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The Application of Quartic Trigonometric B-spline for Solving Second Order Singular Boundary Value Problems

Research Article

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Abstract. In this paper, a quartic trigonometric B-spline collocation approach is described and presented for the numerical solution of the second order singular boundary value problems. Several numerical examples are discussed to exhibit the feasibility and capability of the technique. The unknown coefficients C_i, $i = -4, -3, \dots, n-1$ are obtained through optimization. The maximum errors (L_∞) and norm errors (L_2) are also computed for different space size steps to assess the performance of the proposed technique. The rate of convergence is discussed numerically to be of fourth-order. The numerical solutions are contrasted with both analytical and other existing numerical solutions that exist in the literature. The comparison shows that the quartic trigonometric B-spline method is superior as it yields more accurate solutions.

Keywords. Quartic trigonometric basis functions; Trigonometric collocation method; Singular boundary value problem

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1. Introduction

Singular boundary value problems for ordinary differential equation appear very commonly in the field of science and Engineering specially in electrically conducting solids, electrical potential theory, circular membrane theory, membrane response of a spherical cap, plasma physics, theory of colloids and flow and heat transfer over a stretching sheet etc. These also arise in physiology as well e.g. in the study of various tumor growth problem, steady state oxygen diffusion in a spherical cell with Michaelis-Menton uptake kinetics.

Consider the class of singular two-points boundary value problems [7, 8, 10]

$$x^{-c}(x^c y')' = f(x,y), \quad 0 < x \leq 1, \tag{1.1}$$

$$y'(0) = 0 \text{ and } y(1) = \beta, \tag{1.2}$$

where β is a finite constant and $c \geq 1$. In order to make sure the existence and uniqueness of the solution of above problems, suppose that $f(x,y)$ is continuous, $\frac{\partial f}{\partial y}$ exists and is continuous and $\frac{\partial f}{\partial y} \geq 0$ [8, 10, 15].

Second-order singular boundary value problems have been investigated by many authors [7, 8, 10, 13, 17, 18, 23]. Several methods have been elaborated for the solution of these kinds of problems and were discussed in [14, 21]. Bickley [6] Pioneered the use of cubic splines rather than a global high-order approximation to enhance accuracy for the approximated solution of linear ordinary differential equation. Albasiny and Hoskins [5] obtained spline solutions by solving a tri-diagonal matrix system. Fyfe [9] studied the method introduced by Bickley [6] and carried out an error analysis. It was found that the spline method is more effective than the usual finite difference scheme [18, 21] because it has the flexibility to obtain the solution at any point in the domain with greater accuracy. Several authors [14, 15, 20] discussed the second order singular boundary value problems via Chebyshev polynomial and B-spline methods because of its better approximation than usual finite difference methods. Caglar and Caglar [7, 8] discussed the problem of second order singular ordinary differential equations by using B-spline collocation methods because these methods provide the accurate results than the monomial cubic splines. Continuing with this approach, Goh *et al*. [10] described and presented the quartic B-spline collocation method for the solution of second order singular boundary value problems. A quartic B-spline method was found to be more accurate than cubic B-spline scaling functions.

B-spline functions can be used for the numerical solution of linear and nonlinear differential equations due to their important geometric properties and features. The trigonometric spline function was first introduced by Schoenberg in 1964 [22]. Trigonometric B-spline is a non-polynomial B-spline functions containing trigonometric terms. The derivation and properties of Trigonometric B-spline were found in [16, 25]. Cubic trigonometric B-spline has been used by Hamid [12] for boundary value problem of order two involving ordinary differential equations. The application of cubic trigonometric B-spline to the numerical solution of the hyperbolic problems, generalized nonlinear Klien-Gordon equation and non-classical diffusion problems have been carried out in [1–3, 26, 27]. These trigonometric B-spline methods provide the better accuracy than the usual finite difference methods. Gupta and Kumar [11] solved a singular boundary value problem by using cubic trigonometric B-spline approach. Heilat *et al*. [13] solved linear system of second order boundary value problems by using extended cubic B-spline. Suardi *et al*. [24] derived a scheme based on cubic B-spline to solve two-point boundary value problems.

In this work, a new quartic trigonometric B-spline technique is described and presented for solving a second order singular boundary value problems. The technique is based on the quartic trigonometric B-spline functions. Some researchers have considered the ordinary B-spline collocation method for solving the proposed problem but, so far as we are aware, not with the quartic trigonometric B-spline collocation method. The values of unknown coefficients C_i, $i=-4,-3,\dots,n-1$ are found via optimization. The order of convergence can be calculated numerically and found to be fourth order. A quartic trigonometric B-spline is used as an interpolating function in the space dimension. The efficiency and applicability of the technique are demonstrated by applying the scheme to several examples. The numerical results demonstrate that this method is superior as it yields more accurate solutions than ordinary cubic B-spline collocation methods [7,8] and quartic B-spline collocation approach [10].

This paper is organized as follows: Quartic trigonometric B-spline collocation method is described in Section 2. A numerical method of solving second order boundary value problem is presented in Section 3. In Section 4, the values of unknown coefficients C_i, $i=-4,-3,\dots,n-1$ are obtained via optimization. Numerical examples and discussions are considered in Section 5. Finally, the concluded remarks are presented in Section 6.

2. Description of Quartic Trigonometric B-spline Collocation Method

The trigonometric B-spline basis of order 1 can be obtained by following formula [27]

$$TB_i^1(x)=\begin{cases}1 & x\in[x_i,x_{i+1}),\\ 0 & \text{otherwise.}\end{cases} \tag{2.1}$$

The trigonometric B-spline basis of order $k>1$ can be calculated from the following recursive formula [27]

$$TB_i^k=\frac{\sin\left(\frac{x-x_i}{2}\right)}{\sin\left(\frac{x_{i+k-1}-x_i}{2}\right)}TB_i^{k-1}(x)+\frac{\sin\left(\frac{x_{i+k}-x}{2}\right)}{\sin\left(\frac{x_{i+k}-x_{i+1}}{2}\right)}TB_{i+1}^{k-1}(x). \tag{2.2}$$

For $k=5$ calculating degree upto 4, the resulting basis $TB_i^5(x)$ is

$$TB_i^5(x)=\frac{1}{\omega}\begin{cases}p^4(x_i), & [x_i,x_{i+1}]\\ p^2(x_i)(p(x_i)q(x_{i+2})+p(x_{i+1})q(x_{i+3}))+p(x_i)p^2(x_{i+1})q(x_{i+4}) & \\ \quad +p^3(x_{i+1})q(x_{i+5}), & [x_{i+1},x_{i+2}]\\ p^2(x_i)q^2(x_{i+3})+p(x_i)q(x_{i+4})(p(x_{i+1})q(x_{i+3})+q(x_{i+4})p(x_{i+2})) & \\ \quad +q(x_{i+5})(p^2(x_{i+1})q(x_{i+3})+q(x_{i+4})p(x_{i+1})p(x_{i+2}) & \\ \quad +q(x_{i+5})p^2(x_{i+2})), & [x_{i+2},x_{i+3}]\\ p(x_i)q^3(x_{i+4})+q^2(x_{i+5})(p(x_{i+2})q(x_{i+4})+p(x_{i+3})q(x_{i+5})) & \\ \quad +p(x_{i+1})q^2(x_{i+4})q(x_{i+5}), & [x_{i+3},x_{i+4}]\\ q^4(x_{i+5}), & [x_{i+4},x_{i+5}]\end{cases} \tag{2.3}$$

where

$$p(x_i)=\sin\left(\frac{x-x_i}{2}\right),\ \ q(x_i)=\sin\left(\frac{x_i-x}{2}\right),\ \ \omega=\sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)\sin(2h)$$

and where $h=(b-a)/n$ and $TB_i^5(x)$ is piecewise trigonometric function of degree 4 with C^3 continuity, non-negativity and partitioning of unity. A quartic trigonometric B-spline treated as a approximation solution of proposed problem, $S_T(x)$ which is a linear combination of the quartic trigonometric B-spline basis over the subinterval $[x_i,x_{i+1}]$ can be defined as

$$S_T(x)=\sum_{i=-4}^{n-1}C_iTB_i^5(x), \tag{2.4}$$

where C_i are the real non-zero coefficients. Equation (2.4) can be simplified to

$$S_T(x_i)=C_{i-4}TB_{i-4}^5(x_i)+C_{i-3}TB_{i-3}^5(x_i)+C_{i-2}TB_{i-2}^5(x_i)+C_{i-1}TB_{i-1}^5(x_i), \tag{2.5}$$

$$\begin{aligned}S_T(x_i)=C_{i-4}&\left[\sin^3\left(\frac{h}{2}\right)\csc(h)\csc\left(\frac{3h}{2}\right)\csc(2h)\right]\\&+C_{i-3}\left[\sin^2\left(\frac{h}{2}\right)\csc(h)\csc(2h)+2\sin\left(\frac{h}{2}\right)\sin(h)\csc\left(\frac{3h}{2}\right)\csc(2h)\right]\\&+C_{i-2}\left[\sin^2\left(\frac{h}{2}\right)\csc(h)\csc(2h)+2\sin\left(\frac{h}{2}\right)\sin(h)\csc\left(\frac{3h}{2}\right)\csc(2h)\right]\\&+C_{i-1}\left[\sin^3\left(\frac{h}{2}\right)\csc(h)\csc\left(\frac{3h}{2}\right)\csc(2h)\right].\end{aligned} \tag{2.6}$$

Here, we only need values up to second derivative to solve the second order boundary value problem. Now, taking the first and second derivative of equation (2.4) and evaluating at x_i, we have

$$S_T'(x_i)=C_{i-4}\left[\frac{-\csc(h)\sec(h)}{2+4\cos(h)}\right]+C_{i-3}[-\csc(2h)]+C_{i-2}[\csc(2h)]+C_{i-1}\left[\frac{\csc(h)\sec(h)}{2+4\cos(h)}\right], \tag{2.7}$$

$$S_T''(x_i)=C_{i-4}[\csc(h)\csc(2h)]+C_{i-3}[-\csc(h)\csc(2h)]+C_{i-2}[-\csc(h)\csc(2h)]+C_{i-1}[\csc(h)\csc(2h)]. \tag{2.8}$$

3. Numerical Solution of Second Order Singular Boundary Value Problems

For the linear case, the equation (1.1) and (1.2) can be written as [10, 15]

$$y''(x)+\frac{c}{x}y'(x)+g(x)y(x)=f(x),\quad 0<x\le 1 \tag{3.1}$$

with the boundary conditions

$$y'(0)=0,\quad y(0)=\beta. \tag{3.2}$$

The above differential equation has a singularity at $x=0$. L'Hospital rule is applied to remove the singularity. Hence the boundary value problem can be changed into the following form

$$\begin{cases}(c+1)y''(x)+g(0)y(x)=f(0), & x=0,\\ y''(x)+\frac{c}{x}y'(x)+g(x)y(x)=f(x), & 0<x\le 1.\end{cases} \tag{3.3}$$

In order to solve the problem, a quartic trigonometric B-spline $S_T(x_i)$, $i = 0,1,2,\dots,n$ is considered to be the solution of above differential equation at $x = x_i$ and we substitute equations (2.6)-(2.8) into equation (3.3), this leads to

$$\begin{cases}(c+1)S''_T(x_i)+g(x_i)S_T(x_i)=f(x_i), & i=0, \\ S''_T(x_i)+\frac{c}{x_i}S'_T(x_i)+g(x_i)S_T(x_i)=f(x_i), & i=1,2,\dots,n,\end{cases} \tag{3.4}$$

$$S'_T(x_i)=0, \quad i=0, \tag{3.5}$$

$$S_T(x_i)=\beta, \quad i=n. \tag{3.6}$$

Consider

$$C_{n-1}=\lambda. \tag{3.7}$$

A system of $(n+3)$ linear equations with $(n+4)$ unknown $C_{-4},C_{-3},\dots,C_{n-1}$ is obtained. This system can be written in matrix equation of the form as

$$AC=F, \tag{3.8}$$

where A is an $(n+3)\times(n+4)$ matrix which can be defined in equation, $C=[C_{-4},C_{-3},\dots,C_{n-1}]^T$ and $F=[0,f(x_0),\dots,f(x_n),\beta]^T$. This system of equations has infinitely many solutions.

$$A=\begin{bmatrix} -a_1 & -a_2 & a_2 & a_1 & 0 & 0 & \dots & \dots & 0 \\ a_3 & a_4 & a_4 & a_3 & 0 & 0 & \dots & \dots & 0 \\ p_1 & q_1 & r_1 & s_1 & 0 & 0 & \dots & \dots & 0 \\ 0 & p_2 & q_2 & r_2 & s_2 & 0 & \dots & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & p_n & q_n & r_n & s_n \\ 0 & \dots & \dots & \dots & 0 & x & y & y & x \\ 0 & \dots & \dots & \dots & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where

$a_1=\left(\dfrac{\csc(h)\sec(h)}{2+\cos(h)}\right)$, $a_2=\csc(2h)$,

$a_3=(c+1)(\csc(h)\csc(2h))+g(x_0)\left(\sin^3\left(\dfrac{h}{2}\right)\csc(h)\csc\left(\dfrac{3h}{2}\right)\csc(2h)\right)$,

$a_4=(c+1)(-\csc(h)\csc(2h))+g(x_0)\left(\sin^2\left(\dfrac{h}{2}\right)\csc(h)\csc(2h)+2\sin\left(\dfrac{h}{2}\right)\sin(h)\csc\left(\dfrac{3h}{2}\right)\csc(2h)\right)$,

$p_i=\csc(h)\csc(2h)-\dfrac{c}{x_i}\left(\dfrac{\csc(h)\sec(h)}{2+\cos(h)}\right)+g(x_i)\left(\sin^3\left(\dfrac{h}{2}\right)\csc(h)\csc\left(\dfrac{3h}{2}\right)\csc(2h)\right)$, $i=1,2,3,\dots,n$

$q_i=-\csc(h)\csc(2h)-\dfrac{c}{x_i}(\csc(2h))+g(x_i)\left(\sin^2\left(\dfrac{h}{2}\right)\csc(h)\csc(2h)+2\sin\left(\dfrac{h}{2}\right)\sin(h)\csc\left(\dfrac{3h}{2}\right)\csc(2h)\right)$,

$\quad i=1,2,3,\dots,n$,

$r_i=-\csc(h)\csc(2h)-\dfrac{c}{x_i}(\csc(2h))+g(x_i)\left(\sin^2\left(\dfrac{h}{2}\right)\csc(h)\csc(2h)+2\sin\left(\dfrac{h}{2}\right)\sin(h)\csc\left(\dfrac{3h}{2}\right)\csc(2h)\right)$,

$\quad i=1,2,3,\dots,n$

$$s_i = \csc(h)\csc(2h) - \frac{c}{x_i}\left(\frac{\csc(h)\sec(h)}{2+\cos(h)}\right) + g(x_i)\left(\sin^3\left(\frac{h}{2}\right)\csc(h)\csc\left(\frac{3h}{2}\right)\csc(2h)\right), \quad i = 1,2,3,\dots,n,$$
$$x = \sin^3\left(\frac{h}{2}\right)\csc(h)\csc\left(\frac{3h}{2}\right)\csc(2h),$$
$$y = \sin^2\left(\frac{h}{2}\right)\csc(h)\csc(2h) + 2\sin\left(\frac{h}{2}\right)\sin(h)\csc\left(\frac{3h}{2}\right)\csc(2h).$$

4. Optimization

In this section, we find the value of λ by minimizing the L_2-norm error. By applying the linear solve built in MATHEMATICA 9 to above linear matrix system (3.8) for the solution of proposed problem. All C_i, $i = -4,-3,\dots,n-2$ can be written in the form of λ. Thus, for each interval $[x_i, x_{i+1}]$, $i = 0,1,\dots,n-1$, we obtain

$$S_T(x,\lambda) = C_{i-4}(\lambda)TB^5_{i-4}(x) + C_{i-3}(\lambda)TB^5_{i-3}(x) + C_{i-2}(\lambda)TB^5_{i-2}(x) + C_{i-1}(\lambda)TB^5_{i-1}(x). \tag{4.1}$$

Suppose our assumption is very close to the exact solution $y(x)$. Thus equation (3.3) can be changed as [4]

$$S''_T(x,\lambda) + \frac{c}{x}S'_T(x,\lambda) + g(x)S_T(x,\lambda) \approx f(x). \tag{4.2}$$

Therefore, we can find the error as

$$E_T(x,\lambda) = S''_T(x,\lambda) + \frac{c}{x}S'_T(x,\lambda) + g(x)S_T(x,\lambda) - f(x). \tag{4.3}$$

The largest error to be found at the midpoint $x_i^* = \frac{x_i + x_{i+1}}{2}$ of each interval $[x_i, x_{i+1}]$, $i = 1,\dots,n$, so approximated error $E_T(x^*,\lambda)$ can be written as

$$E_T(x_i^*,\lambda) = S''_T(x_i^*,\lambda) + \frac{c}{x_i^*}S'_T(x_i^*,\lambda) + g(x_i^*)S_T(x_i^*,\lambda) - f(x_i^*). \tag{4.4}$$

The above equation contains only one variable λ. Now, we crave to reduce the error norm L_2, such that

$$L_2 = \sqrt{\sum_{i=1}^{n} E_T(x_i^*,\lambda)} = 0. \tag{4.5}$$

From the equation (4.5), we have the values of λ and all the other unknowns $C_{-4}, C_{-3}, \dots, C_{n-2}$. Thus, the solution at each knots x_i can be approximated from equation (2.4).

5. Numerical Results and Discussions

In this section, several numerical examples are considered to demonstrate the competency of the proposed quartic trigonometric spline collocation approach. Numerical results obtained by the proposed method are compared with existing techniques in the literature such as Ravi Kanth and Reddy [20,21], Caglar and Caglar [7,8], Joan Goh *et al.* [10], Gupta and Kumar [11] and with the analytical solution at knots $x = x_i$ using different values of n. It was establish that proposed technique in contrast with these methods is more accurate.

Problem 1. Consider the Bessel's equation of order 0 [10]

$$y''(x)+\frac{1}{x}y'(x)+y(x)=0,\quad y'(0)=0,\quad y(1)=1.$$

The analytic solution of the problem is $\frac{J_0(x)}{J_0(1)}$.

The maximum norm(L_∞) and Euclidean norm(L_2) are calculated using the following formula

$$L_\infty=\|S_T(x_i)-y(x_i)\|_\infty=\max|S_T(x_i)-y(x_i)|,$$

$$L_2=\|S_T(x_i)-y(x_i)\|_2=\sqrt{\sum_i[S_T(x_i)-y(x_i)]^2}\ ,\ \text{respectively.}$$

The numerical order of convergence, R of the present method, is calculated by following formula [1,3]

$$R=\frac{\log(L_\infty(n_i))-\log(L_\infty(n_{i+1}))}{\log(n_{i+1})-\log(n_i)}.$$

Table 1 shows the maximum norm and Euclidean norm at different step size h. It concluded that the results obtained by proposed method is more accurate than the the cubic-B-spline method [7] and quartic B-spline method [10]. The order of convergence is found to be fourth. Figure 1 depicts the comparison of approximated solution with analytical solution at $h=0.05$.

Table 1. Maximum errors (L_∞) and norm errors (L_2) for Problem 1 at different h.

h	Cubic B-spline [7]		Quartic B-spline [10]		Present method		Order of Convergence
	L_∞	L_2	L_∞	L_2	L_∞	L_2	R
0.2	...	...	...	...	7.34E-07	1.24E-06	...
0.1	1.14E-04	2.68E-04	1.67E-06	1.89E-06	5.43E-08	1.09E-07	3.756169
0.05	2.82E-05	9.20E-05	2.04E-07	2.34E-07	2.99E-09	9.22E-09	4.181910
0.025	...	...	...	...	1.79E-10	8.03E-10	4.063657
0.0125	...	...	...	...	1.12E-11	7.12E-11	3.999537
0.02	4.49E-06	2.29E-05	1.42E-08	1.62E-08	7.29E-11	3.67E-10	...
0.01	1.12E-06	8.05E-06	1.44E-09	1.64E-09	4.90E-12	3.56E-11	...

Table 2. Comparison of the approximated solutions with the exact solutions for Problem 2 when $h=0.05$ at different knots.

x	Exact	Cubic B-spline [7]	HFDM [21]	Cubic spline [20]	Quartic B-spline [10]	Present method
0.0	3.257206	3.256908	3.257208	3.256912	3.257206	3.257205
0.2	3.331322	3.331030	3.331323	3.331033	3.331321	3.331321
0.4	3.560864	3.560589	3.560864	3.560592	3.560863	3.560863
0.6	3.968246	3.968011	3.968247	3.968013	3.968246	3.968246
0.8	4.593706	4.593551	4.593706	4.593551	4.593706	4.593706
1.0	5.500000	5.500000	5.50000	5.500000	5.500000	5.500000

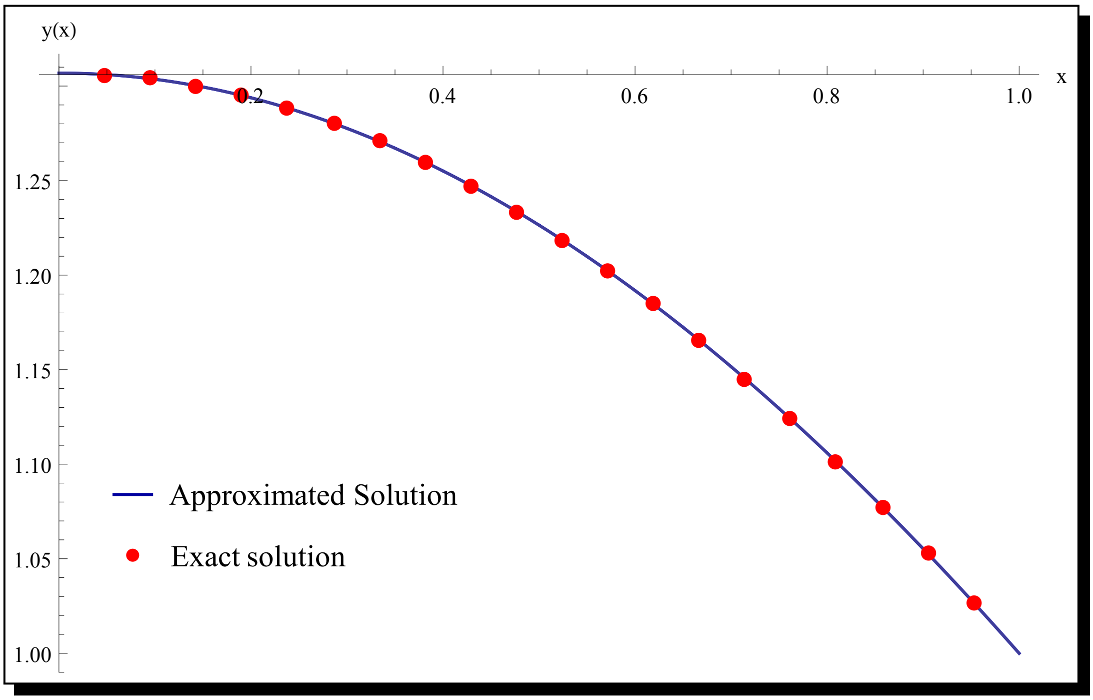

Figure 1. Comparison graph of approximated solution and exact solution for Problem 1 when $h = 0.05$.

Problem 2. Consider the second order singular differential equation [7, 10]

$$y''(x) + \frac{2}{x} y'(x) - 4y(x) = -2, \quad 0 < x \le 1$$

with boundary conditions

$$y'(0) = 0, \quad y(1) = 5.5.$$

The analytical solution is $y(x) = 0.5 + \frac{5\sinh 2x}{x\sinh 2}$. The numerical approximations calculated for each knot x_i, when $n = 20$ are tabulated in Table 2 and compared the results with methods developed in [7, 10, 20, 21]. The order of convergence can be calculated to be fourth numerically which is tabulated in Table 3. The comparison of approximated solutions with analytical solutions at different knots when $n = 20$ is displayed in Figure 2.

Table 3. The maximum norm errors, Euclidean norm and order of convergence at different step size h for Problem 2.

h	Present method		Order of Convergence
	L_∞	L_2	R
0.2	5.61E-05	9.08E-05	. . .
0.1	2.79E-06	6.97E-06	4.331805
0.05	1.68E-07	6.30E-07	4.049642
0.025	1.04E-08	5.57E-08	4.013696
0.0125	6.49E-10	4.91E-09	4.003998

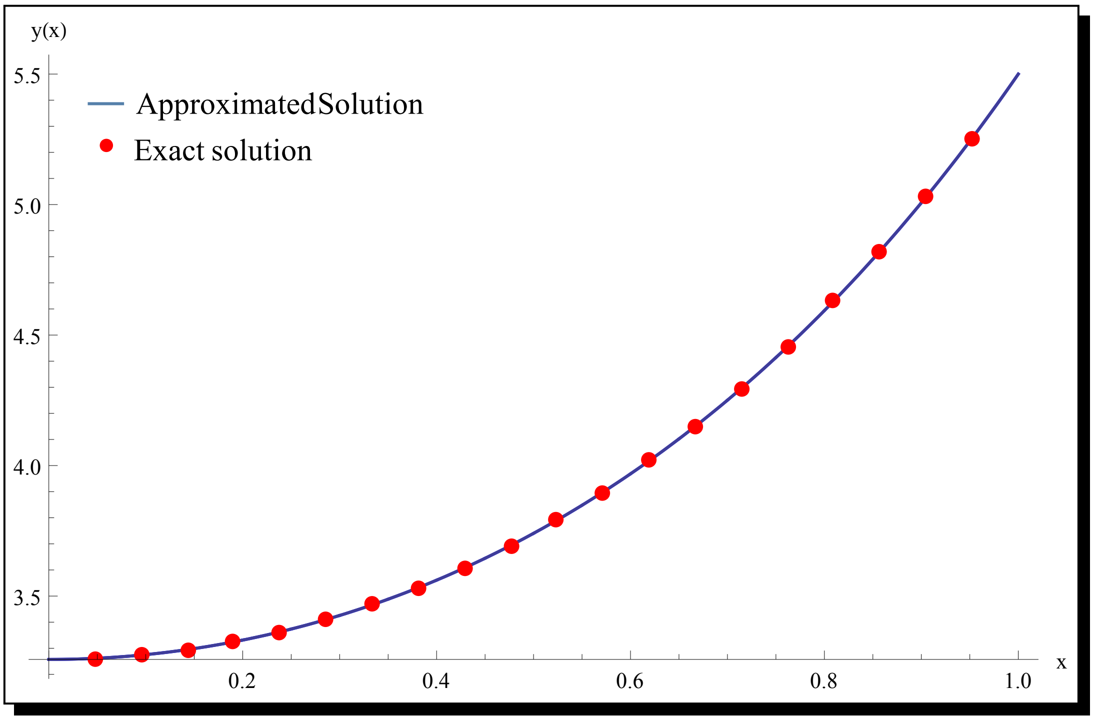

Figure 2. Comparison graph of approximated solution and exact solution for Problem 2 when $h = 0.05$.

Problem 3. Consider the second order singular boundary value problem [7, 10]

$$y''(x) + \frac{1}{x} y'(x) = \left(\frac{8}{8 - x^2}\right)^2, \quad 0 < x \le 1$$

$$y'(0) = 0, \quad y(1) = 0.$$

The analytical solution is $y(x) = \log\left(\frac{7}{8-x^2}\right)$. The comparison of approximated solution of second order singular boundary value problem with exact solution at different knots and the maximum errors, norm errors and order of convergence of the proposed method is tabulated in Table 3. Figure 3 shows the numerical solution and exact solution at $n = 20$.

Table 4. Comparison of approximate solution with the exact solutions for Problem 3 when $h = 0.05$ and order of convergence at different h.

x	Exact	Present method	Error	h	Present method L_∞	Present method L_2	Order of convergence R
0.0	–0.267063	–0.267063	3.46E-08	0.2	8.46E-06	1.45E-05	. . .
0.25	–0.251376	–0.251376	2.98E-08	0.1	6.55E-07	1.32E-06	3.691739
0.50	–0.203565	–0.203565	2.59E-08	0.05	3.46E-08	1.11E-07	4.241060
0.75	–0.121249	–0.121249	1.66E-08	0.025	2.04E-09	9.71E-09	4.086573
1.00	0	6.51E-19	–6.51E-19	0.0125	1.25E-10	8.53E-10	4.031099

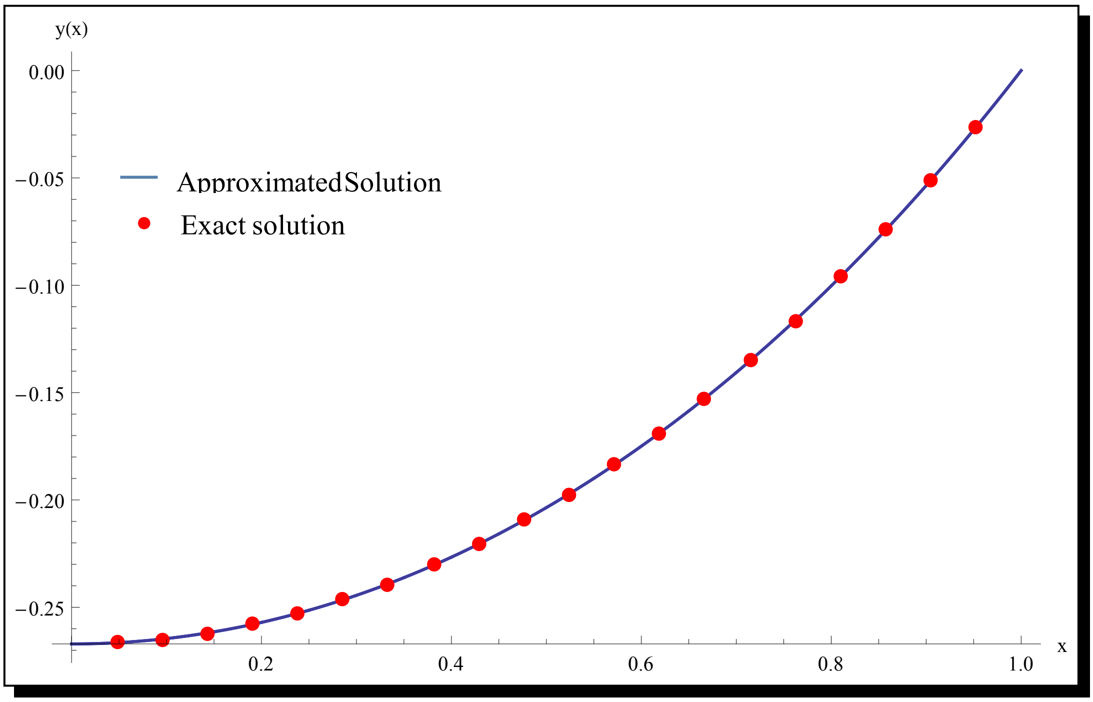

Figure 3. Comparison graph of approximated solution and exact solution for Problem 3 when $h = 0.05$.

Problem 4.

$$y''(x) + \frac{1}{x} y'(x) = \frac{\pi}{2x}\left(\sin\frac{\pi x}{2} + \frac{\pi x}{2}\cos\frac{\pi x}{2}\right),$$

with boundary conditions

$$y'(0) = y(1) = 0.$$

The analytical solution is $y(x) = -\cos\frac{\pi x}{2}$. Table 5 shows the maximum error norm (L_∞) which is calculated at different values of step size h and compared the results with modified hierarchy basis method [19] and cubic trigonometric B-spline method [11]. Figure 4 depicts the comparison of numerical solutions with analytical solutions at $h = 1/32$. It is concluded that the present quartic trigonometric B-Spline method is more accurate than the methods developed in [11, 19].

Table 5. The maximum error norm (L_∞) for Problem 4 at different values of h.

h	Modified hierarchy basis method [19]	Cubic TB-spline [11]	Present method	Order of convergence
	L_∞	L_∞	L_∞	R
1/16	. . .	. . .	4.09E-08	. . .
1/32	3.07E-02	2.88E-06	2.39E-09	4.099220
1/64	1.34E-02	4.96E-07	1.46E-10	4.034051
1/128	6.20E-03	9.30E-08	9.00E-12	4.018103

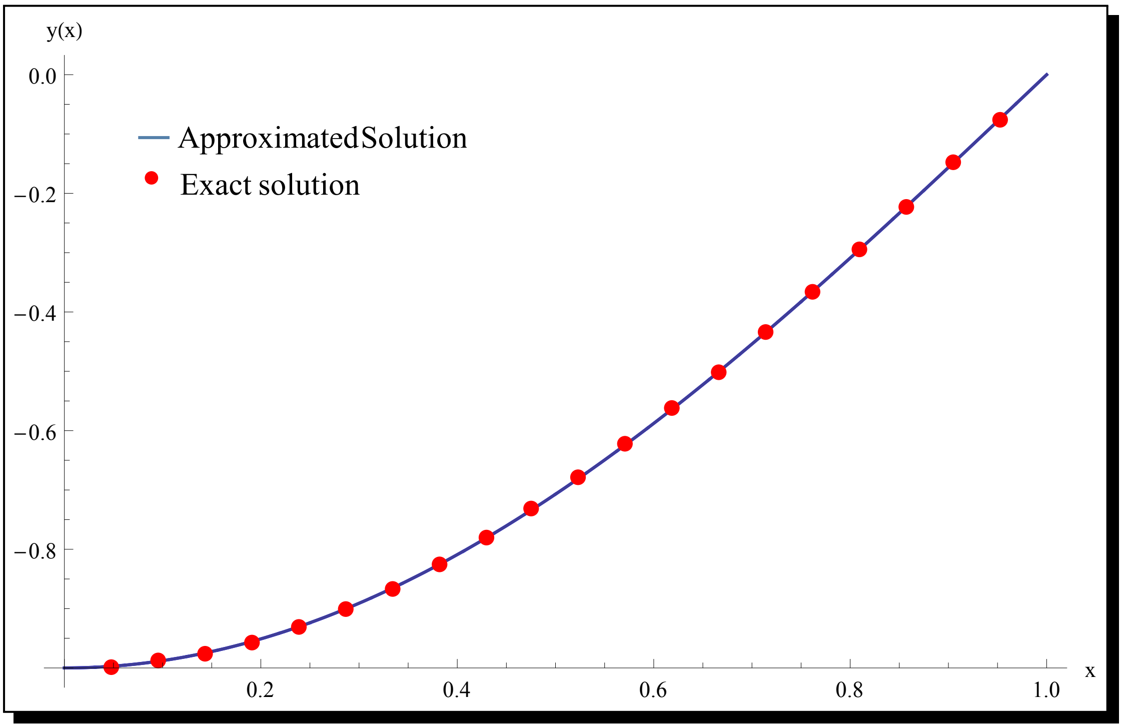

Figure 4. A approximated solution and exact solution for Problem 4 when $h = 1/32$.

6. Concluding Remarks

In this paper a numerical approach grounded on quartic trigonometric B-spline functions has been utilized to solve the second order singular boundary value problems. The quartic trigonometric B-spline method used in this paper is simple and straight forward to apply. The numerical results reported in the Tables $1-5$ and depicted in the graphs illustrated the applicability and accuracy of the method when compared with other available methods like finite difference method [21], cubic spline methods [20], cubic and quartic B-spline collocation methods [7, 8, 10] and cubic trigonometric B-spline method [11] or compare favorably with them to say the least.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

The authors wrote, read and approved the final manuscript.

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