dots + z_{N+4}\Phi(-N-4) + z_{N+5}\Phi(-N-5) \\ &\quad + z_{N+6}\Phi(-N-6)] = h^3 b_0. \end{aligned} \tag{3.8}$$

Since the support of the basis function $\Phi(x)$ is $(-7,7)$, $\Phi'(x)$, $\Phi''(x)$, and $\Phi'''(x)$, are zero outside the interval $(-7,7)$; also by (2.4) and (2.13), we get

$$\begin{aligned} &z_{-6}\Phi'''(6) + z_{-5}\Phi'''(5) + z_{-4}\Phi'''(4) + z_{-3}\Phi'''(3) + z_{-2}\Phi'''(2) + z_{-1}\Phi'''(1) \\ &\quad + z_0\Phi'''(0) + z_1\Phi'''(-1) + z_2\Phi'''(-2) + z_3\Phi'''(-3) + z_4\Phi'''(-4) + z_5\Phi'''(-5) \\ &\quad + z_6\Phi'''(-6)\ - a_0h^3 z_0\Phi(0) = h^3 b_0. \end{aligned} \tag{3.9}$$

If $\Phi_j''' = \Phi'''(j)$, then

$$\begin{aligned} &z_{-6}\Phi_6''' + z_{-5}\Phi_5''' + z_{-4}\Phi_4''' + z_{-3}\Phi_3''' + z_{-2}\Phi_2''' + z_{-1}\Phi_1''' + z_0(\Phi_0''' - a_0h^3) \\ &\quad + z_1\Phi_{-1}''' + z_2\Phi_{-2}''' + z_3\Phi_{-3}''' + z_4\Phi_{-4}''' + z_5\Phi_{-5}''' + z_6\Phi_{-6}''' = h^3 b_0. \end{aligned} \tag{3.10}$$

For $q_0 = \Phi_0''' - a_0h^3$, we get (3.5). This completes the proof. □

Theorem 3.2. *For $j = 1,2,3,\dots,N$, the system* (3.4) *is equivalent to*

$$\begin{aligned} &z_{-6}\Phi_{j+6}''' + z_{-5}\Phi_{j+5}''' + \dots + z_0\Phi_j''' + z_1(\Phi_{j-1}''' - a_jh^3\Phi_{j-1}) + z_2(\Phi_{j-2}''' - a_jh^3\Phi_{j-2}) + \dots \\ &\quad + z_{N-1}(\Phi_{j-N+1}''' - a_jh^3\Phi_{j-N+1}) + z_N(\Phi_{j-N}''' - a_jh^3\Phi_{j-N}) + z_{N+1}\Phi_{j-N-1}''' + \dots \\ &\quad + z_{N+6}\Phi_{j-N-6}''' = h^3 b_j. \end{aligned} \tag{3.11}$$

Proof. Expending (3.4) gives

$$\Big\{z_{-6}\Phi'''\left(\frac{x_j-x_{-6}}{h}\right)+z_{-5}\Phi'''\left(\frac{x_j-x_{-5}}{h}\right)+z_{-4}\Phi'''\left(\frac{x_j-x_{-4}}{h}\right)+\dots$$
$$+z_{N+4}\Phi'''\left(\frac{x_j-x_{N+4}}{h}\right)+z_{N+5}\Phi'''\left(\frac{x_j-x_{N+5}}{h}\right)+z_{N+6}\Phi'''\left(\frac{x_j-x_{N+6}}{h}\right)\Big\}$$
$$-a_0h^3\Big\{z_{-6}\Phi\left(\frac{x_j-x_{-6}}{h}\right)+z_{-5}\Phi\left(\frac{x_j-x_{-5}}{h}\right)+z_{-4}\Phi'''\left(\frac{x_j-x_{-4}}{h}\right)+\dots$$
$$+z_{N+4}\Phi\left(\frac{x_j-x_{N+4}}{h}\right)+z_{N+5}\Phi\left(\frac{x_j-x_{N+5}}{h}\right)+z_{N+6}\Phi\left(\frac{x_j-x_{N+6}}{h}\right)\Big\}=h^3b_j. \tag{3.12}$$

For $x_j=jh$, $j=1,2,3,\dots,N$, we have

$$z_{-6}\Phi'''(j+6)+z_{-5}\Phi'''(j+5)+\dots+z_{N+5}\Phi'''(j-N-5)+z_{N+6}\Phi'''(j-N-6)$$
$$-a_jh^3\{z_{-6}\Phi(j+6)+z_{-5}\Phi(j+5)+\dots+z_{N+5}\Phi(j-N-5)+z_{N+6}\Phi(j-N-6)\}$$
$$=h^3b_j. \tag{3.13}$$

If $\Phi_j'''=\Phi'''(j)$ for $j=1,2,3,\dots,N$, then

$$z_{-6}(\Phi_{j+6}'''-a_jh^3\Phi_{j+6})+z_{-5}(\Phi_{j+5}'''-a_jh^3\Phi_{j+5})+\dots$$
$$+z_{N+5}(\Phi_{j-N-5}'''-a_jh^3\Phi_{j-N-5})+z_{N+6}(\Phi_{j-N-6}'''-a_jh^3\Phi_{j-N-6})=h^3b_j. \tag{3.14}$$

As the derivatives of Φ are zero at the points non-existent in the interval $(-7,7)$ then the use of (2.3) and (2.13) gives (3.11). From (3.5) and (3.11), we get (3.15) consisting of $(N+1)$ equations having $(N+13)$ unknowns i.e. z_i

$$AZ=D, \tag{3.15}$$

where A, Z and D are the matrices having orders $(N+1)\times(N+13)$, $N+13$, and $N+1$, respectively are given by

$$A=\begin{bmatrix}
\Phi_6''' & \Phi_5''' & \Phi_4''' & \Phi_3''' & \Phi_2''' & \Phi_1''' & q_0 & \Phi_{-1}''' & \Phi_{-2}''' & \Phi_{-3}''' & \Phi_{-4}''' & \Phi_{-5}''' & \Phi_{-6}''' & \dots & 0 & 0 & 0\\
0 & \Phi_6''' & \Phi_5''' & \Phi_4''' & \Phi_3''' & \Phi_2''' & \Phi_1''' & q_1 & \Phi_{-1}''' & \Phi_{-2}''' & \Phi_{-3}''' & \Phi_{-4}''' & \Phi_{-5}''' & \dots & 0 & 0 & 0\\
0 & 0 & \Phi_6''' & \Phi_5''' & \Phi_4''' & \Phi_3''' & \Phi_2''' & \Phi_1''' & q_2 & \Phi_{-1}''' & \Phi_{-2}''' & \Phi_{-3}''' & \Phi_{-4}''' & \dots & 0 & 0 & 0\\
0 & 0 & 0 & \Phi_6''' & \Phi_5''' & \Phi_4''' & \Phi_3''' & \Phi_2''' & \Phi_1''' & q_3 & \Phi_{-1}''' & \Phi_{-2}''' & \Phi_{-3}''' & \dots & 0 & 0 & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \Phi_{-5}''' & \Phi_{-6}''' & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \Phi_{-4}''' & \Phi_{-5}''' & \Phi_{-6}'''
\end{bmatrix},$$

$$Z=(z_{-6},z_{-5},z_{-4},z_{-3},z_{-2},\dots,z_{N+6})$$

and

$$D=(b_0h^3,b_1h^3,b_2h^3,b_3h^3,\dots,b_Nh^3),$$

where $\Phi_j'''=\Phi'''(j)$ and $q_j=\Phi_0'''-a_jh^3$. □

3.2 Adjustment of Boundary Conditions

The determination of unique solution for the system of equations require twelve more boundary conditions. An improvement of results with the proper adjustment of these boundary conditions can be made.

Case 1. In case of letting $z'_0 = 0$, we obtain two conditions with the use of given boundary conditions

$$z_0 = y_r, \quad z'_0 = 0, \quad z_N = y_l. \tag{3.16}$$

Ten conditions are still required for the determination of unknowns. These conditions are defined by considering septic polynomial defining boundary conditions of order eight for the unique solution of (3.15) that is specified at the left and right end points $z_{-5}, z_{-4}, z_{-3}, z_{-2}, z_{-1}$ and $z_{N+1}, z_{N+2}, z_{N+3}, z_{N+4}, z_{N+5}$, respectively.

Since

$$z_{-i} = q(-x_i), \quad i = 1,2,3,4,5, \tag{3.17}$$

where

$$q(x_i) = \sum_{j=1}^{8} \binom{8}{j} (-1)^{j+1} z(x_{i-j}). \tag{3.18}$$

As with reference to (3.1), $Z(x_i) = z_i$ for $i = 1,2,3,4,5$ then substituting x_i by $-x_i$ gives

$$q(-x_i) = \sum_{j=1}^{8} \binom{8}{j} (-1)^{j+1} z_{-i+j}. \tag{3.19}$$

Hence at the left end following boundary conditions can be used:

$$\sum_{j=0}^{8} \binom{8}{j} (-1)^{j} z_{-i+j} = 0, \quad i = 5,4,3,2,1. \tag{3.20}$$

For the right end similarly, we can explain $z_i = q(-x_i)$, $i = N+1, N+2, N+3, N+4, N+5$, and

$$q(x_i) = \sum_{j=1}^{8} \binom{8}{j} (-1)^{j+1} z_{i-j}. \tag{3.21}$$

Thus, at the right end we have following boundary conditions:

$$\sum_{j=0}^{8} \binom{8}{j} (-1)^{j} z_{i-j} = 0, \quad i = N+1, N+2, N+3, N+4, N+5. \tag{3.22}$$

That eventually make a required system of $(N+13)$ equations having $(N+13)$ unknowns. Among this system of $(N+13)$ equations, $(N+1)$ equations are employed by (3.5) and (3.11), two of them by the boundary conditions (3.16) and remaining ten equations from (3.20) and (3.22).

$$BZ = R, \quad \text{where } B = (B_0^T, A^T, B_1^T)^T, \tag{3.23}$$

A is employed with the use of (3.16), and B_0 and B_1 are given by (3.16), (3.20), (3.22).

$$B_0 = \begin{bmatrix} 0 & 1 & -8 & -28 & -56 & 70 & -56 & -28 & -8 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -8 & -28 & -56 & 70 & -56 & -28 & -8 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & -8 & -28 & -56 & 70 & -56 & -28 & -8 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -8 & -28 & -56 & 70 & -56 & -28 & -8 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -8 & -28 & -56 & 70 & -56 & -28 & -8 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

in B_0 matrix the first five rows are obtained by using (3.20) and the sixth row is given by (3.16) i.e.

$$z_0 = y_r$$

and

$$B_1 = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 1 & -8 & -28 & -56 & 70 & -56 & -28 & -8 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & -8 & -28 & -56 & 70 & -56 & -28 & -8 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & -8 & -28 & -56 & 70 & -56 & -28 & -8 & 1 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & -8 & -28 & -56 & 70 & -56 & -28 & -8 & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & -8 & -28 & -56 & 70 & -56 & -28 & -8 & 1 & 0 \end{bmatrix},$$

$$Z = (z_{-6}, z_{-5}, z_{-4}, \dots, z_{N+4}, z_{N+5}, z_{N+6})^T,$$
$$R = (0,0,0,0,0,y_r,D^T,y_l,0,0,0,0,0)^T. \tag{3.24}$$

Case 2. In this case we express the given boundary condition $z'_0 = 0$ in the following way. By using (3.3), we have

$$Z'(x_j) = \frac{1}{h}\sum_{i=-6}^{N+6} z_i \Phi'\left(\frac{x_j - x_i}{h}\right). \tag{3.25}$$

As we defined earlier $x_j = jh$ if we put $j=0$ we get $x_0 = 0$; the above equation can be written as

$$Z'(0) = \frac{1}{h}\sum_{i=-6}^{N+6} z_i \phi'(-i). \tag{3.26}$$

Since the boundary condition $z'_0 = Z'(0) = 0$,

$$\sum_{i=-6}^{N+6} z_i \phi'(-i) = 0 \tag{3.27}$$

i.e.

$$\phi'(6)z_{-6} + \phi'(5)z_{-5} + \phi'(4)z_{-4} + \phi'(3)z_{-3} + \phi'(2)z_{-2} + \phi'(1)z_{-1} + \phi'(0)z_0$$
$$+ \phi'(-1)z_1 + \phi'(-2)z_2 + \phi'(-3)z_3 + \phi'(-4)z_4 + \phi'(-5)z_5 + \phi'(-6)z_6 = 0.$$

By using (2.13) we can express above equation as

$$\frac{1}{1458945}z_{-6} + \frac{1013}{1458945}z_{-5} + \frac{9568}{291789}z_{-4} + \frac{455191}{1458945}z_{-3} + \frac{436447}{486315}z_{-2} + \frac{1262514}{1458945}z_{-1}$$
$$-\frac{1262514}{1458945}z_1 - \frac{436447}{486315}z_2 - \frac{455191}{1458945}z_3 - \frac{9568}{291789}z_4 - \frac{1013}{1458945}z_5 - \frac{1}{1458945}z_6 = 0. \tag{3.28}$$

We eventually obtain a new system of $(N+13)$ equations with $(N+13)$ unknowns. Among the system of $(N+13)$ equations, in which $(N+1)$ are employed with the use of (3.5) and (3.11), two of them are obtained by (3.16) and ten equations come from (3.20) for $i = 2,3,4,5$, (3.22), and (3.28):

$$BZ = R, \tag{3.29}$$

here $B = (\mathbb{B}^T, A^T, B_1^T)^T$, A is given by (3.16), and $\mathbb{B}$ and B_1 are obtained by (3.16), (3.20), (3.22) and (3.28)

$$\mathbb{B} = \begin{bmatrix} 0 & 1 & -8 & -28 & -56 & 70 & -56 & -28 & -8 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -8 & -28 & -56 & 70 & -56 & -28 & -8 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & -8 & -28 & -56 & 70 & -56 & -28 & -8 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -8 & -28 & -56 & 70 & -56 & -28 & -8 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -8 & -28 & -56 & 70 & -56 & -28 & -8 & 1 & \dots & 0 & 0 \\ \frac{1}{1458945} & \frac{1013}{1458945} & \frac{9568}{291789} & \frac{455191}{1458945} & \frac{436447}{486315} & \frac{420838}{486315} & 0 & -\frac{420838}{486315} & -\frac{436447}{486315} & -\frac{455191}{1458945} & -\frac{9568}{291789} & -\frac{1013}{1458945} & -\frac{1}{1458945} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{3.30}$$

in matrix $\mathbb{B}$ the first fours rows are employed from (3.20) for $i = 2,3,4,5$, fifth row is obtained from (3.28) and the last row of $\mathbb{B}$ comes from (3.16). The matrix B_1 remains the same as discussed in Case I and the matrices Z and R are given by

$$Z = (z_{-6}, z_{-5}, \dots, z_{N+5}, z_{N+6})^T, \tag{3.31}$$

$$R = (0,0,0,0,y'(0), y_r, D^T, y_l, 0,0,0,0,0). \tag{3.32}$$

4. Error Estimation

This section is consisted of the discussion for the properties of approximation of proposed scheme. As the scheme (2.2) regenerates seven degree polynomial then in accordance with proposed scheme of Dyn *et al*. [4] possesses approximation order eight. Thus the collocation algorithm (3.1) treated with the end points septic precision possess the approximation power $O(h^2)$. Main result for estimation of error is presented by the given proposition.

Proposition 4.1. *Let $y(x) \in c^8[0,1]$ be the exact solution and using* (3.22) *or* (3.29)*, z_i are calculated with 8th order boundary condition at the end points then*

$$\|\mathrm{err}(x)\|_\infty = \|Z^j - y^j\|_\infty = O(h^{2-j}), \quad j = 0,1,2, \tag{4.1}$$

for j being the order of derivative.

Proof. As the order of approximation of the proposed scheme is eight then with the use of (4.12), we can write for small h and smooth function $y(x)$ as

$$\begin{aligned} y'''(x_j) = \frac{2^3}{362880h^3}\{&-y(x_j-6h) - 245y(x_j-5h) - 3800y(x_j-4h) - 7279y(x_j-3h) \\ &+ 11571y(x_j-2h) + 15126y(x_j-h) - 15126y(x_j+h) - 11571y(x_j+2h) \\ &+ 7279y(x_j+3h) + 3800y(x_j+4h) + 245y(x_j+5h) + y(x_j+6h)\}. \end{aligned} \tag{4.2}$$

This can be written as

$$\begin{aligned} y'''(x_j) = \frac{2^3}{362880h^3}\{&-y_{j-6} - 245y_{j-5} - 3800y_{j-4} - 7279y_{j-3} + 11571y_{j-2} + 15126y_{j-1} \\ &- 15126y_{j+1} - 11571y_{j+2} + 7279y_{j+3} + 3800y_{j+4} + 245y_{j+5} + y_{j+6}\}. \end{aligned} \tag{4.3}$$

Similarly, we have

$$\begin{aligned} Z'''(x_j) = \frac{2^3}{362880h^3}\{&-z_{j-6} - 245z_{j-5} - 3800z_{j-4} - 7279z_{j-3} + 11571z_{j-2} + 15126z_{j-1} \\ &- 15126z_{j+1} - 11571z_{j+2} + 7279z_{j+3} + 3800z_{j+4} + 245z_{j+5} + z_{j+6}\}. \end{aligned} \tag{4.4}$$

If we define error function $e(x) = Z(x) - y(x)$ and error vectors at the joint points by

$$e(x_j) = Z(x_j) - y(x_j + jh), \quad -6 \le j \le N+6, \tag{4.5}$$

or equivalently $e_j = Z_j - y(x_j + jh)$, $-6 \le j \le N+6$, then this implies

$$\left.\begin{aligned} e'_j &= Z'_j - y'(x+jh), \\ e''_j &= Z''_j - y''(x+jh), \\ e'''_j &= Z'''_j - y'''(x+jh). \end{aligned}\right\} \tag{4.6}$$

By subtracting (4.3) from (4.4), we get

$$
\begin{aligned}
Z'''(j)-y'''(j)=\frac{2^3}{362880h^3}\{&-1(z_{j-6}-y_{j-6})-245(z_{j-5}-y_{j-5})-3800(z_{j-4}-y_{j-4})\\
&+7279(z_{j-3}-y_{j-3})+11571(z_{j-2}-y_{j-2})+15126(z_{j-1}-y_{j-1})\\
&-15126(z_{j+1}-y_{j+1})-11571(z_{j+2}-y_{j+2})+7279(z_{j+3}-y_{j+3})\\
&+3800(z_{j+4}-y_{j+4})+245(z_{j+5}-y_{j+5})+(z_{j+6}-y_{j+6})\}.
\end{aligned}
\tag{4.7}
$$

This implies

$$
\begin{aligned}
e'''(x_j)=\frac{2^3}{362880h^3}\{&-e_{j-6}-245e_{j-5}-3800e_{j-4}-7279e_{j-3}+11571e_{j-2}+15126e_{j-1}\\
&-15126e_{j+1}-11571e_{j+2}+7279e_{j+3}+3800e_{j+4}+245e_{j+5}+e_{j+6}\}.
\end{aligned}
\tag{4.8}
$$

By Lemma 2.4, we get the following expressions:

$$
\begin{aligned}
e_j'''=\frac{1}{h^3}\{&\phi_6'''e_{j-6}+\phi_5'''e_{j-5}+\phi_4'''e_{j-4}+\phi_3'''e_{j-3}+\phi_2'''e_{j-2}+\phi_1'''e_{j-1}+\phi_0'''e_j\\
&+\phi_{-1}'''e_{j+1}+\phi_{-2}'''e_{j+2}+\phi_{-3}'''e_{j+3}+\phi_{-4}'''e_{j+4}+\phi_{-5}'''e_{j+5}+\phi_6'''e_{j+6}\}+O(h^8),
\end{aligned}
\tag{4.9}
$$

where $j=0,1,2,3,\dots,N$. By subtracting (2.1) from (3.2), we get

$$
Z_j'''-y_j'''=a_j(Z_i-y_j). \tag{4.10}
$$

This implies

$$
e_j'''=a_je_j,\quad 0\le i\le N. \tag{4.11}
$$

Using (4.9), we get

$$
\begin{aligned}
&\phi_6'''e_{j-6}+\phi_5'''e_{j-5}+\phi_4'''e_{j-4}+\phi_3'''e_{j-3}+\phi_2'''e_{j-2}+\phi_1'''e_{j-1}+\phi_0'''e_j\\
&\quad+\phi_{-1}'''e_{j+1}+\phi_{-2}'''e_{j+2}+\phi_{-3}'''e_{j+3}+\phi_{-4}'''e_{j+4}+\phi_{-5}'''e_{j+5}+\phi_6'''e_{j+6}=0
\end{aligned}
\tag{4.12}
$$

where $j=0,1,2,3,\dots,N$ and $q_j=\phi_0''-h^2a_j$.

As $0\le x\le 1$ and $x_j=jh$, $j=0,1,2,3,\dots,N$ so $e_0,e_1,\dots,e_N$ are not zero however $e_{-6},e_{-5},\dots,e_{-1}$ and $e_{N+1},e_{N+2},\dots,e_{N+6}$ are zero as they lie away from the interval $[0,1]$. Thus equation (4.9) equal to

$$
(B+O(h^6)E=0, \tag{4.13}
$$

where the matrix $B+O(h^6)$ is achieved by removing the first and four columns and rows of the coefficient matrix B, where

$$
E=(e_{-6},e_{-5},e_{-4},\dots,e_{N+5},e_{N+6})^T. \tag{4.14}
$$

By using (4.9)

$$
(B+O(h^6)E=O(h^8)\|Z(x_j)-y(x_j\|, \tag{4.15}
$$

$$
O(h^8\|E\|=O(h^8). \tag{4.16}
$$

Hence, the matrix $B+O(h^6)$ will be invariant for small h and from linear algebra we get standard result, so we have

$$
\|E\|\le\left(\frac{\|B^{-1}\|}{1-O(h^6)}O(h^8)\right)=O(h^2). \tag{4.17}
$$

This completes the proof. □

5. Numerical Example

This section is consisted of analyzing a numerical collocation algorithm by using 8-point approximating subdivision scheme for the solution of 2-point third order boundary value problem. Absolute errors are calculated to find variation in between the analytic solution and approximate solution. Results are tabulated to make possible comparisons.

Example 1. Let

$$y'''(x) = y(x) - 3e^x, \quad 0 < x < 1, \tag{5.1}$$

be the given boundary value problem with the given boundary conditions $y'(0) = 0$, $y(1) = 0$, $y(0) = 1$. The analytical solution of the considered problem is given by

$$y(x) = (1-x)e^x. \tag{5.2}$$

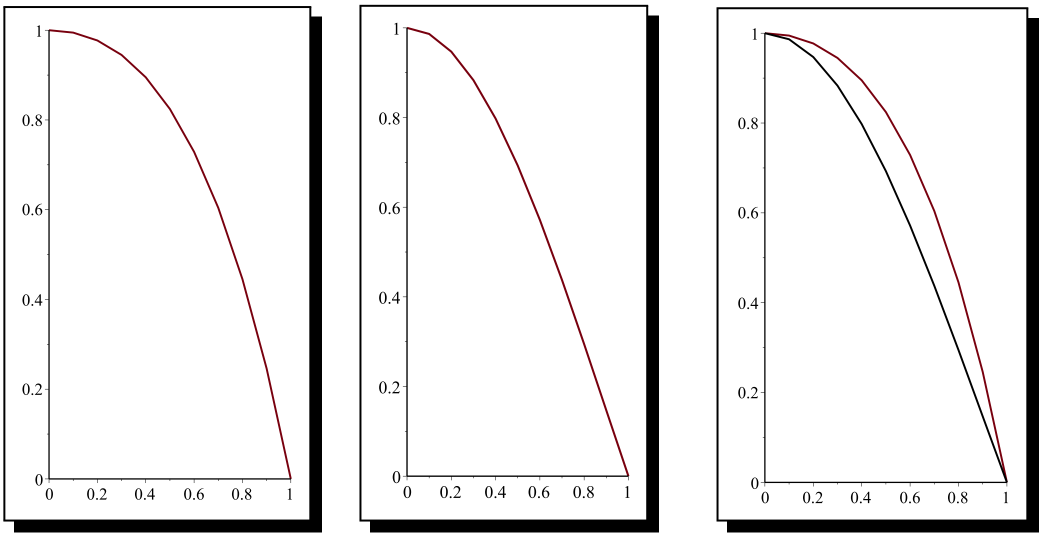
The solution of collocation algorithm for $N = 10$ as $Z_j = \sum_{i=-6}^{16} z_i \Phi(j-i)$, the values of $z_{-6}, z_{-5}, \dots, z_{15}, z_{16}$ with the use of (3.29) are

$$\begin{aligned}&0.721722429952530, 0.804605935885371, 0.873027594213488, 0.927257336555298,\\ &0.966849405211294, 0.991293461459553, 1, 0.992284825950610, 0.967352389985740,\\ &0.924277777206709, 0.861987145334778, 0.779236407816646, 0.674587957432587,\\ &0.546385310621080, 0.392725231459815, 0.211427220964437, 0, -.244395385899840,\\ &-.524986211510930, -.845432838399004, -1.20987797460736, -1.62300079925686,\\ &-2.09069536880952\end{aligned} \tag{5.3}$$

Boundary adjustments given in equation (3.28) gives the approximate solution of (3.29) presented in Table 1 with the absolute errors calculated. These solutions are graphically represented in Figure 1. Figure 1(a) represents the analytic solution and the approximate solution is given by Figure 1(b). The comparison between the analytic and approximate solution is represented by Figure 1(c).

Table 1. Solutions and errors estimation of Example 1

x_i	Analytic sol Y	Approximate sol Z by (3.29)	Absolute err by (3.29)
0	1	1	0
0.1	0.9946538262	0.992284825950610	0.014766299
0.2	0.9771222064	0.967352389985740	0.028679549
0.3	0.9449011656	0.924277777206709	0.040878161
0.4	0.8950948188	0.861987145334778	0.050483997
0.5	0.8243606355	0.779236407816646	0.056590557
0.6	0.7288475200	0.674587957432587	0.058259925
0.7	0.6041258121	0.546385310621080	0.054507376
0.8	0.4451081856	0.392725231459815	0.026518309
0.9	0.2459603111	0.211427220964437	0.03453309014
1	0	0	0



(a) Analytic solution (b) Approximate solution (c) Analytic and approximate solution

Figure 1. Comparison between analytic and approximate solution

6. Conclusion and Future Work

In this research we present stationary 8-point approximating subdivision scheme to get an approach to the approximate solution of third order linear boundary value problem. It is observed that the selection of the scheme and adjustment of the boundary conditions effect the accuracy of the approximate solution. The proper choice for the selection of scheme and an appropriate setting of boundary conditions may cause flexibility for the improvement of the results. The proposed approximating scheme give more accurate results in comparison to the interpolating scheme and also to that of finite difference method of order two. The boundary conditions of the proposed scheme are brought into adjustment with the use of septic polynomial at the end points that make a system of equations for the solution of unknowns. Improvement in the approximation order of the scheme with the appropriate setting of boundary conditions is expected to give future research insight.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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