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Opial-type Inequalities for Generalized Integral Operators With Special Kernels in Fractional Calculus

Research Article

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Abstract. In this article, we originate some new Opial-type inequalities on fractional calculus involving generalized Riemann-Liouville fractional integral, the Riemann-Liouville k-fractional integral, the (k,r) fractional integral of the Riemann-type and the generalized fractional integral operator involving Hypergeometric function in its kernel. As special case of our general results we obtain the results of Farid *et al*. [7].

Keywords. Opial-type inequalities; Generalized Riemann-Liouville fractional integral operator; Riemann-Liouville k-fractional integral; (k,r) fractional integral of the Riemann-type

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1. Introduction

The subject of fractional calculus achieve a significant popularity during last few decades, due to its demonstrated applications in the fields of science and engineering. It provide several potentially useful tools for solving differential and integral equations. Now a days the applications of fractional calculus include fluid flow, rheology, dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, electrical networks, probability and statistics, control theory of dynamical systems, Optics and signal processing, and so on. Many mathematician originate Hardy-type inequalities for different fractional order integrals and derivatives.

One of the most useful tools in the study of many quantitative as well as qualitative properties of solutions of differential and integral equations is integral inequalities. Among these, the so-called Opial inequality have attracted continual interest of researchers and have proven to be very important in many situations. This inequality [13] (see also [10, p. 114]) was came into sight in 1960 and is stated as:

Theorem 1.1. *Let $f \in C^1[0,h]$ satisfies $f(0) = f(h) = 0$ and $f(x) > 0$ on $(0,h)$, then*

$$\int_0^h |f(x)f'(x)|dx \leq \frac{h}{4}\int_0^h |f'(x)|^2 dx.$$

The constant $\frac{h}{4}$ is the best choice.

It is one of the fundamental inequalities among integrals and an indispensable tool for the study of L_p spaces. It used to prove the Minkowski inequality, which is the triangle inequality in the space $L_p(\mu)$ and also to establish that $L_q(\mu)$ is the dual space of $L_p(\mu)$ for $p \in [1,\infty)$.

In recent years, Opial's inequality has been further generalized to many different aspects, for instance, to integral inequalities involving higher order derivatives of the given function [4] and to integral inequalities involving many functions of multiple variables [6] in which the integrands are product of powers of the given functions and their derivatives. Agarwal, Alzer and Pang [1–3] study the Opial-type inequalities involving ordinary derivatives and their applications in differential and difference equations.

We say that a function $g : [a,b] \to \mathbb{R}$ belongs to the class $U(f,k)$ if it admits the representation

$$|g(t)| \leq \int_a^t k(t,\tau)|f(\tau)|d\tau,$$

where f is a continuous function and k a non-negative kernel, $f(t) > 0$ implies $g(t) > 0$ for every $x \in [a,b]$. We also assume that all integrals under consideration exist and that they are finite.

We start with the generalized L_p space given in [12] defined as follows:

Definition 1.2. The space $L_{p,r}[a,b]$ is defined as a space of continuous real valued function $h(y)$ on $[a,b]$, such that

$$\left(\int_a^b |h(t)|^p t^r dt\right)^{\frac{1}{p}} < \infty,$$

where $1 \leq p < \infty$, and $r \geq 0$.

Next we give the well known definition of Riemann-Liouville fractional integrals (see [9]).

Definition 1.3. Let $[a,b]$ be a finite interval on $\mathbb{R}$. The left and right sided Riemann-Liouville fractional integrals $I^{\alpha}_{a^+}f$ and $I^{\alpha}_{b^-}f$ of order $\alpha > 0$ are defined as:

$$I^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau, \quad t > a,$$

and

$$I^{\alpha}_{b^-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1}f(\tau)d\tau, \quad t < b,$$

respectively. Here Γ represents usual Gamma function which is defined as:

$$\Gamma(t) = \int_0^{\infty} \tau^{t-1}\exp^{-\tau} d\tau.$$

Definition 1.4. If $f \in L_{1,r}[a,b]$, then the generalized *Riemann-Liouville fractional integral* $I^{\alpha,r}_a$ of order α and $r \geq 0$ is defined by

$$I^{\alpha,r}_a f(t) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (t^{r+1}-\tau^{r+1})^{\alpha-1}\tau^r f(\tau)d\tau, \quad \alpha > 0. \tag{1.1}$$

Following definition of k-fractional integral operator of Riemann-type is introduced in [11] and is defined as:

Definition 1.5. If $f \in L_1[a,b]$ and $k > 0$, then Riemann-Liouville k-fractional integral $I^{\alpha}_{k,a}$ of order α is defined by

$$I^{\alpha}_{k,a}f(t) = \frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-\tau)^{\frac{\alpha}{k}-1}f(\tau)d\tau, \quad \alpha > 0, t \in [a,b], \tag{1.2}$$

where Γ_k is the Gamma k-function which is defined as:

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1}\exp^{-\frac{t^k}{k}} dt.$$

Next (k,r) fractional integral operator given in [14] is defined as follows:

Definition 1.6. If $f \in L_{1,r}[a,b]$ and $k > 0$, then the Riemann-Liouville (k,r)-fractional integral $I^{\alpha,r}_{k,a}$ of order α and $r \geq 0$, is defined as

$$I^{\alpha,r}_{k,a}f(t) = \frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t (t^{r+1}-\tau^{r+1})^{\frac{\alpha}{k}-1}\tau^r f(\tau)d\tau, \alpha > 0, \quad t \in [a,b]. \tag{1.3}$$

Next definition is introduced in [5].

Definition 1.7. Let $\alpha > 0$, $\mu > -1$, $\beta,\eta \in \mathbb{R}$. Then the generalized fractional integral $I^{\alpha,\beta,\eta,\mu}_{\alpha,x}$ of order α, for a real-valued continuous function f is defined by:

$$I^{\alpha,\beta,\eta,\mu}_{\alpha,x}f(x) = \frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_a^x t^{\mu}(x-t)^{\alpha-1}\,{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{x}\right)f(t)dt, \quad x \in [a,b],$$

where, the function ${}_2F_1(\cdot,\cdot,;\cdot)$ appearing in kernel of above operator is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}t^n,$$

and $(a)_n$ is the Pochhammer symbol which is defined as:

$$(a)_n = a(a+1)\dots(a+n-1), (a)_0 = 1.$$

The following Opial-type inequality was given in [8].

Theorem 1.8. *Let $g_1 \in U(f_1,k)$, $g_2 \in U(f_2,k)$. Let $\varphi > 0$, $w \geq 0$ be measurable functions on $[a,x]$, and k a non-negative measurable kernel. Let $s > 1$, $s > q > 0$ and $p \geq 0$. Let $f_1, f_2 \in L_s[a,b]$, then the following inequality holds:*

$$\int_a^x w(t)(|g_1(t)|^p|f_2(t)|^q + |g_2(t)|^p|f_1(t)|^q)dt \leq 2^{1-\frac{q}{s}}\left(\frac{q}{p+q}\right)^{\frac{q}{s}}\left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{s}}\left(\int_a^x [h(t)]^{\frac{s}{s-q}}dt\right)^{\frac{s-q}{s}} \times \left(\int_a^x \varphi(\tau)\left[|f_1(\tau)|^s + |f_2(\tau)|^s\right]d\tau\right)^{\frac{p+q}{s}},$$

where

$$h(t) = w(t)\left[\int_a^t k(t,\tau)^{\frac{s}{s-1}}\varphi(\tau)^{\frac{1}{1-s}}d\tau\right]^{\frac{p(s-1)}{s}}[\varphi(t)]^{-\frac{q}{s}}, \tag{1.4}$$

and

$$d_{\frac{p}{q}} = \begin{cases} 2^{1-\frac{p}{q}}, & 0 \leq p \leq q; \\ 1, & p \geq q. \end{cases} \tag{1.5}$$

The next result is the extreme case of Theorem 1.8.

Theorem 1.9. *Let $g_i \in U(f_1,k_i)$, $\tilde{g}_i \in U(f_2,k_i)$, $(i = 1,2)$. Let $w \geq 0$ be measurable function on $[a,x]$ and $p, q_1, q_2 \geq 0$ and $f_1, f_2 \in Ł_\infty[a,b]$, then the following inequality*

$$\int_a^x w(t)\left[|g_1(t)|^{q_1}|\tilde{g}_2(t)|^{q_2}|f_1(t)|^p + |g_2(t)|^{q_2}|\tilde{g}_1(t)|^{q_1}|f_2(t)|^p\right]dt$$
$$\leq \|w\|_\infty \int_a^x \left(\int_a^t k_1(t,\tau)d\tau\right)^{q_1}\left(\int_a^t k_2(t,\tau)d\tau\right)^{q_2}dt\ \frac{1}{2}\left[\|f_1\|_\infty^{2(q_1+p)} + \|f_1\|_\infty^{2q_2} + \|f_2\|_\infty^{2q_2} + \|f_2\|_\infty^{2(q_1+p)}\right]$$

holds.

The counter part of Theorem 1.8 is as follows:

Theorem 1.10. *Let $g_i \in U(f_i,k)$, $(i = 1,2)$. Let $\varphi > 0$, $w \geq 0$ be measurable functions on $[a,x]$ and k a non-negative measurable kernel. Let $s < 0$, $q > 0$ and $p \geq 0$. Let $f_1, f_2 \in L_s[a,b]$, each of which is of fixed sign a.e. on $[a,b]$, with $\frac{1}{f_1}, \frac{1}{f_2} \in L_s[a,b]$, then the following inequality holds:*

$$\int_a^x w(t)(|g_1(t)|^p|f_2(t)|^q + |g_2(t)|^p|f_1(t)|^q)dt$$
$$\geq 2^{1-\frac{q}{s}}\left(\frac{q}{p+q}\right)^{\frac{q}{s}}\left(c_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{s}}\left(\int_a^x [h(t)]^{\frac{s}{s-q}}dt\right)^{\frac{s-q}{s}}\left(\int_a^x \varphi(\tau)\left[|f_1(\tau)|^s + |f_2(\tau)|^s\right]d\tau\right)^{\frac{p+q}{s}},$$

where $h(t)$ is defined by (1.4) *and*

$$c_{\frac{p}{q}} = \begin{cases} 1, & 0 \leq p \leq q; \\ 2^{1-\frac{p}{q}}, & p \geq q. \end{cases} \tag{1.6}$$

The rest of the paper is organized in the following way: In Section 2, we present the Opial-type inequalities for the generalized Riemann-Liouville fractional integral operator. Section 3 deals with the results for the Riemann-Liouville k-fractional integral operator. Section 4 includes the consequences for the (k,r) fractional integral of the Riemann-type. Section 5 consists of Opial-type inequalities for generalized fractional integral operator. Moreover, we deduce in particular the results of [7] from our general results.

2. Opial-type Inequalities for the Generalized Riemann-Liouville Fractional Integral Operator

This section consists of Opial-type inequalities for generalized Riemann-Liouvile fractional integral operator defined by (1.1).

Our first main result is presented in the following theorem.

Theorem 2.1. *Let $r \geq 0$, $I_a^{\alpha,r}$ be the generalized Riemann-Liouville fractional integral operator $s > 1$, $0 < q < s$ and $p \geq 0$. If $\varphi > 0$, $w \geq 0$ are measurable function on $[a,x]$ and $f_1, f_2 \in L_{s,r}[a,b]$, then the inequality*

$$\begin{aligned}
&\int_a^x w(t)\left[|I_a^{\alpha,r} f_1(t)|^p |f_2(t)|^q + |I_a^{\alpha,r} f_2(t)|^p |f_1(t)|^q\right] dt \\
&\leq 2^{1-\frac{q}{s}} \left(\frac{q}{p+q}\right)^{\frac{q}{s}} \left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{s}} \left(\int_a^x [h(t)]^{\frac{s}{s-q}} dt\right)^{\frac{s-q}{s}} \left[\int_a^x \varphi(\tau)[|f_1(\tau)|^s + |f_2(\tau)|^s] d\tau\right]^{\frac{p+q}{s}} \qquad (2.1)
\end{aligned}$$

holds, where

$$h(t) = w(t)\varphi(t)^{-\frac{q}{s}} \left[\int_a^t \left[\frac{(r+1)^{1-\alpha}(t^{r+1}-\tau^{r+1})^{\alpha-1}\tau^r}{\Gamma(\alpha)}\right]^{\frac{s}{s-1}} \varphi(\tau)^{\frac{1}{1-s}} d\tau\right]^{\frac{p(s-1)}{s}}, \qquad (2.2)$$

and $d_{\frac{p}{q}}$ is defined by (1.5).

Proof. Applying Theorem 1.8 with $g_1(t) = I_a^{\alpha,r} f_1(t)$, $g_2(t) = I_a^{\alpha,r} f_2(t)$ and kernel

$$\widetilde{k}(t,\tau) = \begin{cases} \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}(t^{r+1}-\tau^{r+1})^{\alpha-1}\tau^r, & a \leq \tau \leq t; \\ 0, & t < \tau < b, \end{cases} \qquad (2.3)$$

we get the required inequality (2.1). □

The upcoming result is the extreme case of the Theorem 2.1.

Theorem 2.2. *Let $I_a^{\alpha,r}$ be the generalized Riemann-Liouville fractional integral operator. Let $w \geq 0$ be measurable function on $[a,x]$, $p, q_1, q_2 \geq 0$ and $f_1, f_2 \in L_\infty[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\int_a^x w(t)\left[|I_a^{\alpha,r} f_1(t)|^{q_1} |\widetilde{I}_a^{\alpha,r} f_2(t)|^{q_2} |f_1(t)|^p + |I_a^{\alpha,r} f_1(t)|^{q_2} |\widetilde{I}_a^{\alpha,r} f_2(t)|^{q_1} |f_2(t)|^p\right] \\
&\leq \|w\|_\infty \left(\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}\right)^{q_1+q_2} \int_a^x \left(\int_a^t (t^{r+1}-\tau^{r+1})^{\alpha-1}\tau^r d\tau\right)^{q_1+q_2} dt \\
&\quad \times \frac{1}{2}\left[\|f_1\|_\infty^{2(q_1+p)} + \|f_1\|_\infty^{2q_2} + \|f_2\|_\infty^{2q_2} + \|f_2\|_\infty^{2(q_1+p)}\right]. \qquad (2.4)
\end{aligned}$$

Proof. Applying Theorem 1.9 with $g_1(t) = g_2(t) = I_a^{\alpha,r} f_1(t)$, $\widetilde{g}_1(t) = \widetilde{g}_2(t) = \widetilde{I}_a^{\alpha,r} f_2(t)$ and kernel $\widetilde{k}(t,\tau)$ presented in (2.3), we get the required inequality (2.4). □

The counter part of Theorem 2.1 for the case $s < 0$ is given in the following result.

Theorem 2.3. *Let $r \ge 0$, $s < 0$, $q > 0$ and $p \ge 0$. If $\varphi > 0$, $w \ge 0$ are measurable function on $[a,x]$, and $f_1, f_2 \in L_{s,r}[a,b]$ each of which is of fixed sign a.e on $[a,b]$, with $\frac{1}{f_1}, \frac{1}{f_2} \in L_{s,r}[a,b]$, then the following inequality holds:*

$$\int_a^x w(t)\left[|I_a^{\alpha,r} f_1(t)|^p |f_2(t)|^q + |I_a^{\alpha,r} f_2(t)|^p |f_1(t)|^q\right] dt$$
$$\ge 2^{1-\frac{q}{s}} \left(\frac{q}{p+q}\right)^{\frac{q}{s}} \left(c_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{s}} \left(\int_a^x [h(t)]^{\frac{s}{s-q}} dt\right)^{\frac{s-q}{s}} \left[\int_a^x \varphi(\tau)[|f_1(\tau)|^s + |f_2(\tau)|^s] d\tau\right]^{\frac{p+q}{s}}, \quad (2.5)$$

where $c_{\frac{p}{q}}$ and $h(t)$ are defined in (1.6) and (2.2), respectively.

Proof. Applying Theorem 1.10 with $g_1(t) = I_a^{\alpha,r} f_1(t)$ and $g_2(t) = I_a^{\alpha,r} f_2(t)$ and kernel $\widetilde{k}(t,\tau)$ presented in (2.3), we get inequality (2.5). □

Remark 2.4. In particular, if we select $r = 0$, in Theorems 2.1, 2.2 and 2.3, we arrive at [7, Corollary 3.10, Corollary 3.14 and Theorem 3.11].

3. Opial-type Inequalities for the Riemann-Liouville k-fractional Integral Operator

This section deals Opial-type inequalities for the Riemann-Liouvile k-fractional integral defined by (1.2). The first result for k-fractional integral operator of the Riemann-type is described in the following theorem.

Theorem 3.1. *Let $k > 0$, $s > 1$, $0 < q < s$ and $p \ge 0$. If $\varphi > 0$, $w \ge 0$ are measurable function on $[a,x]$, and $f_1, f_2 \in L_s[a,b]$, then the following inequality holds:*

$$\int_a^x w(t)\left[|I_{k,a}^{\alpha} f_1(t)|^p |f_2(t)|^q + |I_{k,a}^{\alpha} f_2(t)|^p |f_1(t)|^q\right] dt$$
$$\le 2^{1-\frac{q}{s}} \left(\frac{q}{p+q}\right)^{\frac{q}{s}} \left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{s}} \left(\int_a^x [h(t)]^{\frac{s}{s-q}} dt\right)^{\frac{s-q}{s}} \left[\int_a^x \varphi(\tau)[|f_1(\tau)|^s + |f_2(\tau)|^s] d\tau\right]^{\frac{p+q}{s}}, \quad (3.1)$$

where

$$h(t) = w(t)\varphi(t)^{-\frac{q}{s}} \left[\int_a^t \left[\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\right]^{\frac{s}{s-1}} \varphi(\tau)^{\frac{1}{1-s}} d\tau\right]^{\frac{p(s-1)}{s}}, \quad (3.2)$$

and $d_{\frac{p}{q}}$ is defined by (1.5).

Proof. Applying Theorem 1.8 with $g_1(t) = I_{k,a}^{\alpha} f_1(t)$, $g_2(t) = I_{k,a}^{\alpha} f_2(t)$ and kernel

$$\bar{k}(t,\tau) = \begin{cases} \frac{1}{k\Gamma_k(\alpha)}(t-\tau)^{\frac{\alpha}{k}-1}, & a \le \tau \le t; \\ 0, & t < \tau < b, \end{cases} \quad (3.3)$$

we get the required inequality (3.1). □

The extreme case of Theorem 3.1 is as follows:

Theorem 3.2. *Let $k>0$, $I^{\alpha}_{k,a}f_i \in U(f_1,k_i)$, $\tilde{I}^{\alpha}_{k,a}f_i \in U(f_2,k_i)$, $(i=1,2)$. Let $w \geq 0$ be measurable function on $[a,x]$, $p,q_1,q_2 \geq 0$ and $f_1,f_2 \in L_{\infty}[a,b]$, then the following inequality holds:*

$$\int_a^x w(t)\left[|I^{\alpha}_{k,a}f_1(t)|^{q_1}|\tilde{I}^{\alpha}_{k,a}f_2(t)|^{q_2}|f_1(t)|^p + |I^{\alpha}_{k,a}f_1(t)|^{q_2}|\tilde{I}^{\alpha,r}_{k,a}f_2(t)|^{q_1}|f_2(t)|^p\right]$$

$$\leq \|w\|_{\infty}\left(\frac{1}{k\Gamma_k(\alpha)}\right)^{q_1+q_2}\int_a^x\left(\int_a^t (t-\tau)^{\frac{\alpha}{k}-1}d\tau\right)^{q_1+q_2}dt$$

$$\times\frac{1}{2}\left[\|f_1\|_{\infty}^{2(q_1+p)}+\|f_1\|_{\infty}^{2q_2}+\|f_2\|_{\infty}^{2q_2}+\|f_2\|_{\infty}^{2(q_1+p)}\right]. \tag{3.4}$$

Proof. Applying Theorem 1.9 with $g_1(t)=g_2(t)=I^{\alpha}_{k,a}f_1(t)$, $\tilde{g}_1(t)=\tilde{g}_2(t)=\tilde{I}^{\alpha}_{k,a}f_2(t)$, and kernel $\bar{k}_1(t,\tau)=\bar{k}_2(t,\tau)=\bar{k}(t,\tau)$ given in (3.3), we get inequality (3.4). □

The counter part of Theorem 3.1 for the case $s<0$ is given in the following result.

Theorem 3.3. *Let $k>0$, $s<0$, $q>0$ and $p\geq 0$. If $\varphi>0$, $w\geq 0$ are measurable function on $[a,x]$. Let $f_1,f_2\in L_s[a,b]$ each of which is of fixed sign a.e. on $[a,b]$, with $\frac{1}{f_1},\frac{1}{f_2}\in L_s[a,b]$, then the following inequality holds:*

$$\int_a^x w(t)[|I^{\alpha}_{k,a}f_1(t)|^p|f_2(t)|^q+|I^{\alpha}_{k,a}f_2(t)|^p|f_1(t)|^q]dt$$

$$\geq 2^{1-\frac{q}{s}}\left(\frac{q}{p+q}\right)^{\frac{q}{s}}\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{s}}\left(\int_a^x [h(t)]^{\frac{s}{s-q}}dt\right)^{\frac{s-q}{s}}\left[\int_a^x\varphi(\tau)[|f_1(\tau)|^s+|f_2(\tau)|^s]d\tau\right]^{\frac{p+q}{s}},$$

where $c_{\frac{p}{q}}$ and $h(t)$ are defined in (1.6) and (3.2), respectively.

Proof. Applying Theorem 1.10 with $g_1(t)=I^{\alpha}_{k,a}f_1(t)$ and $g_2(t)=I^{\alpha}_{k,a}f_2(t)$ and kernel $\bar{k}(t,\tau)$ given in (3.3), we get the required result. □

Remark 3.4. In particular, if we choose $k=1$ in Theorems 3.1, 3.2 and 3.3, we turn up [7, Corollary 3.10, Corollary 3.14 and Theorem 3.11].

4. Opial-type Inequalities for the Generalized Riemann-Liouville (k,r)-fractional Integral Operator

The first result for the generalized the Riemann-Liouville (k,r)-fractional integral operator (1.3) is as follows:

Theorem 4.1. *Let $r\geq 0$, $k>0$, $s>1$, $0<q<s$ and $p\geq 0$. If $\varphi>0$, $w\geq 0$ are measurable function on $[a,x]$ and $f_1,f_2\in L_{s,r}[a,b]$, then the following inequality holds:*

$$\int_a^x w(t)[|I^{\alpha,r}_{k,a}f_1(t)|^p|f_2(t)|^q+|I^{\alpha,r}_{k,a}f_2(t)|^p|f_1(t)|^q]dt$$

$$\leq 2^{1-\frac{q}{s}}\left(\frac{q}{p+q}\right)^{\frac{q}{s}}\left(d_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{s}}\left(\int_a^x [h(t)]^{\frac{s}{s-q}}dt\right)^{\frac{s-q}{s}}\left[\int_a^x\varphi(\tau)[|f_1(\tau)|^s+|f_2(\tau)|^s]d\tau\right]^{\frac{p+q}{s}}, \tag{4.1}$$

where

$$h(t)=w(t)\varphi(t)^{-\frac{q}{s}}\left[\int_a^t\left[\frac{(r+1)^{1-\frac{\alpha}{k}}(t^{r+1}-\tau^{r+1})^{\frac{\alpha}{k}-1}\tau^r}{k\Gamma_k(\alpha)}\right]^{\frac{s}{s-1}}\varphi(\tau)^{\frac{1}{1-s}}d\tau\right]^{\frac{p(s-1)}{s}}, \tag{4.2}$$

and $d_{\frac{p}{q}}$ *is defined by* (1.5).

Proof. Applying Theorem 1.8 with $g_1(t)=I^{\alpha,r}_{k,a}f_1(t)$ and $g_2(t)=I^{\alpha,r}_{k,a}f_2(t)$ and kernel

$$\widehat{k}(t,\tau)=\begin{cases}\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}(t^{r+1}-\tau^{r+1})^{\frac{\alpha}{k}-1}\tau^r, & a\le\tau\le t;\\ 0, & t<\tau<b,\end{cases} \tag{4.3}$$

we get the required inequality (4.1). □

Example 4.2. For $k=1$, $r=0$, $w(t)=1$, $\varphi(t)=1$, $s=2$ and $a=0$ we have

$$h(t)=\left[\frac{t^{2\alpha-1}}{(\Gamma(\alpha))^2(2\alpha-1)}\right]^{\frac{p}{2}}$$

where $0<q<2$ and $p\ge 0$. If $f_1,f_2\in L_2[0,b]$, then for all $x\in[0,b]$ we obtain

$$\int_0^x\left[|I^{\alpha}_{0^+}f_1(t)|^p|f_2(t)|^q+|I^{\alpha}_{0^+}f_2(t)|^p|f_1(t)|^q\right]dt$$

$$\le 2^{1-\frac{q}{2}}\left(\frac{q}{p+q}\right)^{\frac{q}{2}}\left(d_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{2}}\left(\int_0^x\left[\frac{t^{2\alpha-1}}{(\Gamma(\alpha))^2(2\alpha-1)}\right]^{\frac{p}{2-q}}dt\right)^{\frac{2-q}{2}}\left[\int_0^x[|f_1(\tau)|^2+|f_2(\tau)|^2]d\tau\right]^{\frac{p+q}{2}}.$$

The extreme case of Theorem 4.1 is as follows:

Theorem 4.3. *Let* $r\ge 0, k>0$, $I^{\alpha,r}_{k,a}$ *be the* (k,r) *fractional integral of Riemann-type. Let* $w\ge 0$ *be measurable function on* $[a,x]$, $p,q_1,q_2\ge 0$ *and* $f_1,f_2\in L_\infty[a,b]$, *then the following inequality holds:*

$$\int_a^x w(t)\left[|I^{\alpha,r}_{k,a}f_1(t)|^{q_1}|\widetilde{I}^{\alpha,r}_{k,a}f_2(t)|^{q_2}|f_1(t)|^p+|I^{\alpha,r}_{k,a}f_1(t)|^{q_2}|\widetilde{I}^{\alpha,r}_{k,a}f_2(t)|^{q_1}|f_2(t)|^p\right]$$

$$\le\|w\|_\infty\left(\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\right)^{q_1+q_2}\int_a^x\left(\int_a^t(t^{r+1}-\tau^{r+1})^{\frac{\alpha}{k}-1}\tau^r d\tau\right)^{q_1+q_2}dt$$

$$\times\frac{1}{2}\left[\|f_1\|_\infty^{2(q_1+p)}+\|f_1\|_\infty^{2q_2}+\|f_2\|_\infty^{2q_2}+\|f_2\|_\infty^{2(q_1+p)}\right]. \tag{4.4}$$

Proof. Applying Theorem 1.9 with $g_1(t)=g_2(t)=I^{\alpha,r}_{k,a}f_1(t)$, $\widetilde{g}_1(t)=\widetilde{g}_2(t)=\widetilde{I}^{\alpha,r}_{k,a}(t)f_2$, and kernel $\widehat{k}_1(t,\tau)=\widehat{k}_2(t,\tau)=\widehat{k}(t,\tau)$ given by (4.3), we get inequality (4.4). □

The counter part of Theorem 4.1 is given by the following result

Theorem 4.4. *Let* $r\ge 0$, $k>0$, $s<0$, $q>0$ *and* $p\ge 0$. *If* $\varphi>0$, $w\ge 0$ *are measurable function on* $[a,x]$. *Let* $f_1,f_2\in L_{s,r}[a,b]$ *each of which is of fixed sign a.e. on* $[a,b]$, *with* $\frac{1}{f_1},\frac{1}{f_2}\in L_{s,r}[a,b]$, *then the following inequality holds:*

$$\int_a^x w(t)\left[|I^{\alpha,r}_{k,a}f_1(t)|^p|f_2(t)|^q+|I^{\alpha,r}_{k,a}f_2(t)|^p|f_1(t)|^q\right]dt$$

$$\geq 2^{1-\frac{q}{s}}\left(\frac{q}{p+q}\right)^{\frac{q}{s}}\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{s}}\left(\int_a^x [h(t)]^{\frac{s}{s-q}}dt\right)^{\frac{s-q}{s}}\left[\int_a^x \varphi(\tau)[|f_1(\tau)|^s+|f_2(\tau)|^s]d\tau\right]^{\frac{p+q}{s}},$$

where $c_{\frac{p}{q}}$ *and* $h(t)$ *are defined by* (1.6) *and* (4.2)*, respectively.*

Proof. Applying Theorem 1.10 with $g_1(t)=I^{\alpha,r}_{k,a}f_1(t)$ and $g_2(t)=I^{\alpha,r}_{k,a}f_2(t)$ and kernel $\widehat{k}(t,\tau)$ given in (4.3), we get the required result. □

Remark 4.5. In particular, corresponding to $k=1$ and $r=0$, Theorems 4.1, 4.3 and 4.4, becomes [7, Corollary 3.10, Corollary 3.14 and Theorem 3.11], respectively.

5. Opial-type Inequalities for Generalized Fractional Integral Operator involving Gauss hypergeometric Function

This section includes generalized Opial-type inequalities for the fractional integral operator with a particular kernel involving Gauss Hypergeometric function.

The Opial-type inequality for generalized fractional integral operator is given as follows:

Theorem 5.1. *Let* $I^{\alpha,\beta,\eta,\mu}_{a,t}$ *be the generalized fractional integral operator and let* $s>1$, $0<q<s$ *and* $p\geq 0$. *If* $\varphi>0$, $w\geq 0$ *are measurable function on* $[a,x]$ *and* $f_1,f_2\in L_s[a,b]$, *then the following inequality holds:*

$$\int_a^x w(t)\left[|I^{\alpha,\beta,\eta,\mu}_{a,t}f_1(t)|^p|f_2(t)|^q+|I^{\alpha,\beta,\eta,\mu}_{a,t}f_2(t)|^p|f_1(t)|^q\right]dt$$
$$\leq 2^{1-\frac{q}{s}}\left(\frac{q}{p+q}\right)^{\frac{q}{s}}\left(d_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{s}}\left(\int_a^x [h(t)]^{\frac{s}{s-q}}dt\right)^{\frac{s-q}{s}}\left[\int_a^x \varphi(\tau)[|f_1(\tau)|^s+|f_2(\tau)|^s]d\tau\right]^{\frac{p+q}{s}}, \tag{5.1}$$

where

$$h(t)=w(t)\left[\int_a^t\left[\frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\tau^{\mu}(t-\tau)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{\tau}{t}\right)\right]^{\frac{s}{s-1}}\varphi(\tau)^{\frac{1}{1-s}}d\tau\right]^{\frac{p(s-1)}{s}}\varphi(t)^{-\frac{q}{s}}, \tag{5.2}$$

and $d_{\frac{p}{q}}$ *is defined by* (1.5).

Proof. Applying Theorem 1.8 with $g_1(t)=I^{\alpha,\beta,\eta,\mu}_{a,t}f_1(t)$, $g_2(t)=I^{\alpha,\beta,\eta,\mu}_{a,t}f_2(t)$ and kernel

$$\check{k}(t,\tau)=\begin{cases}\frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\tau^{\mu}(t-\tau)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{\tau}{t}\right), & a\leq\tau\leq t;\\ 0, & t<\tau<b,\end{cases} \tag{5.3}$$

we get inequality (5.1). □

Example 5.2. For $w(t)=1$, $\varphi(t)=1$, $\beta=-\alpha$, $\mu=0$, $a=0$ and $s=2$, we have ${}_2F_1\left(0,-\eta;\alpha;1-\frac{\tau}{t}\right)=1$ and

$$h(t)=\left[\frac{t^{2\alpha-1}}{(\Gamma(\alpha))^2(2\alpha-1)}\right]^{\frac{p}{2}}$$

where $0<q<2$, $p\geq 0$. If $f_1,f_2\in L_2[0,b]$, then we obtain

$$\int_0^x\left[|I^{2,-2,3}_{0^+,t}f_1(t)|^p|f_2(t)|^q+|I^{2,-2,3}_{0^+,t}f_2(t)|^p|f_1(t)|^q\right]dt$$

$$\leq 2^{1-\frac{q}{2}}\left(\frac{q}{p+q}\right)^{\frac{q}{2}}\left(d_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{2}}\left(\int_0^x\left[\frac{t^{2\alpha-1}}{(\Gamma(\alpha))^2(2\alpha-1)}\right]^{\frac{p}{2-q}}dt\right)^{\frac{2-q}{2}}\left[\int_0^x[|f_1(\tau)|^2+|f_2(\tau)|^2]d\tau\right]^{\frac{p+q}{2}}.$$

The extreme case of Opial-type inequality presented in Theorem 5.1 is as follows:

Theorem 5.3. *Let $I_{a,t}^{\alpha,\beta,\eta,\mu}$ be the generalized fractional integral operator involving Gauss Hypergeometric function in its kernel. Let $w\geq 0$ be measurable function on $[a,x]$, $p,q_1,q_2\geq 0$ and $f_1,f_2\in L_\infty[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\int_a^x w(t)\left[|I_{a,t}^{\alpha,\beta,\eta,\mu}f_1(t)|^{q_1}|\widetilde{I}_{a,t}^{\alpha,\beta,\eta,\mu}f_2(t)|^{q_2}|f_1(t)|^p+|I_{a,t}^{\alpha,\beta,\eta,\mu}f_1(t)|^{q_2}|\widetilde{I}_{a,t}^{\alpha,\beta,\eta,\mu}f_2(t)|^{q_1}|f_2(t)|^p\right]\\&\quad\leq\|w\|_\infty\left(\frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\right)^{q_1+q_2}\int_a^x\left(\int_a^t\tau^\mu(t-\tau)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{\tau}{t}\right)d\tau\right)^{q_1+q_2}dt\\&\quad\quad\times\frac{1}{2}\left[\|f_1\|_\infty^{2(q_1+p)}+\|f_1\|_\infty^{2q_2}+\|f_2\|_\infty^{2q_2}+\|f_2\|_\infty^{2(q_1+p)}\right].\end{aligned}\tag{5.4}$$

Proof. Applying Theorem 1.9 with $g_1(t)=g_2(t)=I_{a,t}^{\alpha,\beta,\eta,\mu}f_1(t)$, $\widetilde{g}_1(t)=\widetilde{g}_2(t)=\widetilde{I}_{a,t}^{\alpha,\beta,\eta,\mu}f_2(t)$ and kernel $\check{k}_1(t,\tau)=\check{k}_2(t,\tau)=\check{k}(t,\tau)$ given in (5.3), we get inequality (5.4). □

The counter part of Theorem 5.1 for the case $s<0$ is given in the following result.

Theorem 5.4. *Let $s<0$, $q>0$ and $p\geq 0$. If $\varphi>0$, $w\geq 0$ are measurable functions on $[a,x]$, and $f_1,f_2\in L_s[a,b]$ each of which is of fixed sign a.e on $[a,b]$, with $\frac{1}{f_1},\frac{1}{f_2}\in L_s[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\int_a^x w(t)[|I_{a,t}^{\alpha,\beta,\eta,\mu}f_1(t)|^p|f_2(t)|^q+|I_{a,t}^{\alpha,\beta,\eta,\mu}f_2(t)|^p|f_1(t)|^q]dt\\&\geq 2^{1-\frac{q}{s}}\left(\frac{q}{p+q}\right)^{\frac{q}{s}}\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{s}}\left(\int_a^x[h(t)]^{\frac{s}{s-q}}dt\right)^{\frac{s-q}{s}}\left[\int_a^x\varphi(\tau)[|f_1(\tau)|^s+|f_2(\tau)|^s]d\tau\right]^{\frac{p+q}{s}},\end{aligned}\tag{5.5}$$

where $c_{\frac{p}{q}}$ and $h(t)$ are defined by (1.6) and (5.2), respectively.

Proof. Applying Theorem 1.10 with $g_1(t)=I_{a,t}^{\alpha,\beta,\eta,\mu}f_1(t)$, $g_2(t)=I_{a,t}^{\alpha,\beta,\eta,\mu}f_2(t)$ and kernel given in (5.3), we get inequality (5.5). □

Remark 5.5. In Particular, for $\beta=-\alpha$ and $\mu=0$, in Theorems 5.1, 5.3 and 5.4, we arrive at [7, Corollary 3.10, Corollary 3.14 and Theorem 3.11].

6. Conclusion

This article includes more general results on Opial-type integral inequalities for fractional calculus operators. Our outcomes in special cases yields some of the recent results on Opial-type inequalities presented in [7].

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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