Communications in Mathematics and Applications
Vol. 9, No. 3, pp. 393–409, 2018
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v9i3.803

http://www.rgnpublications.com

***Proceedings*:**
3rd International Conference on Pure and Applied Mathematics
Department of Mathematics, University of Sargodha, Sargodha, Pakistan
November 10-11, 2017

Numerical Solution of Modified Forms of Camassa-Holm and Degasperis-Procesi Equations via Quartic B-Spline Collocation Method

Research Article

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Abstract. In this paper, a collocation finite difference scheme based on Quartic B-spline function is developed for solving non-linear modified Camassa-Holm and Degasperis-Procesi equations. A finite difference scheme and Quartic B-spline function are used to discretize the time and spatial derivatives, respectively. The obtained numerical results are compared with the exact analytical solutions and some methods existing in literature. The numerical solutions of proposed non-linear equations are acquired without any linearization technique. The convergence of the method is proved of order $(\Delta t + h^2)$. The efficiency of the proposed scheme is demonstrated through illustrative examples. The presented scheme is realized to be a very reliable alternate method to some existing schemes for such physical problems.

Keywords. Non-linear modified Camassa-Holm and Degasperis-Procesi equations; Quartic B-spline collocation method; Convergence

MSC. 65M70; 35G31; 65Z05; 65D05; 65D07; 65L20

Received: February 3, 2018 **Revised:** March 9, 2018 **Accepted:** March 31, 2018

1. Introduction

Most of our real life-problems are associated with non-linear models occurring in various fields of science and engineering, specially in plasma physics, fluid mechanics, plasma wave, chemical physics and solid state physics etc. To calculate a numerical or theoretical solution of such structures is a challenging assignment. In the previous decades, to evaluate the exact and numerical solutions of such structures both physicists as well as mathematicians have dedicated their significant determination by employing various dominant strategies.

Non-linear equations also cover the cases named surface waves in compressible fluids, acoustic waves in an harmonic crystal, hydromagnetic waves in cold plasma, etc. The core motivation for searching numerical solutions of non-linear equations is their extensive application in various physical models. A significant physical model known as b-equation can be stated as follows:

$$u_t + 2\nu u_x - u_{xxt} + (b+1)u^2 u_x = b u_x u_{xx} + u u_{xxx}, \quad L_1 \le x \le L_2,\ 0 \le t \le T, \tag{1.1}$$

where $u = u(x,t)$ with the initial condition,

$$u(x,0) = h(x), \quad L_1 \le x \le L_2, \tag{1.2}$$

and the boundary conditions,

$$u(L_1,t) = g_1(t),\ u(L_2,t) = g_2(t),\ u_x(L_1,t) = g_3(t), \quad 0 \le t \le T. \tag{1.3}$$

1.1 Modified Camassa-Holm Equation

When $b = 2$, equation (1.1) becomes Camassa-Holm (CH) equation and substituting $b = 2$, $\nu = 0$, we obtain an equation of the following form

$$u_t - u_{xxt} + 3u^2 u_x = 2u_x u_{xx} + u u_{xxx}, \quad L_1 \le x \le L_2,\ 0 \le t \le T, \tag{1.4}$$

which is known as *Modified Camassa-Holm* (MCH) equation. The non-linear Camassa-Holm equation arises in fluid dynamics which is dimensionless and integrable equation. It was introduced by Camassa and Holm as a bi-Hamiltonian model for waves in shallow water for the paramter $\nu > 0$ and the solitary wave solutions were smooth solitons. When $\nu = 0$, this equation has the peakon solutions i.e. solitons with a sharp peak so with a discontinuity at the peak in the wave slope. The shallow water waves and the interaction of two peakons are displayed in Figure 1.1. Wazwaz [13] applied the sine-cosine as well as tanh methods to investigate the exact solitary wave solutions of MCH equation. The exact solution of MCH equation is

$$u(x,t) = -2\,\mathrm{sech}^2\left(\frac{x}{2} - t\right). \tag{1.5}$$

1.2 Modified Degaperis-Procesi Equation

The equation (1.1) becomes Degaperis-Procesi equation by setting $b = 3$. The equation (1.1) takes the following form when $b = 3$, $\nu = 0$

$$u_t - u_{xxt} + 4u^2 u_x = 3u_x u_{xx} + u u_{xxx}, \quad L_1 \le x \le L_2,\ 0 \le t \le T, \tag{1.6}$$

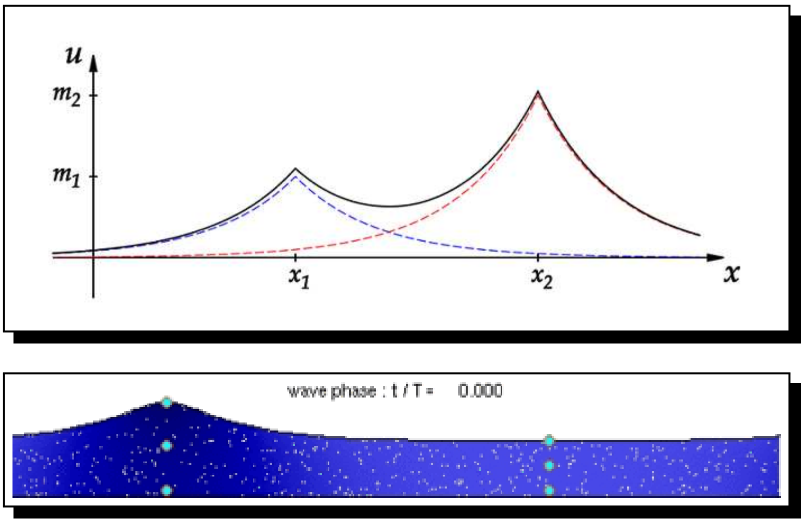


Figure 1.1. Interaction of two peakons and shallow water waves.

which is known as *modified Degaperis-Procesi* (MDP) equation. In mathematical physics, it is a non-linear *partial differential equation* (PDE) which models the propagation of nonlinear dispersive waves. It was discovered by Degasperis and Procesi in search for integrable equations that Camassa-Holm and *Degaperis-Procesi* (DP) equations are the only integrable cases of equation (1.1) has been confirmed utilizing various integrability tests. It was later discovered that (with $\nu > 0$) the DP equation plays a similar role in water wave theory as the CH equation due to its mathematical properties, for instance, the wind waves that arise on the free surface of bodies of water and ocean waves as shown in Figure 1.2. Wazwaz [14] investigated the solutions of both MCH and MDP equations by implementing extended tanh method. The exact solution of MDP equation is

$$u(x,t) = \frac{-15}{8}\operatorname{sech}^2\left(\frac{x}{2} - \frac{5t}{4}\right). \tag{1.7}$$

Figure 1.2. Wind waves and ocean waves.

Various dominant techniques have been developed in literature for solving such non-linear structures. Yildirim [16] applied the variational iteration method to solve both MCH and MDP equations. Ganji et al. [7] implemented *Adomian Decomposition Method* (ADM) to obtain the solitary wave type solutions of both the equations. Abbasbandy [4] utilized *Homotopy Analysis Method* (HAM) for the solution of MCH equation. Yusufoglu [18] computed solitary solutions of MCH and MDP equations by implementing Exp-function method. Zhang et al. [21] exploited auxiliary equation method to compute the solution of both MCH and MDP equations. Zhang *et al*. [19] applied *Homotopy Perturbation Method* (HPM) to compute the solutions of MCH and MDP equations. Yousif *et al*. [17] applied *Homotopy Perturbation Method* (HPM) for solving the MCH and MDP equations. Manafian *et al*. [9] computed solitary wave solutions of both the equations using (G'/G) expansion method. Zhang *et al*. [20] wrote a note on solitary wave solutions of the non-linear generalized Camassa-Holm equation. There are various approximation techniques which have been examined by many researchers such as finite element, finite difference, spline interpolation etc. Spline interpolation method is one of the most effective approximation method on account of its simplicity. The main advantage of using the proposed *Quartic B-Spline Method* (QuBSM) is that it is able to approximate the analytical curve up to certain smoothness. Therefore, it has the flexibility to get the approximation at any point in the domain with more accurate results as compared to the usual finite difference method.

In this study, a collocation finite difference approach based on quartic B-spline is presented for the numerical solution of MCH and MDP equations with initial and boundary conditions. A usual finite difference scheme is formulated to discretize the time derivative. Quartic B-spline is taken as an interpolation function in the space dimension. Some researchers have utilized the methods named *Variational Iteration method* (VIM) [16], *Adomain Decomposition Method* (ADM) [7], *Homotopy Perturbation Method* (HPM) [19] to solve the MCH and MDP equations but so far as we are aware not with quartic B-spline collocation method. The convergence of the proposed method is established. The feasibility of the proposed method is verified by test problems and the approximated solutions are found to be in good agreement with the exact solutions. It can be concluded that our method furnishes more accurate results as compared to the existing techniques.

The current study is systemized as follows: In Section 2, QuBSM is formulated and implemented to non-linear MCH and MDP equations. In Section 3, convergence of the method is established. In Section 4, two numerical cases of MCH and MDP equations are considered to demonstrate the accuracy and feasibility of the method. In Section 5, concluding remarks of the whole picture are presented.

2. Materials and Methods

This section presents *Quartic B-Spline Basis Function* (QuBSBF) and execution of QuBSM to solve the MCH and MDP equations.

2.1 Quartic B-Spline basis Functions

The grid region $[L_1, L_2] \times [0, T]$ is discretized in such a way that we achieve equally divided mesh with grid points (x_m, t_k) where $x_m = a + mh$, $t_k = k\Delta t$ and $m = 0, 1, \dots, n$, $k = 0, 1, \dots, N$. Here h and Δt denote spatial size and time step, respectively. The QuBSBF can be stated as:

$$B_m^4(x) = \frac{1}{h^4} \begin{cases} (x - x_{m-2})^4 & x \in [x_{m-2}, x_{m-1}] \\ \left((x - x_{m-2})^4 - 5(x - x_{m-1})^4\right) & x \in [x_{m-1}, x_m] \\ \left((x - x_{m-2})^4 - 5(x - x_{m-1})^4 + 10(x - x_m)^4\right) & x \in [x_m, x_{m+1}] \\ \left((x_{m+3} - x)^4 - 5(x_{m+2} - x)^4\right) & x \in [x_{m+1}, x_{m+2}] \\ (x_{m+3} - x)^4 & x \in [x_{m+2}, x_{m+3}] \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

where $B_{m-1}(x) = B_0(x - (m-1)h)$ and $m = 2, 3 \dots$. The quartic B-spline basis function is depicted in Figure 2.1.

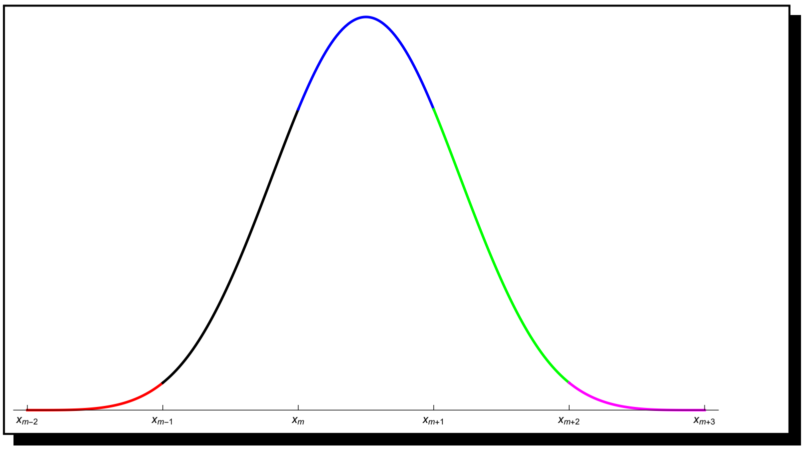


Figure 2.1. Quartic B-spline basis

Our approach for one-dimensional MCH and MDP equations utilizing collocation method with QuBSBF is to find an approximate solution as [1–4, 10, 15, 17, 22]:

$$U_m^k(x,t) = \sum_{m=-2}^{n+1} D_m^k(t) B_m^4(x), \tag{2.2}$$

where $D_m^k(t)$ are to be determined for the approximation $U_m^k(x,t)$ to the exact solution $u(x,t)$ at the point (x_m, t_k).

Utilizing equations (2.1) and (2.2), the values of U_m^k and its derivatives at the nodes $x = x_m$ can be written as:

$$\begin{cases} U_m^k = D_{m-2}^k + 11D_{m-1}^k + 11D_m^k + D_{m+1}^k \\ (U_x)_m^k = \frac{-4}{h} D_{m-2}^k - \frac{12}{h} D_{m-1}^k + \frac{12}{h} D_m^k + \frac{4}{h} D_{m+1}^k \\ (U_{xx})_m^k = \frac{12}{h^2} D_{m-2}^k - \frac{12}{h^2} D_{m-1}^k - \frac{12}{h^2} D_m^k + \frac{12}{h^2} D_{m+1}^k \\ (U_{xxx})_m^k = \frac{24}{h^3} D_{m-2}^k - \frac{72}{h^3} D_{m-1}^k + \frac{72}{h^3} D_m^k - \frac{24}{h^3} D_{m+1}^k. \end{cases} \tag{2.3}$$

The equation (2.2) and boundary conditions given in equation (1.3) are used to obtain the approximate solution at end points as:

$$\begin{cases} U(x_0,t_{k+1}) = D^k_{-2} + 11D^k_{-1} + 11D^k_0 + D^k_1 = g_1(t_{k+1}) \\ U_x(x_0,t_{k+1}) = \frac{-4}{h}D^k_{-2} + \frac{-12}{h}D^k_{-1} + \frac{12}{h}D^k_0 + \frac{4}{h}D^k_1 = g_3(t_{k+1}) \\ U(x_n,t_{k+1}) = D^k_{n-2} + 11D^k_{n-1} + 11D^k_n + D^k_{n+1} = g_2(t_{k+1}). \end{cases} \tag{2.4}$$

2.2 Implementation of the Method to MCH and MDP Equations

By using finite difference scheme for time derivative and temporal discretization, the equation (1.1) for $v = 0$ can be written as:

$$\frac{U^{k+1}_m - U^k_m}{\Delta t} - \frac{(U_{xx})^{k+1}_m - (U_{xx})^k_m}{\Delta t} + \frac{\varphi^{k+1}_m + \varphi^k_m}{2} = 0, \tag{2.5}$$

where k and $k+1$ describe successive time levels and

$$\varphi^k_m = (\varphi(x_m,t_k,U^k_m,(U_x)^k_m,(U_{xx})^k_m,(U_{xxx})^k_m)) = (b+1)(U^2U_x)^k_m - b(U_xU_{xx})^k_m - (UU_{xxx})^k_m.$$

A slight simplification implies

$$2U^{k+1}_m - 2(U_{xx})^{k+1}_m + \Delta t\varphi^{k+1}_m = (\psi(x))^k_m, \tag{2.6}$$

where

$$(\psi(x))^k_m = 2U^k_m - (U_{xx})^k_m - 2(U_{xx})^k_m - \Delta t\varphi^k_m.$$

Since the initial condition is known so we may construct second order approximation at the first time level [10] by applying taylor series as follows:

$$u^1_m = u^0_m + \Delta t(u_t)^0_m + \frac{(\Delta t)^2}{2!}(u_{tt})^0_m + O(\Delta t)^3. \tag{2.7}$$

By using initial condition and equation (1.1) for $v = 0$, the values of $(u_t)^0_m$ and $(u_{tt})^0_m$ are computed as under:

$$(u_t)^0_m = [u_{xxt} - \varphi(u)]^0_m, (u_{tt})^0_m = [u_{xxtt} - (\varphi(u))_t]^0_m,$$

where

$$\varphi(u)^0_m = (b+1)(U^2U_x)^0_m - b(U_xU_{xx})^0_m - (UU_{xxx})^0_m.$$

Substituting these two values into equation (2.7), we obtain

$$u^1_m = u^0_m + \Delta t[u_{xxt} - \varphi(u)]^0_m + \frac{(\Delta t)^2}{2!}[u_{xxtt} - (\varphi(u))_t]^0_m + O(\Delta t)^3, \tag{2.8}$$

which gives the first order approximation.

Theorem 2.1. *The current procedure to discretize equation* (1.1) *is of first order convergence in time direction.*

Proof. Suppose U^k_m be the approximate solution of the exact solution $u(x_m,t_k)$ at time $t = t_k$ and local truncation error in equation (2.6) is $e_k = U^k_m - u(x_m,t_k)$. By applying Lemma [5], we have

$$e_{N+1} \le \mu_k(\Delta t)^2, \quad k \ge 2. \tag{2.9}$$

By utilizing equation (2.7) for $k = 1$, we obtain

$$e_1 \le \mu_1(\Delta t)^3. \tag{2.10}$$

Choosing $\mu = \max\{\mu_1, \mu_2, \ldots, \mu_N\}$ and taking global error $E_{N+1} = \sum_{k=1}^{N} e_k$ at $(N+1)^{\text{th}}$ time level, we obtain the following expression:

$$\begin{aligned}
|E_{N+1}| = |E_{N+1}| &= \left|\sum_{k=1}^{N} e_k\right| \le \sum_{k=1}^{N} |e_k| \\
&\le \mu_1(\Delta t)^3 + \sum_{k=2}^{N} \mu_k(\Delta t)^2 \\
&\le N\mu(\Delta t)^2 \\
&\le N\mu(T/N)\Delta t) \\
&= C\Delta t,
\end{aligned}$$

where $\Delta t \le (T/N)$ and $C = \mu T$ which implies first order convergence in time direction. □

Equation (2.5) becomes MCH and MDP equation when $b = 2$ and $b = 3$ are taken, respectively. Using equation (2.3) into equation (2.6), we obtain

$$pD_{m-2}^{k+1} + qD_{m-1}^{k+1} + qD_m^{k+1} + pD_{m+1}^{k+1} + h^2\varphi(u_m^{k+1}) = h^2\psi^k(x_m), \tag{2.11}$$

where

$$p = \frac{2h^2 - 24}{\Delta t}, \quad q = \frac{22h^2 + 24}{\Delta t}.$$

The above relation generates $n+1$ non-linear equations in $n+4$ unknowns D_m^{k+1} at the time level t_{k+1} i.e. $D_m^{k+1} = \left(D_{-2}^{k+1}, D_{-1}^{k+1}, D_0^{k+1}, D_1^{k+1}, \ldots, D_{n+1}^{k+1}\right)$. Eliminate the unknowns $D_{-2}^{k+1}, D_{-1}^{k+1}$ and D_{n+1}^{k+1} by using the boundary conditions given in equations (2.4) and (2.11). Thus, a system of order $(n+1)\times(n+1)$ can be written as follows:

$$LD_m^{k+1} + h^2M_m^{k+1} = h^2R_m^k, \tag{2.12}$$

where

$$L = \begin{pmatrix} \frac{864}{7\Delta t} & 0 & 0 & 0 & & & & \\ x^* & y & x & 0 & & 0 & & \\ x & y & y & x & & & & \\ & & & & x & y & y & x \\ & 0 & & & 0 & 0 & \frac{288}{\Delta t} & \frac{288}{\Delta t} \end{pmatrix}, \quad D_m^{k+1} = \begin{bmatrix} D_0^{k+1} \\ D_1^{k+1} \\ \vdots \\ D_n^{k+1} \end{bmatrix}, \quad M_m^{k+1} = \begin{bmatrix} \varphi_0^{k+1} \\ \varphi_1^{k+1} \\ \vdots \\ \varphi_n^{k+1} \end{bmatrix}, \quad R_m^k = \begin{bmatrix} \psi_0^* \\ \psi_1^* \\ \psi_2^k \\ \vdots \\ \psi_{n-1}^k \\ \psi_n^* \end{bmatrix}$$

and

$$\begin{aligned}
\eta &= \frac{hg_2(t_{k+1})y}{56} - \frac{g_1(t_{k+1})y}{14} - \frac{11g_2(t_{k+1})hx}{56} - \frac{3g_1(t_{k+1})x}{8}, \\
\tau &= \frac{hg_2(t_{k+1})x}{56} - \frac{xg_1(t_{k+1})}{14},
\end{aligned}$$

$$x^* = \frac{264+66h^2}{7\Delta t},$$

$$\psi_0^* = \psi_0^k + \frac{\eta}{h^2}, \quad \psi_1^* = \psi_1^k + \frac{\tau}{h^2}, \quad \psi_n^* = \psi_n^k - \frac{g_3(t_{k+1})}{h^2}.$$

3. Convergence of the Method

Suppose $U_m^k(x) = \sum_{m=-2}^{n+1} D_m^k(t) B_m^4(x)$ be the quartic B-spline approximation to the exact solution $u_m^k(x)$. Due to computational round off error assume that $U_m^{*k}(x) = \sum_{m=-2}^{n+1} D_m^{*k}(t) B_m^4(x)$ be the computed spline approximation to $U_m^k(x)$ where $D_m^{*k} = \left(D_0^{*k}, D_1^{*k}, \dots, D_n^{*k}\right)^T$. Therefore, we must estimate the errors $\|u_m^k(x) - U_m^{*k}(x)\|_\infty$ and $\|U_m^{*k}(x) - U_m^k(x)\|_\infty$ separately to estimate the error $\|u_m^k(x) - U_m^k(x)\|_\infty$. Putting $U_m^{*k}(x)$ into equation (2.12), we obtain

$$LD_m^{*k+1} + h^2 M_m^{*k+1} = h^2 R_m^{*k}. \tag{3.1}$$

Subtracting equation (2.12) and equation (3.1), we have

$$L(D^* - D)_m^{k+1} + h^2 (M^* - M)_m^{k+1} = h^2 (R^* - R)_m^k. \tag{3.2}$$

First we need to recall the following theorem.

Theorem 3.1. *Suppose that $g(x) \in C^4[L_1, L_2]$ and $g^{(4)}(x) < l^*$ with equally space partition of $[L_1, L_2]$ and step size h. If $S(x)$ be the unique spline function interpolate g(x) at the knots then $\exists$ a constant δ_j such that*

$$\|g^j - S^j\|_\infty \le \delta_j l^* h^{4-j}, \quad j = 0, 1, 2, 3.$$

Proof. See [6] and [8]. □

Applying triangular inequality and Theorem 3.1, the equation (2.6) yields

$$\begin{aligned}
&|\phi^{*k}(x_m) - \phi^k(x_m)| \\
&= \left| \frac{-2}{\Delta t} U_{xx}^{*k}(x_m) + \frac{2}{\Delta t} U^{*k}(x_m) + \varphi(U^{*k}(x_m)) - \left(\frac{-2}{\Delta t} U_{xx}^k(x_m) + \frac{2}{\Delta t} U^k(x_m) + \varphi(U^k(x_m)) \right) \right| \\
&= \left| \frac{-2}{\Delta t} \left(U_{xx}^{*k}(x_m) - U_{xx}^k(x_m) \right) + \frac{2}{\Delta t} \left(U^{*k}(x_m) - U^k(x_m) \right) + \left(\varphi_m^k(U^*) - \varphi_m^k(U) \right) \right| \\
&\le \frac{2}{\Delta t} |U_{xx}^{*k}(x_m) - U_{xx}^k(x_m)| + \frac{2}{\Delta t} |U^{*k}(x_m) - U^k(x_m)| + |\varphi^k(x_m, U^*(x_m)) - \varphi^k(x_m, U(x_m))|.
\end{aligned}$$

Finally, we are able to write

$$\|(R^* - R)_m^k\| \le \frac{2}{\Delta t} \delta_2 l^* h^2 + \frac{2}{\Delta t} \delta_0 l^* h^4 + \beta(|U^{*k}(x_m) - U^k(x_m)|),$$

where $\|\varphi'(z)\| \le \beta, z \in R^3$ ([12, p. 218]).

The above expression can also be written as

$$\|(R^* - R)_m^k\| \le \frac{2\delta_2 l^* h^2}{\Delta t} + \frac{2\delta_0 l^* h^4}{\Delta t} + \beta \delta_0 l^* h^4 \tag{3.3}$$

or

$$\|(R^*-R)_m^k\| \le \beta_1 h^2, \tag{3.4}$$

where $\beta_1 = \frac{2\delta_2 l^*}{\Delta t} + \frac{2\delta_0 l^* h^2}{\Delta t} + \beta\delta_0 l^* h^2$.

Now, applying Jacobian to the non-linear term on L.H.S of equation (2.12), we obtain

$$h^2\|(M^*-M)_m^{k+1}\| = h^2\left(\frac{\partial\varphi(\xi_1)}{\partial u} J(D^*-D)_m^{k+1}\right), \tag{3.5}$$

where $\xi_1 \in (0,1)$ and J is Jacobian given as:

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & & & & \\ 1 & 11 & 11 & 1 & & 0 & & \\ \cdots & \cdots & \cdots & \cdots & & & & \\ & & & & \cdots & \cdots & \cdots & \\ & & 0 & & 1 & 11 & 11 & 1 \\ & & & & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Substituting equation (3.5) into equation (3.2), the following expression yields

$$W(D^*-D)_m^{k+1} = h^2(R^*-R)_m^k, \tag{3.6}$$

where $W = L + h^2\frac{\partial\varphi(\xi_1)}{\partial u}J$.

Since matrix W is strictly diagonally dominant so non-singular, W^{-1} exists, hence equation (3.6) implies

$$(D^*-D)_m^{k+1} = h^2 W^{-1}(R^*-R)_m^k. \tag{3.7}$$

Taking norm on both sides and using equation (3.4),

$$\|(D^*-D)_m^{k+1}\|_\infty \le h^2\|W^{-1}\|\beta_1 h^2 \tag{3.8}$$

or

$$\|(D^*-D)_m^{k+1}\|_\infty \le \|W^{-1}\|\beta_1 h^4. \tag{3.9}$$

Suppose that γ_m is the sum of m^{th} row of matrix $W = [\nu_{m,i}]$, then we have

$$\begin{cases} \gamma_0 = \frac{864}{7\Delta t}, & \text{if} \quad m = 0 \\ \gamma_1 = \frac{264+66h^2}{7\Delta t}, & \text{if} \quad m = 1 \\ \gamma_m = 24h^2(\frac{2}{\Delta t} + \frac{\partial\varphi}{\partial u}), & \text{if} \quad 2 \le m \le n-1 \\ \gamma_n = \frac{576}{\Delta t}, & \text{if} \quad m = n. \end{cases} \tag{3.10}$$

From the literature of matrices, we have

$$\sum_{m=0}^{n} \nu_{i,m}^{-1}\gamma_m = 1,$$

where $\nu_{i,m}^{-1}$ are the elements of W^{-1} for $i = 0,1,\ldots,n$. Therefore,

$$\|W^{-1}\| = \sum_{m=0}^{n} |\nu_{i,m}^{-1}| \le \frac{1}{min(\gamma_m)} = \frac{1}{h^2 \upsilon_l} \le \frac{1}{h^2|\upsilon_l|}, \tag{3.11}$$

where l is some index between 0 and n. Substituting equation (3.11) into equation (3.9) implies the relation

$$\|(D^* - D)_m^{k+1}\|_\infty \le \beta_1 h^4 \frac{1}{h^2 v_l} = \beta_2 h^2, \tag{3.12}$$

where $\beta_2 = \frac{\beta_1}{v_l}$ is some finite constant.

Lemma 3.2. *The quartic B-spline satisfy*

$$\left| \sum_{m=-2}^{n+1} B_m(x) \right| \le 34, \quad 0 \le x \le 1. \tag{3.13}$$

Proof. We know that

$$\left| \sum_{m=-2}^{n+1} B_m(x) \right| \le \sum_{m=-2}^{n+1} |B_m(x)|. \tag{3.14}$$

At any knot x_m, we have

$$\begin{aligned} \sum_{m=-2}^{n+1} |B_m(x)| &= |B_{m-2}(x)| + |B_{m-1}(x)| + |B_m(x)| + |B_{m+1(x)}| \\ &= 1 + 11 + 11 + 1 \le 24. \end{aligned} \tag{3.15}$$

Also in each subinterval $x_{m-1} \le x \le x_m$,

$$B_m(x_m) = 11, \quad B_{m-1}(x_{m-1}) = 11, \quad B_{m+1}(x_m) = 1, \quad B_{m-2}(x_{m-1}) = 11.$$

Hence in each subinterval $x_{m-1} \le x \le x_m$,

$$\begin{aligned} \sum_{m=-2}^{n+1} |B_m(x)| &= |B_{m-2}(x)| + |B_{m-1}(x)| + |B_m(x)| + |B_{m+1}(x)| \\ &\le 11 + 11 + 11 + 1 = 34, \end{aligned} \tag{3.16}$$

which completes the proof. □

Since

$$U_m^{*(k+1)}(x) - U_m^{k+1}(x) = \sum_{m=-2}^{n+1} (D^* - D)_m^{k+1} B_m(x). \tag{3.17}$$

Applying norm on both sides and using equations (3.12) and (3.13), we have

$$\|U_m^{*(k+1)}(x) - U_m^{k+1}(x)\| \le \sum_{m=-2}^{n+1} |B_m(x)| \, \|(D^* - D)_m^{k+1}\| \le 34\beta_2 h^2. \tag{3.18}$$

Theorem 3.3. *Let $u^{k+1}(x_m)$ be the exact solution of equation* (1.1) *with the boundary conditions equation* (1.3) *and let $U^{*(k+1)}(x_m)$ be the B-spline approximation to $u^{k+1}(x_m)$ then the method has second order convergence and we have*

$$\|u_m^{k+1}(x) - U_m^{k+1}(x)\| \le \epsilon h^2, \tag{3.19}$$

where $\epsilon = \delta_0 l^* h^2 + 34\beta_2$ is finite.

Proof. From Theorem 3.1, we have:

$$\|u_m^{k+1}(x) - U_m^{*(k+1)}(x)\| \le \delta_0 l^* h^4. \tag{3.20}$$

By using equation (3.19), equation (3.20) and triangular inequality, we obtain the following relation

$$\begin{aligned}\|u_m^{k+1}(x)-U_m^{k+1}(x)\| &= \|u_m^{k+1}(x)-U_m^{*(k+1)}(x)+U_m^{*(k+1)}(x)-U_m^{k+1}(x)\| \\ &\le \|u_m^{k+1}(x)-U_m^{*(k+1)}(x)\| + \|U_m^{*(k+1)}(x)-U_m^{k+1}(x)\| \\ &\le \delta_0 l^* h^4 + 34\beta_2 h^2 \\ &= \epsilon h^2,\end{aligned}$$

where $\epsilon = \delta_0 l^* h^2 + 34\beta_2$. □

Now if $U^{k+1}(x,t)$ be the approximate solution by our numerical process to the exact solution $u^{k+1}(x,t)$ then

$$\|u_m^{k+1}(x,t)-U_m^{k+1}(x,t)\| \le \rho(\Delta t + h^2), \tag{3.21}$$

where ρ is constant which demonstrate convergence of order $(\Delta t + h^2)$ in time and spatial direction.

4. Numerical Results and Discussion

In this section, the quartic B-spline method is implemented for solving both MCH and MDP equations with an initial and boundary conditions given in equations (1.2)-(1.3). We carry out from equation (2.12) by QuBSM and intel®Core™i7-3520M CPU @ 2.90 GHz with 4GB RAM with operating system (WINDOWS 10). The numerical implementaion is performed in MATLAB R2015b. Some numerical examples are presented to verify the accuracy, capability and efficiency of QuBSM. The approximate results are compared with the exact solution and some methods existing in literature at (x_m,t_k) taking particular step sizes h and Δt. Exact and numerical solutions are displayed in different figures at various time levels which shows that our results are in good agreement with the exact solution. Absolute errors can be calculated by

$$\text{Absolute Error} = |U_m - u_{excm}|. \tag{4.1}$$

4.1 Numerical Test Cases

Example 4.1. Consider MCH equation (1.4) and exact solution (1.5) with constraints given in equations (1.2) and (1.3) in the domain $[-15,15]$.

We compare the numerical results computed by QuBSM with the methods named *Variational Iteration Method* (VIM) [16], *Adomain Decomposition Method* (ADM) [7], *Homotopy Perturbation Method* (HPM) [19] at various nodal points and time levels which are tabulated in Table 4.1. Moreover, the absolute errors computed by the proposed method are given in Table 4.1 and a comparison shows that our method provides more accurate results as compared to others. Figures 4.1 and 4.2 illustrate the graphs of exact and approximate solutions at various time levels. It can be concluded that our results are in good agreement with the exact solution and more accurate as compare to the methods given in [16], [7], [19].

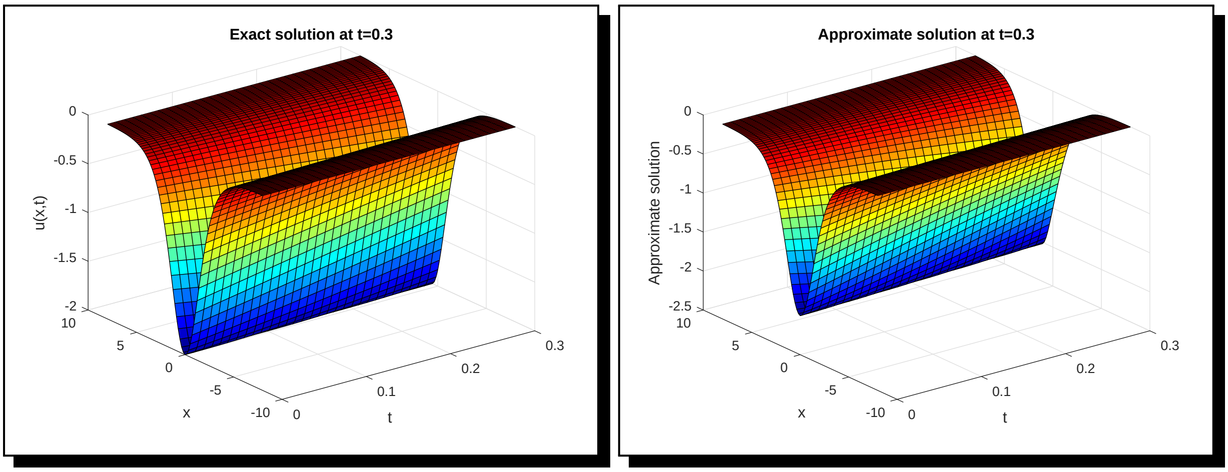
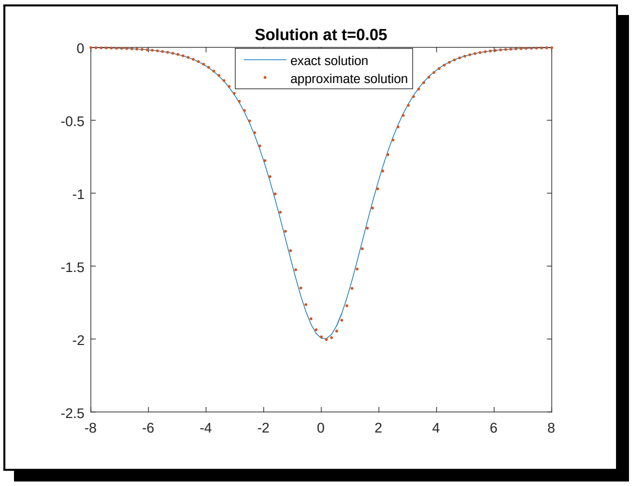
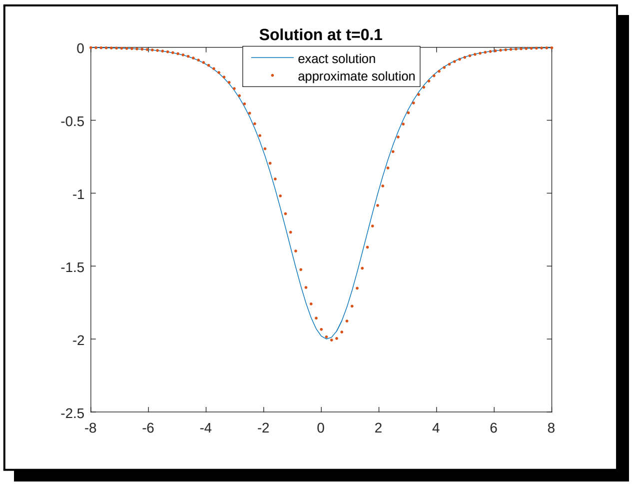


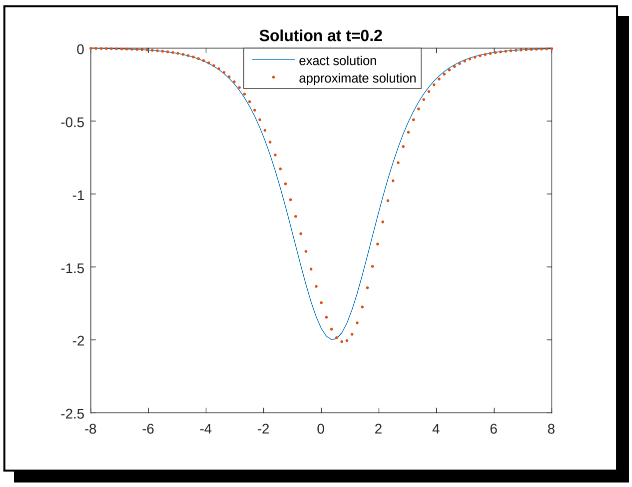
Figure 4.1. Space-time graphs of exact and approximate solutions of MCH equation for t ∈ [0,0.3].



(a) at time $t = 0.05$.



(b) at time $t = 0.1$.



(c) at time $t = 0.2$.

Figure 4.2. Exact and approximate solutions of MCH equation for various values of t.

Table 4.1. Comparison of absolute errors of MCH equation computed by QuBSM with existing methods at different time levels.

x	t	Exact solution	Approximate solution	QuBSM	VIM[16]	ADM[7]	HPM[19]
6	0.05	−0.02179598	−0.02213088	3.349E-04	2.005E-03	—	—
8		−0.00296375	−0.00300734	4.359E-05	2.807E-04	3.332E-04	3.332E-04
9		−0.00109081	−0.00110677	1.596E-05	—	1.229E-04	1.230E-04
10		−0.00040136	−0.00040721	5.860E-06	3.817E-05	4.521E-05	4.530E-05
12		−0.00005432	−0.00005511	7.900E-07	5.170E-06	—	—
6	0.10	−0.02407444	−0.02495921	8.847E-04	4.226E-03	—	—
8		−0.00327520	−0.00339113	1.159E-04	5.911E-04	7.108E-04	7.109E-04
9		−0.00120550	−0.00124798	4.248E-05	—	2.623E-04	2.624E-04
10		−0.00044356	−0.00045916	1.560E-05	8.035E-05	9.659E-05	9.664E-05
12		−0.00006004	−0.00006214	2.100E-06	1.088E-05	—	—
8	0.15	−0.00361934	−0.00384320	2.238E-04	—	1.139E-03	1.139E-03
9		−0.00133224	−0.00141432	8.208E-05	—	4.203E-04	4.203E-04
10		−0.00049020	−0.00052035	3.014E-05	—	1.547E-04	1.548E-04
8	0.20	−0.00399961	−0.00437611	3.765E-04	—	1.624E-03	1.624E-03
9		−0.00147230	−0.00161041	1.381E-04	—	5.992E-04	5.993E-04
10		−0.00054175	−0.00059249	5.073E-05	—	2.207E-04	2.207E-04

Example 4.2. Consider MDP equation (1.6) and exact solution (1.7) with conditions given in equations (1.2) and (1.3) in the domain $[-15,15]$.

We compare the numerical results computed by QuBSM with the methods introduced in [16], [7] and [19] at various knots and time levels given in Table 4.2. For comparison purpose, the absolute errors obtained by the proposed QuBSM are listed in Table 4.2. The graphical representation of exact and approximate solutions is given in Figures 4.3 and 4.4. It can be concluded that our technique furnishes more accurate and improved results as compared to the methods given in [16], [7] and [19].

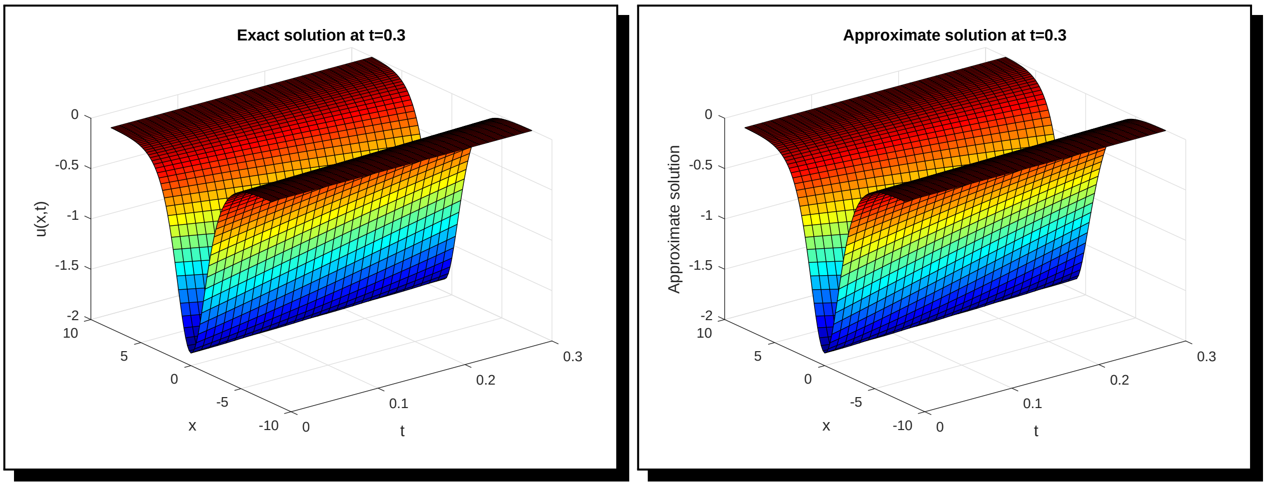
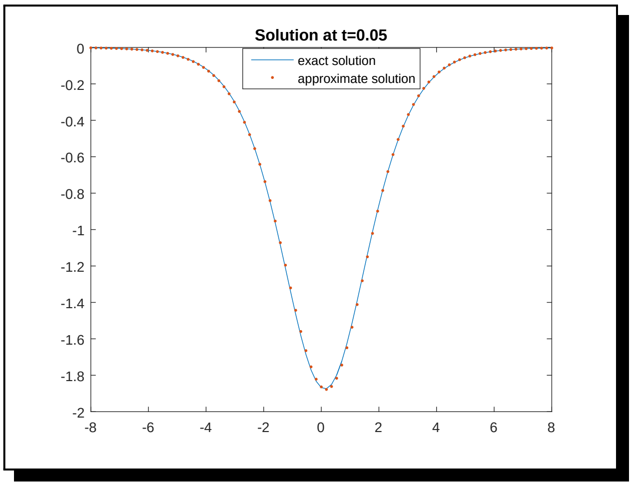
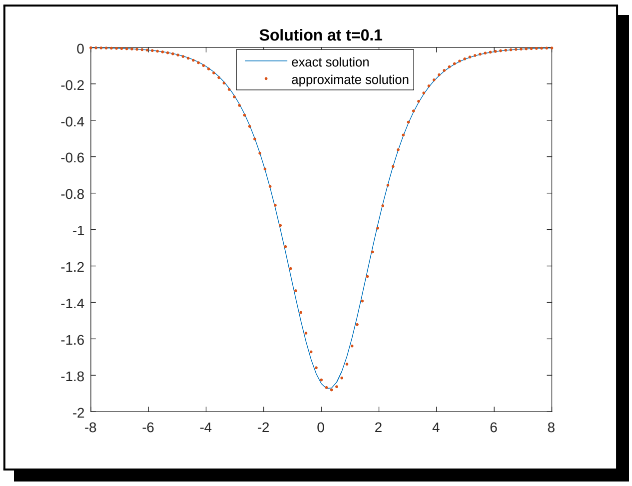


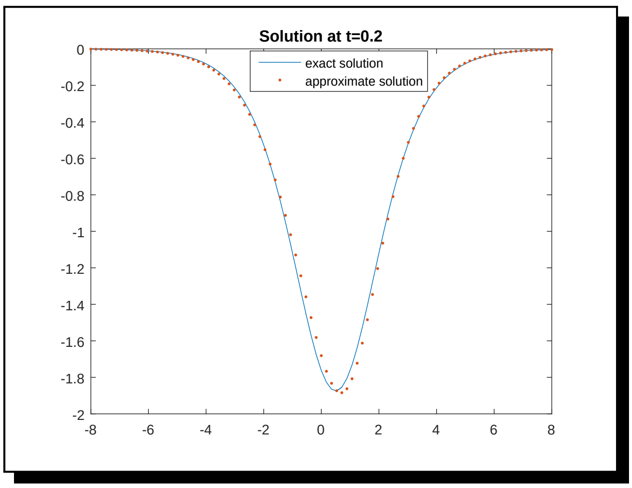
Figure 4.3. Space-time graphs of exact and approximate solutions of MDP equation for $t \in [0, 0.3]$.



(a) at time $t = 0.05$.



(b) at time $t = 0.1$.



(c) at time $t = 0.2$.

Figure 4.4. Exact and approximate solutions of MDP equation for various values of t.

Table 4.2. Comparison of absolute errors of MDP equation computed by QuBSM method with existing methods at different time levels.

x	t	Exact solution	Approximate solution	QuBSM	VIM[16]	ADM[7]	HPM[19]
6	0.05	−0.02094811	−0.02049908	4.490E-04	2.005E-03	—	—
8		−0.00284880	−0.00278568	6.312E-05	2.807E-04	3.332E-04	3.332E-04
9		−0.0010485	−0.00102520	2.332E-05	—	1.229E-04	1.230E-04
10		−0.00038580	−0.00037720	8.590E-06	3.817E-05	4.521E-05	4.530E-05
12		−0.00005222	−0.00005105	1.160E-06	5.169E-05	—	—
6	0.10	−0.02371963	−0.02281592	9.037E-04	4.226E-03	—	—
8		−0.00322779	−0.00310009	1.276E-04	5.911E-04	7.108E-04	7.109E-04
9		−0.0011880	−0.00114089	4.720E-05	—	2.623E-04	2.624E-04
10		−0.00043716	−0.00041976	1.740E-05	8.036E-05	9.659E-05	9.664E-05
12		−0.00005917	−0.00005682	2.350E-06	1.088E-05	—	—
8	0.15	−0.0036571	−0.00346392	1.932E-04	—	1.139E-03	1.139E-03
9		−0.0013462	−0.00127476	1.461E-05	—	4.203E-04	4.203E-04
10		−0.0004953	−0.00046901	2.635E-05	—	1.547E-04	1.548E-04
8	0.20	−0.0041435	−0.00388505	2.585E-04	—	1.624E-03	1.624E-03
9		−0.0015253	−0.00142971	9.568E-05	—	5.992E-04	5.993E-04
10		−0.0005613	−0.00052601	3.529E-05	—	2.207E-04	2.207E-04

5. Concluding Remarks

In this study, we implement quartic B-spline method for solving non-linear MCH and MDP equations with initial and boundary conditions given in equations (1.2)-(1.3). The time derivative is replaced by finite difference approach and quartic B-spline is used to interpolate the space derivatives. It can be observed that sometimes the accuracy of solution may reduce due to time truncation errors of time derivative term. The obtained results given in Tables 4.1–4.2 and Figures 1.1–4.4 are more accurate and reliable. A comparison between the absolute errors at various time levels and knots shows that our method is competent and more accurate as compared to other researchers. The order of convergence is established both in time and space direction. An advantage of using QuBSM mentioned in this paper is that it has the ability to provide the accurate solutions at any intermediate point in space direction. Moreover, it is straightforward and simple to apply with smaller storage. It can be concluded that QuBSM provides more accurate results as compares to the methods named *Variational Iteration Method* (VIM) [16], *Adomain Decomposition Method* (ADM) [7], *Homotopy Perturbation Method* (HPM) [19].

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M. Abbas, A.A. Majid, A.I. Ismail and A. Rashid, Numerical method using cubic B-spline for a strongly coupled reaction-diffusion system, *PLOS ONE* **9** (1) (2014), e83265, DOI: 10.1371/journal.pone.0083265.

[2] M. Abbas, A.A. Majid, A.I. Ismail and A. Rashid, Numerical method using cubic trigonometric B-spline technique for non-classical diffusion problem, *Abstr. Appl. Anal.* Article ID 849682 (2014), 10 pages.

[3] M. Abbas, A.A. Majid and A.I. Ismail, The application of cubic trigonometric B-spline to the numerical solution of the hyperbolic problems, *Appl. Math. Comp.* **239** (2014), 74 – 88.

[4] S. Abbasbandy, Solitary wave solution to the modified form Camassa-Holm equation, *Cha. Sol. Frac.* **39** (2009), 428 – 435.

[5] C. Clavero, J.C. Jorge and F. Lisbona, Uniformly convergent scheme on a nonuniform mesh for convection-diffusion parabolic problems, *J. Comput. Appl. Math.* **154** (2003), 415 – 429.

[6] C. DeBoor, On the convergence of odd degree spline interpolation, *J. Approx. Theo.* **1** (1968), 452 – 463.

[7] D.D. Ganji, E.M.M. Sadeghi and M.G. Rahmat, Modified forms of Degaperis-Procesi and Camassa-Holm equations solved by Adomian's Decomposition method and comparison with HPM and exact solution, *Acta. Appl. Math.* **104** (2008), 303 – 311.

[8] C.A. Hall, On error bounds for spline interpolation, *J. Approx. Theo.* **1** (1968), 209 – 218.

[9] J. Manafian, R. Shahabi, M. Asaspour, I. Zamanpour and J. Jalali, Construction of exact solutions to the modified forms of DP and CH equations by analytical methods, *Stat. Opt. Info. Comp.* **3** (2015), 336 – 347.

[10] R. Mohanty, M. Jain and K. George, Fourth order approximation at first time level, linear stability analysis and the numerical solution of multidimenional second-order non-linear hyperbolic equations in polar coordinates, *J. Comp. Appl. Math.* **93** (1998), 1 – 12.

[11] T. Nazir, M. Abbas, A.A. Majid, A.I. Ismail and A. Rashid, The numerical solution of convection-diffusion problem with cubic trigonometric B-splines, *Appl. Math. Model.* 40 (2016), 4586 – 4611.

[12] W. Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill Inc., p. 218 (1976).

[13] A.M. Wazawaz, Solitary wave solutions for modified forms of Degaperis-Procesi and Camassa-Holm equations, *Phy. Lett. A.* **352** (6) (2006), 500 – 504.

[14] A.M. Wazwaz, New solitary wave solutions to the modified forms of Degaperis-Procesi and Camassa-Holm equations, *Appl. Math.* **186** (1) (2007), 130 – 141.

[15] M. Yaseen, M. Abbas, A.I. Ismail and T. Nazir, A cubic trigonometric B-spline collocation approach for the fractional sub-diffusion equations, *Appl. Math. Comp.* **293** (2017), 311 – 319.

[16] A. Yildirim, Variational iteration method for modified Camassa-Holm and Degasperis-Procesi equations, *Int. J. Num. meth. Biomed. Engi.* 26 (2010), 266 – 272.

[17] M.A. Yousif, B.A. Mahmood and F.H. Easif, A new analytical study of modified Camassa-Holm and Degaperis-Procesi equations, *Amr. Jour. Comp. Math.* **5** (2015), 267 – 273.

[18] E. Yusufoglu, New solitary solutions for modified forms of DP and CH equations using Exp-function method, *Cha. Sol.* **39** (2009), 2442 – 2447.

[19] B. Zhang, S. Li and Z. Liu, Homotopy Perturbation method for modified Camassa-Holm and Degaperis-Procesi equations, *Phy. Lett. A* **372** (2008), 1867 – 1872.

[20] L. Zhang and X.T. Wang, A note on solitary wave solutions of the non-linear generalized Camassa-Holm equation, *Int. J. Anal.* **2013** (2013), Article ID 723698, 6 pages, DOI: 10.1155/2013/723698.

[21] B. Zhang, Z. Liu and J. Mao, New exact solutions for MCH and MDP equations by auxiliary equation method, *App. Math. Comp.* **217** (2010), 1306 – 1314.

[22] S.M. Zin, A.A. Majid, A.I. Ismail and M. Abbas, Application of hybrid cubic B-spline collocation approach for solving a generalized non-linear Klien-Gordon equation, *Math. Prob. Engin.* **2014**, Article ID 108560 (2014), 10 pages.