Communications in Mathematics and Applications
Vol. 9, No. 3, pp. 447–458, 2018
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v9i3.834

http://www.rgnpublications.com

***Proceedings*:**
3rd International Conference on Pure and Applied Mathematics
Department of Mathematics, University of Sargodha, Sargodha, Pakistan
November 10-11, 2017

Generalized Subdivision Surface Scheme Based on 2D Lagrange Interpolating Polynomial and its Error Estimation

Research Article

Muhammad Omar[1] and Faheem Khan[2,*]

[1] *Department of Mathematics, COMSATS University Islamabad, Sahiwal Campus, Pakistan*
[2] *Department of Mathematics, University of Sargodha, Sargodha, Pakistan*
***Corresponding author:** fahimscholar@gmail.com

Abstract. This work gives the idea for constructing subdivision rules for surface based on 2D Lagrange interpolating polynomial [13]. In this method, subdivision rules for quad mesh has been obtained directly from the Lagrange interpolating polynomial. We also see that the simple interpolatory subdivision scheme for quadrilateral nets with arbitrary topology is presented by L. Kobbelt [5], can be directly calculated from the proposed generalized formula for subdivision surface refinement rules. Furthermore, some characteristics, applications and error bounds of the proposed work are also discussed.

Keywords. Subdivision; Lagrange polynomial; Interpolation; Error bound

MSC. 65D17; 65D05; 65U07

Received: February 26, 2018 **Accepted:** May 3, 2018

1. Introduction

In 'Subdivision', an initial mesh of vertices and edges are required to start the process of refinement. Subdivision is currently one of the most powerful tool to model free form smooth shapes. In bivariate case, subdivision process generate smooth surfaces, which is important in designing aesthetically pleasing shapes. Subdivision schemes were first introduced by Catmull and Clark [1], and Doo and Sabin [2] in 1978. They gave the idea for genralized tensor product of B-spline of bi-degree three and two respectively. There are two general classes of subdivision schemes, namely, approximating and interpolating schemes. The limit curve of an approximating scheme usually does not pass through the control points of control polygon. As the level of refinement increases, the polygon usually shrinks towards the final limit curve. The interpolating schemes are more attractive than approximating schemes because of their interpolation property. All vertices in the control polygon are located on the limit curve of the interpolation scheme, which facilitates and simplifies the graphics algorithms and engineering designs.

Lian *et al*. [15] generalized the classical binary 4-point and 6-point interpolatory subdivision schemes to a-ary setting for any integer $a \geq 3$. After that they introduced the a-ary 3-point and 5-point interpolatory subdivision schemes for curve design for arbitrary odd integer $a \geq 3$ (see [14]). After that Lian *et al*. [16] investigate both the $2m$-point, a-ary for any $a \geq 2$ and $(2m+1)$-point, a-ary for any odd $a \geq 3$ interpolatory subdivision schemes for curve design. Ko [11] presented explicitly a new formula for the mask of $(2N+4)$-point binary interpolating and approximating subdivision schemes with two parameters. Recently, there has been tremendous progress in construction of subdivision rules and properties as well as their applications in multiresolution representation. The proposed work presents a new observation about bivariate case by using 2D Lagrange interpolating polynomial. In this work, we avoid in finding the mask of subdivision schemes separately, as a result its approach is simple and avoids complex computation when deriving subdivision rules. This work also provides some special cases of the classical subdivision schemes.

In the present paper, Section 2 gives some preliminaries results and a new relation for $(2N+4)$-point n-ary interpolating curve scheme for closed and open polygon to access main result. Section 3 presents the construction of general formula for the surface case using two dimensional Lagrange interpolating polynomial. In Section 4, comparison of the proposed subdivision schemes by estimating the error bounds of the derived schemes. In Section 5, we also give some numerical example for the visual performance the proposed work. Conclusion of the research work is provided in Section 6.

2. Preliminary results

The general form of univariate n-ary subdivision scheme which maps a control polygon $p^k = \{p_i^k\}_{i\in z}$ to refined polygon $p^{k+1} = \{p_i^{k+1}\}_{i\in z}$ is defined by

$$p_{ni+s}^{k+1} = \sum_{j\in z} a_{nj+s} p_{i-j}^k, \quad s = 0,1,2,\cdots,n-1, \tag{2.1}$$

where $\mathbb{Z}$ is the set of integers and $a = \{a_i \mid i \in z\}$ a set of constants is the mask of the subdivision scheme. A necessary condition for the uniform convergence of the subdivision scheme is

$$\sum_{j\in z} a_{nj+s} = 1, \quad s = 0,1,2,\cdots,n-1. \tag{2.2}$$

Let Υ_{2N+1} be the space of all polynomials of degree $\leq 2N+1$, where N is non-negative integer. If $\{L_\mu(x)\}_{\mu=-N}^{N+1}$ is fundamental Lagrange polynomial, then

$$p(x) = \sum_{\mu=-N}^{N+1} p(\mu)L_\mu(x), \quad p \in \Upsilon_{2N+1}, \tag{2.3}$$

where

$$L_\mu(x) = \begin{cases} \prod_{j=-N}^{N+1} \frac{x-j}{\mu-j}, & j \neq k, \\ \delta_{\mu,j}, & \mu, j = -N, \cdots, N+1 \end{cases}$$

and $\delta_{\mu,j}$ is the Kronecker delta symbol defined as

$$\delta_{\mu,j} = \begin{cases} 1, & \mu = j, \\ 0, & \mu \neq j, \end{cases}$$

Using (2.1)-(2.3), Ko [11] gave the general formula for the mask of $(2N+4)$-point binary interpolating symmetric subdivision schemes. After that Mustafa and Najma [17] presented the generalized form for the mask of $(2N+4)$-point n-ary interpolating symmetric subdivision schemes

$$\begin{cases} a_{nj} = \delta_{j,0} - v\xi_2(N,j) \\ a_{nj+s} = \xi_1(N,j,n,s) - a_{n(N+1)+s}\xi_2(N,j) - a_{n(N+1)+t}\xi_3(N,j), \end{cases} \tag{2.4}$$

where

$$\xi_1(N,j,n,s) = \frac{\prod_{b=-N-1}^{N}(ni+s)}{n^{2N+1}(-1)^{-j+N-1}(s+nj)(N-j)!(N+1+j)!},$$

$$\xi_2(N,j) = \frac{(-1)^{j+N}(2N+2)!}{(N-j)!(N+j+1)!(N-j+1)}, \quad \xi_3(N,j) = \frac{(-1)^{j+N+1}(2N+2)!}{(N-j)!(N+j+1)!(N+j+2)}$$

and

$$a_{n(N+1)+s} = a_{-n(N+2)-s}, \tag{2.5}$$

where $a_{n(N+1)+s}$ can be explicitly defined as

$$a_{n(N+1)+s} = \frac{(d-n)(d-2n)\cdots(d-(2Nn+3n))}{(-1)^{2N+3}(n)^{2N+3}(2N+3)!}, \quad d = n(N+1)+s.$$

Here, n is the arity of the subdivision scheme (i.e. for binary subdivision scheme $n=2$, ternary subdivision scheme $n=3$, quaternary subdivision scheme $n=4,\ldots$), N is non-negative integer, $j=-N-1,\cdots,N$, $s=0,1,2,\cdots n-1$ and $t=n-s$. Using (2.4) and the symmetry of the subdivision scheme, $(2N+4)$-point binary interpolating subdivision scheme takes the form

$$p_{2i+\alpha}^{k+1} = \sum_{l=-N-1}^{N+2} a_{2l-\alpha}p_{i+l}^k, N \geq 0, \quad \alpha = 0,1, \tag{2.6}$$

together with the symmetry condition $a_{-2(N+2)-\alpha} = a_{2(N+1)+\alpha}$.

Setting $a_{2(N+2)} = 0$ and $a_{2(N+1)} = v$, the masks $a_{2l-\alpha}$ come from (2.4). Hence, using (2.4)-(2.6) and $n \geq 2$, we have the following form of $2(N+4)$-point n-ary interpolating subdivision scheme

$$p_{ni+\alpha}^{k+1} = \sum_{l=-N-1}^{N+2} a_{nl-\alpha}p_{i+l}^k, \quad \text{where } \alpha = 0,1,2,\cdots,n-1, \tag{2.7}$$

together with the symmetry condition

$$a_{-n(N+1)-\alpha} = a_{n(N+1)+\alpha}, \tag{2.8}$$

and setting $a_{n(N+2)} = 0$, $a_{n(N+1)} = v$.

2.1 Construction of the Schemes for Open Polygon

In interpolating subdivision schemes, it is not possible to refine the end(first and last) edges of the initial open polygon $p^0 = \{p_i^0 : i = 0, \cdots, N\}$ by (2.7). Dealing with open polygons require a well-defined neighborhood of end points. So, it will be adequate to define the auxiliary point $p_{-i}^0 = 2p_0^0 - p_i^0$ to the initial open polygon p^0 as the extrapolatory rule. Therefore, the following rule is defined to refine the open polygon using $2N+4$-points interpolating scheme

$$p_{ni+\alpha}^{k+1} = \sum_{l=-N-1}^{-i-1} (2a_{nl-\alpha} p_0^k - a_{nl-\alpha} p_{-(i+l)}^k) + \sum_{l=-i}^{N+2} a_{nl-\alpha} p_{i+l}^k, \tag{2.9}$$

together with the symmetry condition

$$a_{-n(N+1)-\alpha} = a_{n(N+1)+\alpha} \tag{2.10}$$

where N is non-negative integer, $\alpha = 0, 1, 2, \ldots, n-1$, $i = 0, 1, \ldots, N$ and $n \geq 2$.

Example. To refine an open polygon using the 6-point ternary interpolating subdivision scheme by (2.9), define the auxiliary points $p_{-2}^0 = 2p_0^0 - p_2^0$ and $p_{-1}^0 = 2p_0^0 - p_1^0$ in the initial open polygon p^0. The first two edges p_0p_1 and p_1p_2 of the non-refined polygon $\{p_i^k, i = 0, \ldots, 3^k N\}$ can be refined by putting $N = 1$, $n = 3$ and $i = 0$ in (2.9).

$$p_0^{k+1} = (2a_{-6} + 2a_{-3} + a_0)p_0^k + (a_3 - a_{-3})p_1^k + (a_6 - a_{-6})p_2^k + a_9 p_3^k,$$
$$p_1^{k+1} = (2a_{-7} + 2a_{-4} + a_{-1})p_0^k + (a_2 - a_{-4})p_1^k + (a_5 - a_{-7})p_2^k + a_8 p_3^k,$$
$$p_2^{k+1} = (2a_{-8} + 2a_{-5} + a_{-2})p_0^k + (a_1 - a_{-5})p_1^k + (a_4 - a_{-8})p_2^k + a_7 p_3^k.$$

Remark 2.1. Here, we observe that some well known interpolating schemes can be obtained from our proposed result (2.7).

- By putting $N = 0$ in the proposed scheme (2.7), 4-point interpolatory scheme of Lian [15] is obtained.
- By setting $n = 2$, $N = v = a_4 = 0$ and $a_3 = \frac{-w}{16}$ for $0 < w < 2(\sqrt{5} - 1)$ in (2.7), the following 4-point interpolatory scheme of Dyn *et al*. [3] is obtained

$$\begin{cases} p_{2i}^{k+1} = p_i^k, \\ p_{2i+1}^{k+1} = \frac{8+w}{16}(p_i^k + p_{i+1}^k) - \frac{w}{16}(p_{i-1}^k + p_{i+2}^k). \end{cases}$$

3. Tensor Product of $(2N+4)$-point Interpolating Subdivision Schemes

Let $p_{i,j}^k \in \mathbb{R}^N$, $i, j \in \mathbb{Z}$, $N \geq 2$, be the set of control points, where $k \geq 2$ indicates the subdivision level. We define n-ary subdivision surface as tensor product of n-ary subdivision curves by

$$p_{ni+\alpha.nj+\beta}^{k+1} = \sum_{r=0}^{m} \sum_{s=0}^{m} a_{\alpha,r} a_{\beta,s} p_{i+r,j+s}^k, \quad \alpha, \beta = 0, 1, \cdots, n-1, \tag{3.1}$$

where, $a_{\alpha,r}$ and $a_{\beta,r}$ are the sets of subdivision masks which satisfy (2.2).

For $k=0$, $p^k_{i,j} \in \mathbb{R}^N$, $i,j \in \mathbb{Z}$, represent the initial points. As $k \to \infty$, the process (3.1) defines an infinite set of points in $\mathbb{R}^N$. The diadic mesh points $(\frac{i}{n^k}, \frac{j}{n^k})_{i,j\in\mathbb{Z}}$ are related to $\{p^k_{i,j}\}_{i,j\in\mathbb{Z}}$ in a natural way. The process then defines a scheme whereby $p^{k+1}_{ni+\alpha,nj+\beta}$ replaces the value $p^k_{i+\alpha/n,j+\beta/n}$ at the mesh point $(\frac{i+\alpha/n}{n^k}, \frac{j+\beta/n}{n^k})$ for $\alpha,\beta \in \{0,n\}$, while the values $p^{k+1}_{ni+\alpha,nj+\beta}$ are inserted at the new mesh points $(\frac{ni+\alpha}{n^{k+1}}, \frac{nj+\beta}{n^{k+1}})$ for $\alpha,\beta = 0,1,\cdots,n-1$ (where $\alpha=\beta\neq 0$). Figure 1 illustrates labeling of old and new points formed by subdivision scheme (3.1).

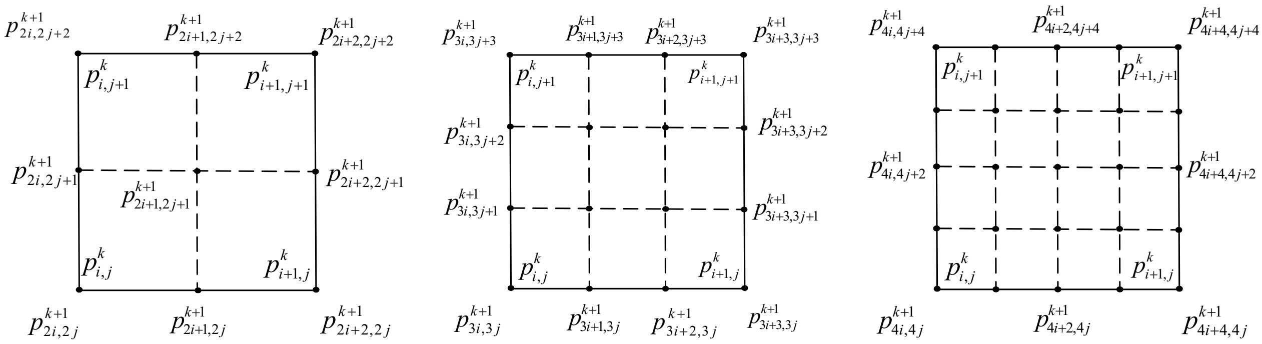


Figure 1. Solid lines show one face of initial polygon whereas doted lines are refined polygons which can be obtained by subdividing one face into four, nine and sixteen new faces using (3.1) for $n=2,3,4$, respectively.

3.1 Construction

Let $\Upsilon_{2\rho+1}$ and $\Upsilon_{2\sigma+1}$ be the space of all polynomials of degrees $\leq 2\rho+1$ and $\leq 2\sigma+1$, respectively. Dahlquist and Bjork [13] presented the Lagrange interpolating polynomial for tensor product

$$p(x,y) = \sum_{\mu=-\rho}^{\rho+1} \sum_{\nu=-\sigma}^{\sigma+1} L_{\mu,\nu}(x,y) p(\mu,\nu), \tag{3.2}$$

where

$$L_{\mu,\nu}(x,y) = \prod_{i=-\rho, i\neq\mu}^{\rho+1} \prod_{j=-\sigma, j\neq\nu}^{\nu+1} \frac{x-i}{\mu-i} \times \frac{x-j}{\nu-j}, \quad \mu=-\rho,\cdots,\rho+1,\ \nu=-\sigma,\cdots,\sigma+1. \tag{3.3}$$

To find the mask of a tensor product scheme, each point in the grid must be accessible from the origin. Tensor product schemes are simply the tensor product of the univariate case.

Mustafa and Najma [17] presented the general formula to generate the masks $\{a_i\}_{i=-2\gamma-3}^{2\gamma+3}$ and $\{a_j\}_{j=-2\sigma-3}^{2\sigma+3}$ for n-ary interpolating subdivision schemes given by

$$\begin{cases} a_{ni} = \delta_{i,0} - v\xi_2(\gamma,j) \\ a_{ni+s_1} = \xi_1(\gamma,i,n,s_1) - a_{n(\gamma+1)+s_1}\xi_2(\gamma,i) - a_{n(\gamma+1)+t_1}\xi_3(\gamma,i) \end{cases} \tag{3.4}$$

and

$$\begin{cases} a_{mj} = \delta_{j,0} - v\eta_2(\sigma,j) \\ a_{mj+s_2} = \eta_1(\sigma,j,n,s_2) - a_{n(\sigma+1)+s_2}\eta_2(\sigma,j) - a_{n(\sigma+1)+t_2}\eta_3(\sigma,j), \end{cases} \tag{3.5}$$

where

$$\xi_1(\rho,i,n,s_1)=\frac{\prod\limits_{b=-(\rho+1)}^{\rho}(nb+s_1)}{(n)^{2\rho+1}(-1)^{-i+\rho-1}(ni+s_1)(\rho-i)!(\rho+i+1)!},$$

$$\eta_1(\sigma,j,m,s_2)=\frac{\prod\limits_{b=-(\sigma+1)}^{\sigma}(mb+s_2)}{(m)^{2\sigma+1}(-1)^{-j+\sigma-1}(mj+s_2)(\sigma-j)!(\sigma+j+1)!},$$

$$\xi_2(\rho,i)=\frac{(-1)^{i+\rho}(2\rho+2)!}{(\rho-i)!(\rho+i+1)!(\rho-i+1)},\quad \eta_2(\sigma,j)=\frac{(-1)^{j+\sigma}(2\sigma+2)!}{(\sigma-j)!(\sigma+j+1)!(\sigma-j+1)},$$

$$\xi_3(\rho,i)=\frac{(-1)^{i+\rho+1}(2\rho+2)!}{(\rho-i)!(\rho+i+1)!(\rho+i+2)},\quad \eta_3(\rho,j)=\frac{(-1)^{j+\sigma+1}(2\sigma+2)!}{(\sigma-j)!(\sigma+j+1)!(\sigma+j+2)}$$

and

$$\begin{cases} a_{n(\rho+1)+s_1}=\frac{(d-n)(d-2n)\cdots(d-(2\rho n+3n))}{(-1)^{2\rho+3}(n)^{2\rho+3}(2\rho+3)!}, & d=n(\rho+1)+s_1,\\ a_{m(\sigma+1)+s_2}=\frac{(e-m)(e-2m)\cdots(e-(2\sigma m+3m))}{(-1)^{2\sigma+3}(2m)^{2\sigma+3}(2\sigma+3)!}, & e=m(\sigma+1)+s_2,\end{cases}\tag{3.6}$$

where, $\delta_{i,0}$ and $\delta_{j,0}$ are the kroneker delta symbols defined by (2.4), n,m are the arities of the subdivision schemes (i.e. $n,m=2,3,4$ represent binary, ternary and quaternary interpolating subdivision schemes, respectively), ρ,σ are non-negative integers, $i=-\rho-1,\cdots,\rho$, $j=-\sigma-1,\cdots,\sigma$, $s_1=1,2,\cdots,n-1$, $s_2=1,2,\cdots,m-1$, $t_1=n-s_1$, and $t_2=m-s_2$ and $a_{n(\rho+1)+s_1}$ and $a_{m(\sigma+1)+s_2}$ are the free parameters.

Since, $a_{i,j}$'s are the tensor product of the mask of univariate scheme (i.e. $a_{i,j}=b_ib_j$), then

$$a_{(ni+s_1,mj+s_2)}=a_{ni+s_1}a_{mj+s_2}.$$

Hence, the tensor product of $(2N+4)$-point interpolating subdivision scheme is

$$p^{k+1}_{ni+\alpha,mj+\beta}=\sum_{l_1=-\rho-1}^{\rho+2}\sum_{l_2=-\sigma-1}^{\sigma+2}a_{(nl_1-\alpha,ml_2-\beta)}p^k_{i+l_1,j+l_2},\tag{3.7}$$

together with the symmetry conditions

$$\begin{cases} a_{-n(\rho+1)-\alpha}=a_{n(\rho+1)+\alpha},\\ a_{-m(\sigma+1)-\beta}=a_{m(\sigma+1)+\beta},\end{cases}\tag{3.8}$$

where $\alpha=0,1,2,\cdots,n-1$, $\beta=0,1,2,\cdots,m-1$, ρ,σ are non-negative integers and $n,m\ge 2$ are the arities of the tensor product. Setting $a_{n(\rho+2)}=a_{m(\sigma+2)}=0$, the masks $a_{nl_1-1-\alpha}$ and $a_{ml_2-1-\beta}$ can be evaluated using (3.4) and (3.5).

Example. Consider the tensor product 4-point DD interpolating subdivision scheme which can be evaluated using (3.7) together with (3.8) as follows.

Let $a(z)$ be the Laurent polynomial defined by

$$a(z)=\frac{1}{16}(-1z^{-3}+9z^{-1}+1+9z^{1}-1z^{3}).\tag{3.9}$$

The Laurent polynomial of the tensor product 4-point binary interpolating scheme can be obtained by taking the tensor product $a(z_1,z_2)=a(z_1)a(z_2)$, where

$$a(z_1)=\frac{1}{16}(-1z_1^{-3}+9z_1^{-1}+1+9z_1^{1}-1z_1^{3}),\quad a(z_2)=\frac{1}{16}(-1z_2^{-3}+9z_2^{-1}+1+9z_2^{1}-1z_2^{3}).$$

So that the suggested tensor product 4-point binary DD interpolating subdivision scheme is

$$\begin{cases} p^{k+1}_{2i,2j} = p^k_{i,j}, \\ p^{k+1}_{2i+1,2j} = \frac{1}{16}(-p^k_{i-1,j} + 9p^k_{i,j} + 9p^k_{i+1,j} - p^k_{i+2,j}), \\ p^{k+1}_{2i,2j+1} = \frac{1}{16}(-p^k_{i,j-1} + 9p^k_{i,j} + 9p^k_{i,j+1} - p^k_{i,j+2}), \\ p^{k+1}_{2i+1,2j+1} = \frac{1}{256}(p^k_{i-1,j-1} - 9p^k_{i,j-1} - 9p^k_{i+1,j-1} + p^k_{i+2,j-1} - 9p^k_{i-1,j} + 81p^k_{i,j} + 81p^k_{i+1,j} - 9p^k_{i+2,j} \\ \qquad -9p^k_{i-1,j+1} + 81p^k_{i,j+1} + 81p^k_{i+1,j+1} - 9p^k_{i+2,j+1} + p^k_{i-1,j+2} - 9p^k_{i,j+2} - 9p^k_{i+1,j+2} + p^k_{i+2,j+2}). \end{cases}$$

The tensor product of the DD interpolating subdivision scheme has the same C^1 continuity because the DD scheme has C^1 continuity. Substituting $n = m = 2$, $\alpha, \beta = 0,1$ and $\rho = \sigma = 0$ in (3.7), the tensor product becomes

$$p^{k+1}_{2i+\alpha,2j+\beta} = \sum_{l_1=-1}^{2} \sum_{l_2=-1}^{2} a_{(2l_1-\alpha,2l_2-\beta)} p^k_{i+l_1,j+l_2}, \tag{3.10}$$

together with the symmetry conditions

$$\begin{cases} a_{-2-\alpha} = a_{2+\alpha}, \\ a_{-2-\beta} = a_{2+\beta}. \end{cases} \tag{3.11}$$

Since $a_{i,j}$'s satisfy $a_{i,j} = b_i b_j$, then

$$a_{(2l_1-\alpha,2l_2-\beta)} = a_{2l_1-\alpha} a_{2l_2-\beta}.$$

Since $a_4 = 0$ for $n = m = 2$. Also, when $v_1 = v_2 = 0$, $w_1 = w_2 = \frac{-1}{16}$, $b_1 = b_2 = 0$ and $s_1, s_2 = 1$ is substituted in (3.7) together with (3.8), the equations (3.10) are satisfied.

Remark 3.1. Some other well known interpolating schemes can be obtained from our proposed result (2.7) which are given below.

- The interpolatory subdivision scheme of Kobbelt [5] is obtained by setting $n = m = 2$, and $\rho, \sigma = 0$ in (3.7) together with (3.9). The end points of this scheme are given by

$$\begin{cases} p^{k+1}_{2i+1,2j} = \frac{8+w}{16}(p^k_{i,j} + p^k_{i+1,j}) - \frac{w}{16}(p^k_{i-1,j} + p^k_{i+2,j}), \\ p^{k+1}_{2i,2j+1} = \frac{8+w}{16}(p^k_{i,j} + p^k_{i,j+1}) - \frac{w}{16}(p^k_{i,j-1} + p^k_{i,j+2}), \end{cases}$$

  and the face points are given by

$$\begin{aligned} p^{k+1}_{2i+1,2j+1} = & \frac{1}{256}(w^2 p^k_{i-1,j-1} - w(8+w)p^k_{i,j-1} - w(8+w)p^k_{i+1,j-1} + w^2 p^k_{i+2,j-1} - w(8+w)p^k_{i-1,j} \\ & + (8+w)^2 p^k_{i,j} + (8+w)^2 p^k_{i+1,j} - w(8+w)p^k_{i+2,j} - w(8+w)p^k_{i-1,j+1} + (8+w)^2 p^k_{i,j+1} \\ & + (8+w)^2 p^k_{i+1,j+1} - w(8+w)p^k_{i+2,j+1} + w^2 p^k_{i-1,j+2} - w(8+w)p^k_{i,j+2} \\ & - w(8+w)p^k_{i+1,j+2} + w^2 p^k_{i+2,j+2}). \end{aligned}$$

- By setting $n = m = 3$, $b_1 = b_2 = 0$, $w_1 = w_2 = a_5 = \frac{-5}{81}$, $u_1 = u_2 = a_4 = \frac{-4}{81}$ in (3.7) together with (3.8), the following 4-point interpolatory tensor product scheme is obtained

$$p^{k+1}_{3i,3j} = p^k_{i,j},$$

$$p^{k+1}_{3i+1,3j} = \frac{1}{6561}(-5p^k_{i-1,j} + 60p^k_{i,j} + 30p^k_{i+1,j} - 4p^k_{i+2,j}),$$

$$
\begin{aligned}
p^{k+1}_{3i+2,3j} &= \frac{1}{6561}(-4p^k_{i-1,j} + 30p^k_{i,j} + 60p^k_{i+1,j} - 5p^k_{i+2,j}),\\
p^{k+1}_{3i,3j+1} &= \frac{1}{6561}(-5p^k_{i,j-1} + 60p^k_{i,j} + 30p^k_{i,j+1} - 4p^k_{i,j+2}),\\
p^{k+1}_{3i+1,3j+1} &= \frac{1}{6561}(25p^k_{i-1,j-1} - 300p^k_{i,j-1} - 150p^k_{i+1,j-1} + 20p^k_{i+2,j-1} - 150p^k_{i-1,j} + 1800p^k_{i,j}\\
&\quad + 900p^k_{i+1,j} - 120p^k_{i+2,j} - 300p^k_{i-1,j+1} + 3600p^k_{i,j+1} + 1800p^k_{i+1,j+1} - 240p^k_{i+2,j+1}\\
&\quad + 25p^k_{i-1,j+2} - 300p^k_{i,j+2} - 150p^k_{i+1,j+2} + 20p^k_{i+2,j+2}),\\
p^{k+1}_{3i+2,3j+1} &= \frac{1}{6561}(20p^k_{i-1,j-1} - 150p^k_{i,j-1} - 300p^k_{i+1,j-1} + 25p^k_{i+2,j-1} - 240p^k_{i-1,j} + 1800p^k_{i,j}\\
&\quad + 3600p^k_{i+1,j} - 300p^k_{i+2,j} - 120p^k_{i-1,j+1} + 900p^k_{i,j+1} + 1800p^k_{i+1,j+1} - 150p^k_{i+2,j+1}\\
&\quad + 16p^k_{i-1,j+2} - 120p^k_{i,j+2} - 240p^k_{i+1,j+2} + 20p^k_{i+2,j+2}),\\
p^{k+1}_{3i,3j+2} &= \frac{1}{6561}(-4p^k_{i,j-1} + 30p^k_{i,j} + 60p^k_{i,j+1} - 5p^k_{i,j+2}),\\
p^{k+1}_{3i+1,3j+2} &= \frac{1}{6561}(20p^k_{i-1,j-1} - 240p^k_{i,j-1} - 120p^k_{i+1,j-1} + 16p^k_{i+2,j-1} - 150p^k_{i-1,j} + 1800p^k_{i,j}\\
&\quad + 900p^k_{i+1,j} - 120p^k_{i+2,j} - 300p^k_{i-2,j+2} + 3600p^k_{i,j+1} + 1800p^k_{i+1,j+1} - 240p^k_{i+2,j+1}\\
&\quad + 25p^k_{i-2,j+1} - 300p^k_{i,j+2} - 150p^k_{i+1,j+2} + 20p^k_{i+2,j+2}),\\
p^{k+1}_{3i+2,3j+2} &= \frac{1}{6561}(16p^k_{i-1,j-1} - 120p^k_{i,j-1} - 240p^k_{i+1,j-1} + 20p^k_{i+2,j-1} - 120p^k_{i-2,j} + 900p^k_{i,j}\\
&\quad + 1800p^k_{i+1,j} - 150p^k_{i+2,j} - 240p^k_{i-2,j+2} + 1800p^k_{i,j+1} + 3600p^k_{i+1,j+1} - 300p^k_{i+2,j+1}\\
&\quad + 20p^k_{i-2,j+1} - 150p^k_{i,j+2} - 300p^k_{i+1,j+2} + 25p^k_{i+2,j+2}).
\end{aligned}
$$

4. Error Bound

In this section, we find the error bound for subdivision surfaces of $(2N+4)$-point n-ary interpolating subdivision schemes. Also, we present the error bounds of binary, ternary and quaternary subdivision surfaces as special cases of the following lemma.

Lemma 4.1. *Given initial control polygon $p^0_{i,j} = p_{i,j}$, $i,j \in \mathbb{Z}$, let the values $p^k_{i,j}$, $k \geq 1$ be defined recursively by subdivision process* (3.1) *together with* (3.2). *Suppose p^k is the piecewise linear interpolant to the values $p^k_{i,j}$ and p^∞ is the limit surface of the subdivision process* (3.1). *If $\xi_s \leq 1$, then the error bound between limit surface and its control polygon after k-fold subdivision is*

$$\|p^k - p^\infty\|_\infty \leq \sigma_\vartheta \left(\frac{(\xi_s)^k}{1-\xi_s}\right), \tag{4.1}$$

where

$$\xi_s = \max_{\alpha,\beta}\left\{\left|\sum_{v=0}^{m} a_{\alpha,t} \sum_{u=0}^{m} a_{\beta,u}\right|,\ \alpha,\beta = 0,1,\ldots,n-1\right\},$$

and

$$\sigma_\vartheta = \max_{\alpha,\beta}\left\{\sum_{q=1}^{3}(\vartheta_q)(\omega^q_{\alpha,\beta}),\ \alpha,\beta = 0,1,\dots,n-1\right\},$$

where

$$\omega^1_{\alpha,\beta} = \left|a_{\beta,0}\sum_{q=1}^{m}a_{\alpha,q} - \frac{\alpha(n-\beta)}{n^2}\right| + \left|a_{\beta,0}\sum_{v=1}^{m-1}\tilde{a}_{\alpha,v}\right|, \quad \omega^2_{\alpha,\beta} = \left|\sum_{q=1}^{m}a_{\beta,q} - \frac{\beta}{n}\right| + \left|\sum_{u=0}^{m}a_{\alpha,u}\sum_{u=0}^{m}\tilde{a}_{\beta,v}\right|,$$

$$\omega^3_{\alpha,\beta} = \left|\sum_{q=1}^{m}a_{\alpha,q}\sum_{q=1}^{m}a_{\beta,q} - \frac{\alpha\beta}{n^2}\right| + \left|\sum_{q=1}^{m}a_{\beta,q}\sum_{v=1}^{m-1}\tilde{a}_{\alpha,v}\right|,$$

together with

$$\begin{cases}\tilde{a}_{\alpha,0} = \sum\limits_{q=1}^{m}a_{\alpha,q} - \frac{\alpha}{n}, & \alpha = 0,1,\dots,n-1,\\ \tilde{a}_{\alpha,l} = \sum\limits_{q=l+1}^{m}a_{\alpha,l}, & l \ge 1\end{cases}$$

and

$$\vartheta_q = \max_{i,j}\|\Delta^0_{i,j,q}\|, \quad q = 1,2,3$$

assuming

$$\begin{cases}\Delta^0_{i,j,1} = p^0_{i+1,j} - p^0_{i,j},\\ \Delta^0_{i,j,2} = p^0_{i,j+1} - p^0_{i,j},\\ \Delta^0_{i,j,3} = p^0_{i+1,j+1} - p^0_{i,j}.s\end{cases}$$

4.1 Error Bounds of $(2N+4)$-point n-ary Interpolating Subdivision Scheme

In this section we have presented error bounds computed by equation (4.1) between limit surface and its control polygon after k-fold subdivision of 4-point binary, ternary, quaternary, quinary and senary interpolating subdivision schemes. It can be seen from Table 1 that error bound of 4-point binary interpolating scheme is less than that of 4-point ternary interpolating scheme and error bound of 4-point ternary interpolating scheme is less than that of 4-point quaternary interpolating scheme and so on at each subdivision level.M̃oreover, we have also given the graphical comparison of 4-points binary, ternary, quaternary, quinary and senary interpolating schemes in Figure 2.

Table 1. Error bounds of 4-point n-ary interpolating schemes

$n \mid k$	1	2	3	4	5	6	7
2	0.0300000	0.0150000	0.0075000	0.0037500	0.0018750	0.0009375	0.0004688
3	0.0166667	0.0055556	0.0018523	0.0006172	0.0002053	0.0000683	0.0000233
4	0.0116667	0.0029173	0.0007291	0.0001823	0.0000456	0.0000111	0.0000021
5	0.0090000	0.0018000	0.0003600	0.0000720	0.0000144	0.0000029	0.0000058
6	0.0073333	0.0012222	0.0002037	0.0000340	0.0000056	0.0000009	0.0000001

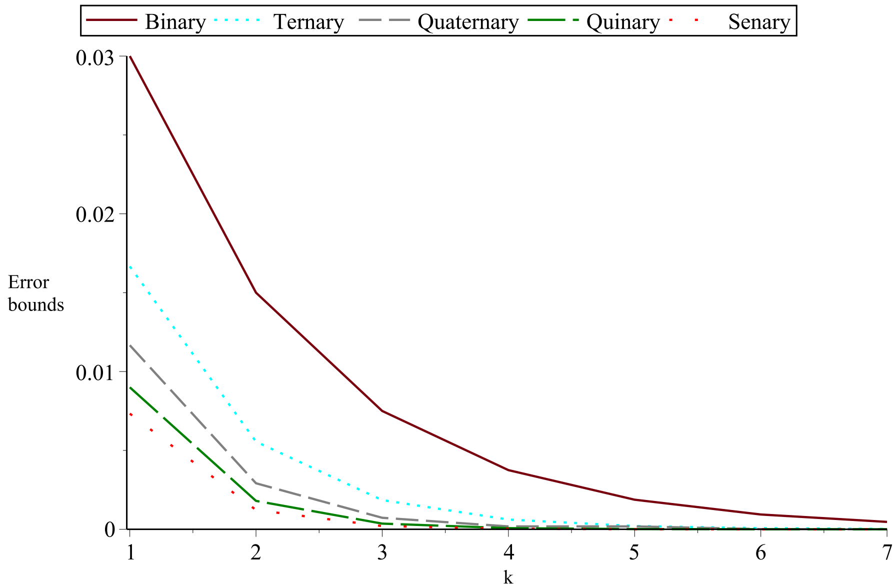
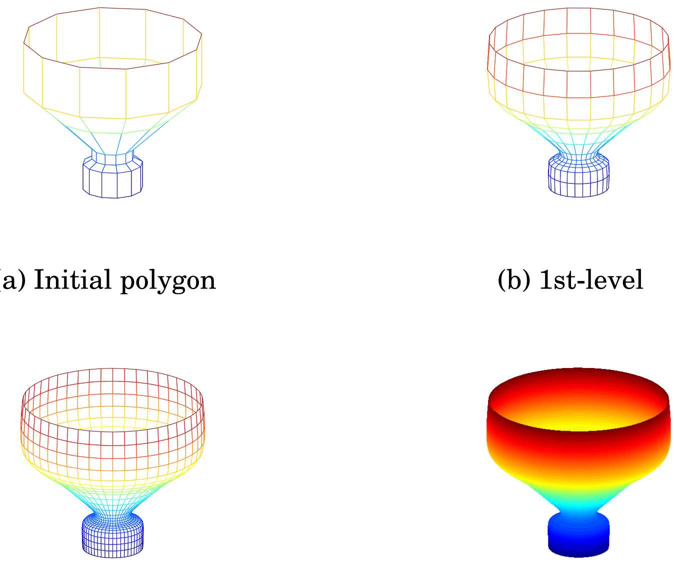


Figure 2. Comparison of the error bounds of 4-point binary, ternary, quaternary, quinary and senary interpolating schemes (after k-fold subdivision)

5. Visual Performance

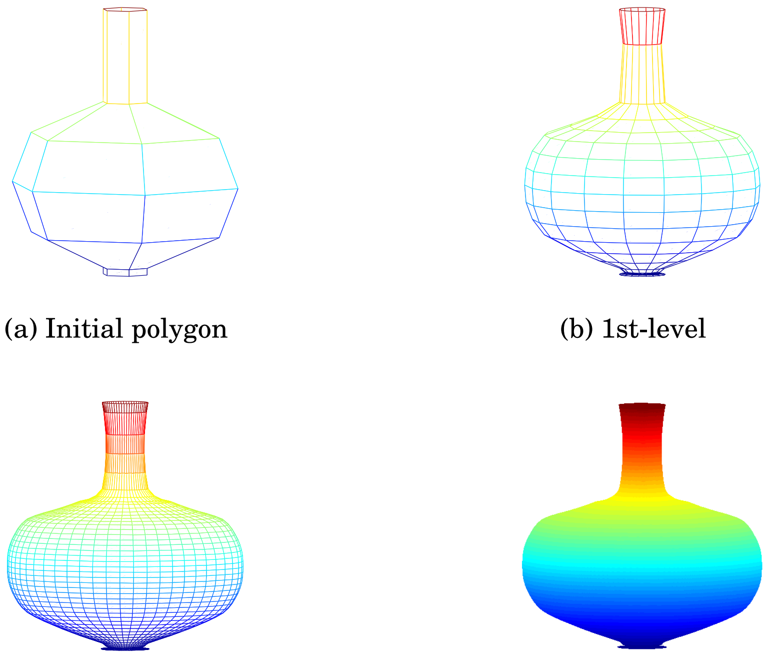
Here, the performance of some of the schemes which are determined from the proposed formulae (3.2)-(3.8) are shown in Figure 4 and Figure 3.



(a) Initial polygon (b) 1st-level

(c) 2nd-level (d) Limit surface

Figure 3. Performance of 4-point binary interpolating subdivision surface scheme: (a), (b), (c) and (d) show the initial polygon, 1st-, 2nd-subdivision levels and limit surface, respectively.



(a) Initial polygon

(b) 1st-level

(c) 2nd-level

(d) Limit surface

Figure 4. Performance of 4-point ternary interpolating subdivision surface scheme: (a), (b), (c) and (d) show the initial polygon, 1st-, 2nd-subdivision levels and limit surface, respectively.

6. Conclusion

In this paper, a general formula for subdivision surface scheme is formulated by using 2D Lagrange interpolating polynomial. This paper presents a new and efficient approach to acquire variety of subdivision schemes for surfaces. Most of the well-known subdivision schemes are special cases of the proposed work. To test out any improvement in the proposed scheme, error bound between the generalized subdivision surface and its control polygon after κ-fold subdivision has been estimated. The bounds are expressed in the form of first order differences of the initial control point sequence and constants.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] E. Catmull and J. Clark, Recursively generated B-spline surfaces on arbitrary topological meshes, *Computer Aided Design* **10**(6) (1978), 350 – 355, DOI: 10.1016/0010-4485(78)90110-0.

[2] D. Doo and M.A. Sabin, Behaviour of recursive subdivision surfaces near extraordinary points, *Computer Aided Design* **10**(6) (1978), 356 – 360, DOI: 10.1016/0010-4485(78)90111-2.

[3] N. Dyn, D. Levin and J. Gregory, A 4-point interpolatory subdivision scheme for curve design, *Computer Aided Geometric Design* **4**(4) (1987), 257 – 268.

[4] A. Weissman, *A 6-point Interpolatory Subdivision Scheme for Curve Design*, M.Sc. Thesis, Tel Aviv University (1990).

[5] L. Kobbelt, Interpolatory subdivision on open quadrilateral nets with arbitrary topology, *Computer Graphics Forum* **15**(3) (1996), 409 – 420, DOI: 10.1111/1467-8659.1530409.

[6] M.F. Hassan, Ivrissimitzis, N. Dogson and M.A. Sabin, An interpolating 4-point C^2 ternary stationary subdivision scheme, *Compt. Aided Grom. Design* **19** (2002), 1 – 18.

[7] M.K. Jena, P. Shunmugaraj and P.C. Das, A non-stationary subdivision scheme for curve interpolation, *ANZIAM Journal* **44** (2003), 216 – 235.

[8] S.W. Choi, B.G. Lee, Y.J. Lee and J. Yoon, Stationary subdivision schemes reproducing polynomials, *Computer Aided Geometric Design* **23** (2006), 351 – 360, DOI: 10.1016/j.cagd.

[9] C. Beccari, G. Casciola and L. Romani, An interpolating 4-point C^2 ternary non-stationary subdivision scheme with tension control, *Computer Aided Geometric Design* **24**(4) (2007), 210 – 219, DOI: 10.1016/j.cagd.

[10] C. Beccari, G. Casciola and L. Romani, A non-stationary uniform tension controlled interpolating 4-point scheme reproducing conics, *Computer Aided Geometric Design* **24** (2007), 1 – 9, DOI: 10.1016/j.cagd.

[11] K.P. Ko, *A study on subdivision scheme-draft*, Dongseo University Busan South Korea, `http://kowon.dongseo.ac.kr/~kpko/publication/2004book.pdf` (2007).

[12] L. Romani, From approximating subdivision schemes for exponential splines to high-performance interpolating algorithms, *Journal of Computational and Applied Mathematics*, **224**(1) (2009), 383 – 396, DOI: 10.1016/j.cam.05.013.

[13] G. Dahlquist and A. Bjork, Numerical methods in scientific computing, *Society for Industrial and Applied Mathematics* **1** (2008), 397, DOI: 10.1137/1.9780898717785.

[14] J.A. Lian, On a-ary subdivision for curve design - II: 3-point and 5-point interpolatory schemes, *Applications and Applied Mathematics* **3**(2) (2008), 176 – 187.

[15] J.A. Lian, On a-ary subdivision for curve design: 4-point and 6-point inerpolatory schemes, *Appl. Math. Int. J.* **3**(1) (2008), 18 – 29.

[16] J.A. Lian, On a-ary subdivision for curve design - III: $2m$-point and $(2m+1)$-point interpolatory schemes, *Applications and Applied Mathematics* **4**(2) (2009), 434 – 444.

[17] G. Mustafa and A.R. Najma, The mask of $(2b+4)$-point n-ary subdivision scheme, *Computing* **90** (2010), 1 – 14, DOI: 10.1007/s00607-010-0108-x.