Communications in Mathematics and Applications
Vol. 9, No. 3, pp. 377–392, 2018
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v9i3.802

http://www.rgnpublications.com

***Proceedings*:**
3rd International Conference on Pure and Applied Mathematics
Department of Mathematics, University of Sargodha, Sargodha, Pakistan
November 10-11, 2017

New Cubic B-spline Approximation for Solving Non-linear Singular Boundary Value Problems Arising in Physiology

Research Article

Muhammad Kashif Iqbal[1,*], Muhammad Abbas[2] and Nauman Khalid[1]

[1] *Department of Mathematics, National College of Business Administration and Economics, Lahore, Pakistan*
[2] *Department of Mathematics, University of Sargodha, Sargodha, Pakistan*
***Corresponding author:** kashifiqbal@gcuf.edu.pk

Abstract. In this article, a new cubic B-spline approximation method is presented for solving second order singular boundary value problems. The proposed numerical technique is based on cubic B-spline collocation method equipped with a new approximation for second order derivative. Several test problems arising in the field of Physiological sciences are considered. The approximate results are compared with the numerical techniques existing in literature. It is found that our new approximation performs superior to current methods due to its simplicity, straight forward interpolation and less computational cost.

Keywords. Cubic B-spline basis functions; Cubic B-spline collocation method; Singular boundary value problems; Quasi-linearization technique

MSC. 34B15; 34B16;74H15; 65L10; 65L11

Received: February 2, 2018 **Accepted:** March 14, 2018

1. Introduction

Singular boundary value problems frequently appear in optimization, estimation and control of dynamic processes such as chemical reactions, inelastic flows, gravity-assisted flows, electrically charged fluid flows, thermal explosions and atomic nuclear reactions. In this work, we consider the following class of 2^{nd} order non-linear boundary value problems having singularity at $x = a$

$$u''(x) + \left(m_1 + \frac{m_2}{x}\right)u'(x) = f(x,u), \quad a \le x \le b \tag{1.1}$$

subject to the following initial/boundary conditions

$$\begin{cases} u'(a) = 0, \quad u(a) = \gamma \\ u'(a) = 0, \quad \alpha_1 u(b) + \alpha_2 u'(b) = \gamma \end{cases} \tag{1.2}$$

where m_i's, α_i's, and γ are constants. To ensure the existence of unique solution to (1.1), subject to the boundary conditions (1.2), we assume that both f and f_u are continuous with $f_u \ge 0$ in the entire domain [10]. With certain linear and non-linear f, the problem (1.1)-(1.2) provides a mathematical model for the distribution of heat sources in the human head, steady state oxygen diffusion in a spherical cell, thermal explosions,the electric double layer in a salt-free solution, the formation of heat and mass transfer within porous catalyst particle, isothermal gas equilibrium and many other mechanisms [12, 14, 20, 21]. Due to its wide range of applications in the natural world, the *singular boundary value problems* (SBVP's) have attracted a significant amount of research work in recent years. The presence of singularities raise difficulties while exploring the numerical or analytical solution to this type of problems.

Russell and Shampine [27] proposed a three point finite difference scheme with Newton's iteration procedure for numerical solution of non-linear SBVP's. A shooting algorithm based on the Taylor series method was developed by Rentrop [26] for solving boundary value problems. The application of a fourth order finite difference scheme to non-linear SBVP's was discussed by Chawla *et al.* [11]. Pandey and Singh [23] employed finite difference scheme for solving second order SBVP's arising in Physiology. Parand *et al.* [24] employed the Hermite function collocation method for numerical solution of second order singular initial value problems. The series solution of non-linear SBVP's by means of He's Variational iteration method was studied in [17, 31]. Recently, Singh and Kumar [28] explored the series solution of a class of non-linear SBVP's using Green's identities with the Adomian decomposition method. Wang *et al.* [30] used reproducing kernel method for solving second order Lane-Emden type equations. Babolian *et al.* [5] investigated the numerical solution of a class of second order non-linear SBVP's by implementing Sinc-Galerkin technique. Mohsenyzadeh *et al*. [22] utilized orthonormal Bernoulli polynomials to construct a new basis for solving second order SBVP's. Singh *et al.* [29] proposed a semi-numerical method for numerical solution of second order SBVP's. The series solution of second order non-linear SBVP's by means of improved differential transformation method was explored in [32].

Spline functions have been extensively used to explore the numerical solution of boundary value problems due to their strong geometrical features and flexibility to find the approximate solution at any point of the domain with high accuracy. Bickley [6] was the first to use cubic spline interpolation for solving ordinary differential equations. Albasiny and Hoskins [4] carried forward Bickley's work and investigated the spline solution to ordinary differential equation by solving a tri-diagonal system of equations. The numerical solution of second order two point linear boundary value problems by means of B-splines of degree three has been investigated in [8]. The non polynomial cubic spline functions were put to use by Rashidinia *et al.* [25] for solving SBVP's arising in Physiology. Caglar [9] employed third degree B-spline collocation method for approximate solution of a class of non-linear boundary value problems which arise in the study of thermal explosions and steady state oxygen diffusion in the sphere-shape cells. Khuri and Sayfy [18] proposed a new numerical technique based on a combination of B-spline collocation and modified Adomian decomposition methods for a non-linear mathematical model describing the distribution of heat sources in human head. Abukhaled *et al.* [3] explored the numerical solution of second order SBVP's arising in Physiology by means of chebyshev polynomials and B-spline basis functions. Goh *et al.* [15, 16] employed extended uniform B-spline with fourth degree blending functions and Quartic B-splines for numerical solution of linear SBVP's.

In this study, the numerical solution to a class of non-linear SBVP's is investigated by means of cubic B-spline functions. The presented approach is based on a new approximation for the second order derivative, which is formulated by proper linear combinations of typical B-spline approximation for $u''(x)$ [1] at the neighbouring knots. This new approximation is novel for second order non-linear SBVP's.

This work is organized as follows: In Section 2, we present some basic results of typical third degree B-spline interpolation. The formulation of new approximation for $u''(x)$ is described in Section 3. In Section 4 the numerical method is demonstrated. An error analysis of the proposed fifth order numerical scheme is presented in section 5 and the numerical results and comparisons are tabulated in Section 6. The comparisons of numerical results exhibit that our proposed technique is very effective and powerful for the numerical solution of two point singular boundary value problems.

2. Cubic B-spline Functions

Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a uniform partition of the interval of $[a, b]$, where n is a positive integer, $x_i = x_0 + ih, i \in \mathbb{Z}$ and $h = \frac{b-a}{n}$. The typical third degree B-spline basis functions are defined as [1, 7]

$$B_i(x) = \frac{1}{6h^3}\begin{cases} (x - x_{i-2})^3, & x \in [x_{i-2}, x_{i-1}] \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & x \in [x_{i-1}, x_i] \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & x \in [x_i, x_{i+1}] \\ (x_{i+2} - x)^3, & x \in [x_{i+1}, x_{i+2}] \\ 0, & \text{otherwise,} \end{cases} \tag{2.1}$$

where $i = -1, 0, 1, 2, \cdots, n+1$. Let $u(x)$ be a sufficiently smooth function, then there exists a unique cubic spline $U(x)$, satisfying the interpolating conditions $U(x_i) = u(x_i)$, for all $i = 0, 1, 2, \cdots, n$, $U'(a) = u'(a)$ and $U'(b) = u'(b)$ such that

$$U(x) = \sum_{i=-1}^{n+1} c_i B_i(x), \tag{2.2}$$

where c_i's are finite constants yet to be determined. For the sake of simplicity we denote, the B-spline approximations at the jth knot, $U(x_j)$, $U'(x_j)$ and $U''(x_j)$ by U_j, m_j and M_j, respectively. The cubic B-spline basis function (2.1) together with (2.2) gives the following relations

$$U_j = \sum_{i=j-1}^{j+1} c_i B_i(x) = \frac{1}{6}(c_{j-1} + 4c_j + c_{j+1}), \tag{2.3}$$

$$m_j = \sum_{i=j-1}^{j+1} c_i B_i'(x) = \frac{1}{2h}(-c_{j-1} + c_{j+1}), \tag{2.4}$$

$$M_j = \sum_{i=j-1}^{j+1} c_i B_i''(x) = \frac{1}{h^2}(c_{j-1} - 2c_j + c_{j+1}). \tag{2.5}$$

Moreover, from (2.3)-(2.5), following relations can be established [13].

$$m_j = u'(x_j) - \frac{1}{180} h^4 u^{(5)}(x_j) + \cdots, \tag{2.6}$$

$$M_j = u''(x_j) - \frac{1}{12} h^2 u^{(4)}(x_j) + \frac{1}{360} h^4 u^{(6)}(x_j) + \cdots. \tag{2.7}$$

From (2.6) and (2.7), we have $\|m_j - u'(x_j)\|_\infty = O(h^4)$ and $\|M_j - u''(x_j)\|_\infty = O(h^2)$. This gives enough motivation to formulate a better approximation for $u''(x)$ [19].

3. The New Approximation for $u''(x)$

In order to formulate a new approximation for $u''(x)$, we use (2.7) to establish the following expression for M_{j-1} at the knot x_j, $j = 1, 2, 3, \cdots, n-1$

$$\begin{aligned} M_{j-1} &= u''(x_{j-1}) - \frac{1}{12} h^2 u^{(4)}(x_{j-1}) + \frac{1}{360} h^4 u^{(6)}(x_{j-1}) + \cdots \\ &= u''(x_j) - h u^{(3)}(x_j) + \frac{5}{12} h^2 u^{(4)}(x_j) - \frac{1}{12} h^3 u^{(5)}(x_j) + \cdots. \end{aligned}$$

Similarly,

$$M_{j+1} = u''(x_j) + h u^{(3)}(x_j) + \frac{5}{12} h^2 u^{(4)}(x_j) + \frac{1}{12} h^3 u^{(5)}(x_j) + \cdots.$$

Let $\widetilde{M}_j$ be the new approximation to $u''(x)$ such that

$$\widetilde{M}_j = A_1 M_j + A_2 M_{j-1} + A_3 M_{j+1}. \tag{3.1}$$

The parameters A_1, A_2 and A_3 are chosen so that the error order of $\widetilde{M}_j$ is as high as possible. This linear combination gives the following three equations

$$A_1 + A_2 + A_3 = 1,$$

$$-A_2+A_3=0,$$

$$-A_1+5A_2+5A_3=0.$$

Hence, $A_1=\frac{5}{6}$ and $A_2=A_3=\frac{1}{12}$, the expression (3.1) takes the following form

$$\tilde{M}_j=\frac{1}{12h^2}\big(c_{j-2}+8c_{j-1}-18c_j+8c_{j+1}+c_{j+2}\big). \tag{3.2}$$

In order to establish the new approximation to $u''(x)$ at the knot x_0, we use four neighbouring values such that

$$\widetilde{M}_0=A_0M_0+A_1M_1+A_2M_2+A_3M_3, \tag{3.3}$$

where

$$M_1=u''(x_0)+hu^{(3)}(x_0)+\frac{5}{12}h^2u^{(4)}(x_0)+\frac{1}{12}h^3u^{(5)}(x_0)+\cdots,$$

$$M_2=u''(x_0)+2hu^{(3)}(x_0)+\frac{23}{12}h^2u^{(4)}(x_0)+\frac{7}{6}h^3u^{(5)}(x_0)+\cdots,$$

$$M_3=u''(x_0)+3hu^{(3)}(x_0)+\frac{53}{12}h^2u^{(4)}(x_0)+\frac{17}{4}h^3u^{(5)}(x_0)+\cdots.$$

Choosing the parameters A_0, A_1, A_2 and A_3 so that the error order is as high as possible, the expression (3.3) gives the following four equations

$$A_0+A_1+A_2+A_3=1,$$

$$A_1+2A_2+3A_3=0,$$

$$-A_0+5A_1+23A_2+53A_3=0,$$

$$A_1+14A_2+51A_3=0.$$

Hence, $A_0=\frac{7}{6}$, $A_1=-\frac{5}{12}$, $A_2=\frac{1}{3}$ and $A_3=-\frac{1}{12}$, the expression (3.3) takes the following form

$$\widetilde{M}_0=\frac{1}{12h^2}\big(14c_{-1}-33c_0+28c_1-14c_2+6c_3-c_4\big). \tag{3.4}$$

Working on the same pattern, the approximation at the knot x_n is given by

$$\widetilde{M}_n=\frac{1}{12h^2}\big(-c_{n-4}+6c_{n-3}-14c_{n-2}+28c_{n-1}-33c_n+14c_{n+1}\big). \tag{3.5}$$

4. Description of the Numerical Method

Applying Quasi-linearlization technique, the non-linear problem (1.1) is converted into the following linear problem

$$u''_{k+1}(x)+\Big(m_1+\frac{m_2}{x}\Big)u'_{k+1}(x)+Y_k(x)u_{k+1}(x)=Z_k(x),\quad a\le x\le b, \tag{4.1}$$

where $Y_k(x)=-\big(\frac{\partial f}{\partial u}\big)_{(x,u_k)}$, $Z_k(x)=f(x,u_k)-\big(\frac{\partial f}{\partial u}\big)_{(x,u_k)}$, $k=0,1,2,\cdots$.

The boundary conditions (1.2) are also transformed as

$$\begin{cases} u'_{k+1}(a)=0 \\ \alpha_1u_{k+1}(b)+\alpha_2u'_{k+1}(b)=\gamma \end{cases} \tag{4.2}$$

In order to remove the singularity at $x = a$, using L'Hospital rule, we modify (1.1) as

$$p(x)u''_{k+1}(x) + q(x)u'_{k+1}(x) + Y_k(x)u_{k+1}(x) = Z_k(x), \quad a \le x \le b, \tag{4.3}$$

where $p(x) = \begin{cases} 1 + m_2, & \text{if } x = a \\ 1, & \text{otherwise} \end{cases}$ and $q(x) = \begin{cases} m_1, & \text{if } x = a \\ m_1 + \frac{m_2}{x}, & \text{otherwise.} \end{cases}$

Let $U(x)$ be the cubic B-spline solution to (4.3), satisfying the interpolating conditions, such that

$$U(x) = \sum_{i=-1}^{n+1} c_i B_i(x). \tag{4.4}$$

Discretizing (4.3) at the knot x_j, $j = 1,2,3,\cdots,n-1$, we obtain

$$p(x_j)U''_{k+1}(x_j) + q(x)U'_{k+1}(x_j) + Y_k(x_j)U_{k+1}(x_j) = Z_k(x_j). \tag{4.5}$$

Using (2.3)-(2.4) and (3.2) in equation (4.5), we have

$$p(x_j)\Big(\frac{c_{j-2} + 8c_{j-1} - 18c_j + 8c_{j+1} + c_{j+2}}{12h^2}\Big) + q(x_j)\Big(\frac{-c_{j-1} + c_{j+1}}{2h}\Big) + Y_k(x_j)\Big(\frac{c_{j-1} + 4c_j + c_{j+1}}{6}\Big) = Z_k(x_j). \tag{4.6}$$

Similarly, discretizing (4.3) at the knots x_0 and x_n, we obtain the following equations

$$p(x_0)\Big(\frac{14c_{-1} - 33c_0 + 28c_1 - 14c_2 + 6c_3 - c_4}{12h^2}\Big) + q(x_0)\Big(\frac{-c_{-1} + c_1}{2h}\Big) + Y_k(x_0)\Big(\frac{c_{-1} + 4c_0 + c_1}{6}\Big) = Z_k(x_0), \tag{4.7}$$

$$p(x_n)\Big(\frac{-c_{n-4} + 6c_{n-3} - 14c_{n-2} + 28c_{n-1} - 33c_n + 14c_{n+1}}{12h^2}\Big) + q(x_n)\Big(\frac{-c_{n-1} + c_{n+1}}{2h}\Big) + Y_k(x_n)\Big(\frac{c_{n-1} + 4c_n + c_{n+1}}{6}\Big) = Z_k(x_n). \tag{4.8}$$

The boundary conditions (4.2), give the following two equations

$$\frac{c_{-1} + 4c_0 + c_1}{6} = 0, \tag{4.9}$$

$$\alpha_1\Big(\frac{c_{n-1} + 4c_n + c_{n+1}}{6}\Big) + \alpha_2\Big(\frac{-c_{n-1} + c_{n+1}}{2h}\Big) = \gamma. \tag{4.10}$$

In this way, we have a system of $n+3$ linear equations (4.6)-(4.10), which can be written in matrix form as

$$Ac = b, \tag{4.11}$$

where A is the coefficients matrix of order $n+3$, $c = [c_{-1} \;\; c_0 \;\; c_1 \cdots c_{n+1}]^T$ and b is a column matrix with $n+3$ entries. For $k = 0$, we start from an initial guess $U_0(x)$ and solve the system (4.11) for c. Substituting c_i's back into (4.4), we obtain the approximate solution $U_{k+1}(x)$. The process is repeated for $k = 1,2,3,\cdots$ until $\max|U_{k+1}(x_j) - U_k(x_j)| \le 10^{-7}$. All the numerical computations are performed in MATHEMATICA 9 using core i7 processor.

5. Error Analysis

Using the cubic B-spline approximations (2.3)-(2.4) and (3.2), the following relations can be established

$$h\left[\frac{1}{6}U'(x_{j-1})+\frac{4}{6}U'(x_j)+\frac{1}{6}U'(x_{j+1})\right]=\frac{1}{2}\left[U(x_{j+1})-U(x_{j-1})\right], \tag{5.1}$$

$$h^2U''(x_j)=\frac{1}{2}\left[7U(x_{j-1})-8U(x_j)+U(x_{j+1})\right]+h\left[U'(x_{j-1})+2U'(x_j)\right]. \tag{5.2}$$

Moreover, from [13], we have

$$h^3U'''(x_{j^+})=12\left[U(x_j)-U(x_{j+1})\right]+6h\left[U'(x_j)+U'(x_{j+1})\right], \tag{5.3}$$

$$h^3U'''(x_{j^-})=12\left[U(x_{j-1})-U(x_j)\right]+6h\left[U'(x_{j-1})+U'(x_j)\right], \tag{5.4}$$

where $U'''(x_{j^+})$ and $U'''(x_{j^-})$ denote the approximate values of $u'''(x_j)$ in $[x_j,x_{j+1}]$ and $[x_{j-1},x_j]$, respectively.

Using the operator notation $E^{\lambda}(U'(x_j))=U'(x_{j+\lambda})$, $\lambda\in\mathbb{Z}$, equation (5.1) can be written as [13]

$$h\left[\frac{1}{6}E^{-1}+\frac{4}{6}+\frac{1}{6}E\right]U'(x_j)=\frac{1}{2}\left[E-E^{-1}\right]u(x_j).$$

Hence

$$hU'(x_j)=3\left(E-E^{-1}\right)\left[E^{-1}+4+E\right]^{-1}u(x_j). \tag{5.5}$$

Using $E=\mathrm{e}^{hD}$, $D\equiv d/dx$, we can get

$$E+E^{-1}=\mathrm{e}^{hD}+\mathrm{e}^{-hD}=2\left[1+\frac{h^2D^2}{2!}+\frac{h^4D^4}{4!}+\frac{h^6D^6}{6!}+\cdots\right],$$

$$E-E^{-1}=\mathrm{e}^{hD}-\mathrm{e}^{-hD}=2\left[hD+\frac{h^3D^3}{3!}+\frac{h^5D^5}{5!}+\frac{h^7D^7}{7!}+\cdots\right].$$

Therefore, equation (5.5) can be expressed as

$$U'(x_j)=\left(D+\frac{h^2D^3}{3!}+\frac{h^4D^5}{5!}+\cdots\right)\left[1+\left(\frac{h^2D^2}{6}+\frac{h^4D^4}{72}+\frac{h^6D^6}{2160}+\cdots\right)\right]^{-1}u(x_j).$$

Simplifying, we obtain

$$U'(x_j)=\left(D-\frac{h^4D^5}{180}+\frac{h^6D^7}{1512}-\cdots\right)u(x_j).$$

Hence

$$U'(x_j)=u'(x_j)-\frac{1}{180}h^4u^{(5)}(x_j)+\cdots. \tag{5.6}$$

Similarly, writing (5.2) in operator notation we get

$$\begin{aligned}h^2U''(x_j)&=\frac{1}{2}\left[7E^{-1}-8+E\right]u(x_j)+h\left[E^{-1}+2\right]u'(x_j)\\&=\left(-3hD+2h^2D^2-\frac{h^3D^3}{2}+\frac{h^4D^4}{6}-\frac{h^5D^5}{40}+\frac{h^6D^6}{180}-\cdots\right)u(x_j)\\&\quad+\left(3h-h^2D+\frac{h^3D^2}{2}-\frac{h^4D^3}{6}+\frac{h^5D^4}{24}-\frac{h^6D^5}{120}-\cdots\right)u'(x_j).\end{aligned}$$

Simplifying the above relation, we have

$$U''(x_j)=u''(x_j)+\frac{1}{60}h^3u^{(5)}(x_j)-\frac{1}{360}h^4u^{(6)}(x_j)+\cdots. \tag{5.7}$$

Using the same method,(5.3) can also be written as

$$U'''(x_j) \approx \frac{1}{2}\left[u'''(x_{j^+}) + u'''(x_{j^-})\right] = u'''(x_j) + \frac{1}{12}h^2 u^{(5)}(x_j) + \cdots. \tag{5.8}$$

Let us define the error term $e(x) = U(x) - u(x)$. Using the relations (5.6)-(5.8) in the Taylor series expansion of $e(x)$ we get

$$e(x_j + \theta h) = \frac{\theta(5\theta - 2)(\theta + 1)}{360} h^5 u^{(5)}(x_j) - \frac{\theta^2}{720} h^6 u^{(6)}(x_j) + \cdots, \tag{5.9}$$

where $\theta \in [0,1]$. From (5.9), it is obvious that the new cubic B-spline approximation is $O(h^5)$ accurate.

6. Numerical Results

In this section, we present the numerical solution to (1.1) using the *new cubic B-spline approximation method* (NCBSM). The accuracy of the proposed technique is tested by the error norms L_∞ and L_2 which are calculated as [2]

$$L_\infty = \|U_j - u_j\|_\infty = \max_j |U_j - u_j|,$$

$$L_2 = \|U_j - u_j\|_2 = \sqrt{\sum_j (U_j - u_j)^2},$$

where U_j and u_j denote the approximate and exact solutions at j^{th} knot respectively.

Problem 1. *Consider the zero*th *order Bessel's equation* [8, 15, 16]

$$u''(x) + \frac{1}{x}u'(x) + u(x) = 0, \quad 0 \le x \le 1$$

subject to the boundary conditions

$$u'(0) = 0, \quad u(1) = 1$$

The analytical exact solution to this homogeneous boundary value problem is $u(x) = \frac{J_0(x)}{J_0(1)}$. Table 6.1 shows the maximum and Euclidian norms corresponding to different values of h. It is clear that the numerical results obtained by the NCBSM are more accurate as compared to the *Cubic B-spline method* (CBSM) [8], *Extended cubic B-spline method* (ECBSM) [16] and the *Quartic B-spline method* (QBSM) [15]. Figure 6.1 displays a very close agreement of the approximate solution with the exact solution for $h = 0.05$.

Table 6.1. Comparison of computational error norms for Problem 1

h	CBSM [8]		ECBSM [16]		QBSM [15]		Proposed method	
	L_∞ norm	L_2 norm	L_∞ norm	L_2 norm	L_∞ norm	L_2 norm	L_∞ norm	L_2 norm
0.1	1.1×10^{-4}	2.7×10^{-4}	1.3×10^{-5}	3.0×10^{-5}	1.7×10^{-6}	1.9×10^{-6}	5.5×10^{-8}	1.1×10^{-7}
0.05	2.8×10^{-5}	9.1×10^{-5}	1.2×10^{-7}	4.6×10^{-7}	2.0×10^{-7}	2.3×10^{-7}	2.6×10^{-9}	8.0×10^{-9}
0.02	4.5×10^{-6}	2.3×10^{-5}	9.8×10^{-9}	4.3×10^{-8}	1.4×10^{-8}	1.6×10^{-8}	5.9×10^{-11}	3.0×10^{-10}
0.01	1.1×10^{-6}	8.0×10^{-6}	$\cdots$	$\cdots$	1.4×10^{-9}	1.6×10^{-9}	3.4×10^{-12}	2.5×10^{-11}

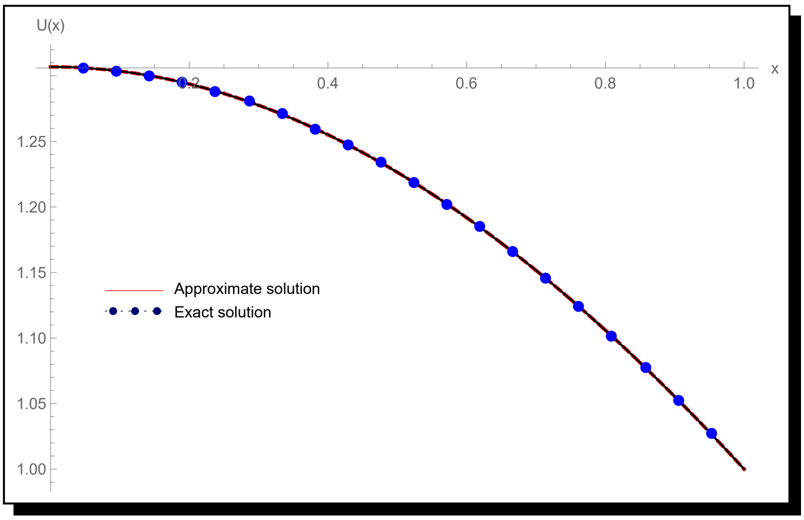


Figure 6.1. Comparison of approximate and exact solution for Problem 1 when $h = 0.05$

Problem 2. *Consider the non-linear SBVP* [29].

$$u''(x) + \frac{\alpha}{x}u'(x) = \beta x^{\beta-2}e^{u(x)}(\beta x^{\beta}e^{u(x)} - \alpha - \beta + 1), \quad 0 \le x \le 1$$

subject to the boundary conditions

$$u(0) = \log\frac{1}{4}, \quad u(1) = \log\frac{1}{5}.$$

The analytical solution to this problem is $u(x) = \frac{1}{4+x^{\beta}}$. The comparison of obtained numerical results with *Adomian decomposition method* (ADM) [29] and a *semi numerical method* (SNM) proposed in [29] is presented in Table 6.2 when $h = 0.05$, $\alpha = 0.5$ and $\beta = 2$. Figure 6.2 depicts a comparison of exact and approximate solutions when $h = 0.05$. The absolute numerical errors for two different values of h are shown in Figure 6.3.

Table 6.2. Comparison of absolute numerical error for Problem 2 when $h = 0.05$, $\alpha = 0.5$ and $\beta = 2$

x	ADM [29]	SNM [29]	CBSM	Proposed method
0.0	0	0	1.07×10^{-14}	2.20×10^{-14}
0.1	$\cdots$	$\cdots$	8.26×10^{-6}	2.53×10^{-9}
0.2	8.44×10^{-5}	6.07×10^{-6}	1.16×10^{-5}	3.45×10^{-9}
0.3	$\cdots$	$\cdots$	1.27×10^{-5}	4.58×10^{-9}
0.4	7.41×10^{-5}	2.17×10^{-6}	1.25×10^{-5}	5.08×10^{-9}
0.5	$\cdots$	$\cdots$	1.12×10^{-5}	5.24×10^{-9}
0.6	3.00×10^{-5}	4.89×10^{-6}	9.29×10^{-6}	4.99×10^{-9}
0.7	$\cdots$	$\cdots$	6.98×10^{-6}	4.33×10^{-9}
0.8	2.03×10^{-5}	9.86×10^{-6}	4.52×10^{-6}	3.23×10^{-9}
0.9	$\cdots$	$\cdots$	2.14×10^{-6}	1.72×10^{-9}
1.0	1.60×10^{-15}	0	0	2.22×10^{-16}

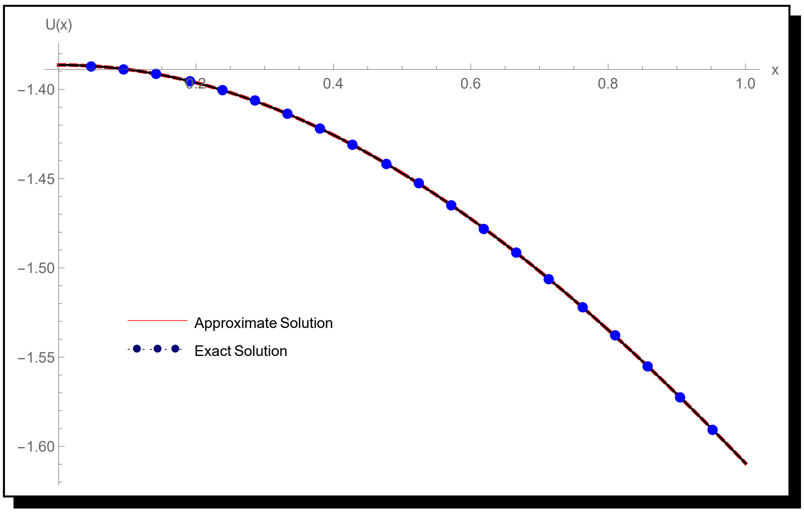


Figure 6.2. Comparison of approximate and exact solution for Problem 2 when $h = 0.05$

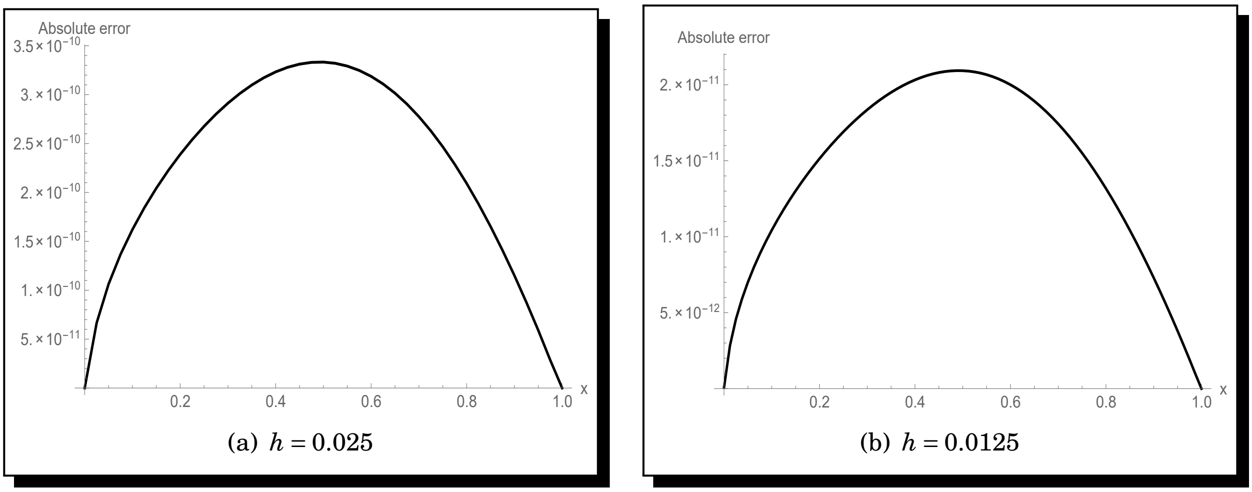


Figure 6.3. Absolute numerical error for Problem 2 using different values of h

Problem 3. *Consider the following non-linear SBVP arising in the study of tumor growth problems [3, 23].*

$$u''(x) + \left(1 + \frac{\alpha}{x}\right)u'(x) = \frac{5x^3(5x^5 e^{u(x)} - x - \alpha - 4)}{4 + x^5}, \quad 0 \le x \le 1$$

subject to the boundary conditions

$$u'(0) = 0, \quad u(1) + 5u'(1) = -5 - \log 5.$$

The exact solution is $u(x) = \log \frac{1}{4+x^5}$. The comparison of numerical results with *finite difference method* (FDM) [23] and *Chebyshev polynomials with cubic B-spline method* (CPCBSM) [3] is displayed in Table 6.3. The exact and approximate solution for $\alpha = 2$ and $h = 1/16$ are shown in Figure 6.4.

Table 6.3. Comparison of absolute errors for Problem 3

α	n	CPCBSM[3]	FDM [23]	Proposed method
0.25	16	7.79×10^{-4}	1.17×10^{-3}	6.55×10^{-5}
	32	1.98×10^{-4}	3.04×10^{-4}	3.25×10^{-6}
	64	4.98×10^{-5}	7.67×10^{-5}	1.73×10^{-7}
1.0	16	7.79×10^{-4}	1.46×10^{-3}	5.46×10^{-5}
	32	1.98×10^{-4}	3.68×10^{-4}	2.55×10^{-6}
	64	4.98×10^{-5}	9.20×10^{-5}	1.17×10^{-7}
8.0	16	2.52×10^{-3}	4.11×10^{-3}	3.03×10^{-5}
	32	6.30×10^{-4}	9.76×10^{-4}	1.10×10^{-6}
	64	1.57×10^{-4}	2.38×10^{-4}	8.70×10^{-9}

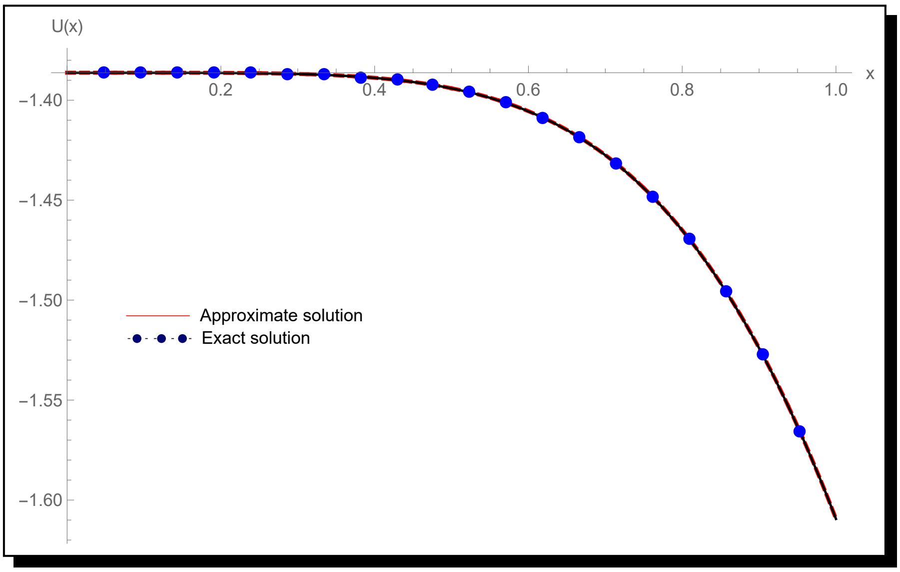


Figure 6.4. Comparison of approximate and exact solution for Problem 3 when $\alpha = 2$ and $h = 1/16$

Problem 4. *Consider the following Emden-Flower type equation* [30].

$$u''(x) + \frac{1}{x}u'(x) = u^3(x) - 3u^5(x), \quad 0 \le x \le 1$$

subject to the initial conditions

$$u(0) = 1, \;\; u'(0) = 0.$$

The exact solution is $u(x) = \frac{1}{\sqrt{1+x^2}}$. Table 6.5 demonstrates a comparison of numerical results with RKM [30] and CBSM. In Figure 6.6, the approximate and exact solutions are displayed when $h = 0.05$. Figure 6.7 depicts that as we decrease the mesh size, the computed results rapidly converge to the exact solution.

Table 6.4. Comparison absolute error for Problem 4 when $h = 0.05$

x	RKM [30]	CBSM	Proposed Method
0.0	0	6.44×10^{-15}	5.77×10^{-15}
0.1	3.38×10^{-7}	1.22×10^{-6}	2.20×10^{-9}
0.2	3.57×10^{-7}	4.12×10^{-6}	4.35×10^{-9}
0.3	4.20×10^{-7}	7.55×10^{-6}	7.57×10^{-9}
0.4	5.53×10^{-7}	1.01×10^{-5}	1.17×10^{-8}
0.5	7.65×10^{-7}	1.10×10^{-5}	1.62×10^{-8}
0.6	1.05×10^{-6}	9.67×10^{-6}	2.03×10^{-8}
0.7	1.39×10^{-6}	6.40×10^{-6}	2.35×10^{-8}
0.8	1.82×10^{-6}	1.65×10^{-6}	2.57×10^{-8}
0.9	2.45×10^{-6}	4.04×10^{-6}	2.66×10^{-8}
1.0	3.77×10^{-6}	1.02×10^{-5}	2.66×10^{-8}

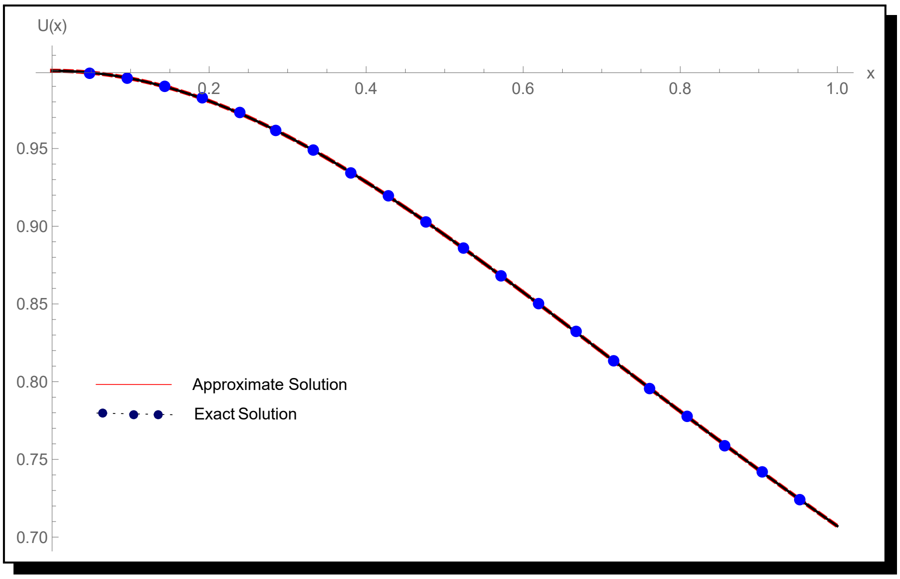


Figure 6.5. Comparison of approximate and exact solution for Problem 4 when $h = 0.05$

Problem 5. *Consider the following non-linear Emden-Flower equation of first kind arising in the study of iso-thermal gas sphere* [17, 28, 32].

$$u''(x) + \frac{2}{x}u'(x) + u^5(x) = 0, \quad 0 \le x \le 1$$

subject to the boundary conditions

$$u'(0) = 0, \quad u(1) = \frac{\sqrt{3}}{2}.$$

The exact solution is $u(x) = \frac{1}{\sqrt{1+\frac{x^2}{3}}}$. The absolute numerical errors are tabulated in Table 6.5 and are compared with VIM [17], Green's functions with improved Adomian decomposition method (GIADM) [28] and improved differential transformation method (IDTM) [32]. It can be noted that the our proposed method performs better than other approaches even with a small value of step size. In Figure 6.6, for $h = 0.05$, the approximate and exact solutions are displayed. The absolute computational error corresponding to two different values of h are plotted in Figure 6.7, which exhibits a rapid convergence of the new cubic B-spline approximation towards the exact solution.

Table 6.5. Comparison of absolute numerical errors for Problem 5 using $h = 0.05$

x	VIM [17]	GIADM [28]	IDTM [32]	Proposed method
0.0	6.3×10^{-3}	3.2×10^{-3}	1.7×10^{-4}	9.4×10^{-8}
0.1	6.3×10^{-3}	3.1×10^{-3}	1.7×10^{-4}	9.0×10^{-8}
0.2	6.1×10^{-3}	3.0×10^{-3}	1.6×10^{-4}	8.6×10^{-8}
0.3	5.9×10^{-3}	2.6×10^{-3}	1.6×10^{-4}	7.9×10^{-8}
0.4	5.5×10^{-3}	2.2×10^{-3}	1.5×10^{-4}	7.0×10^{-8}
0.5	5.1×10^{-3}	1.8×10^{-3}	1.4×10^{-4}	5.9×10^{-8}
0.6	4.5×10^{-3}	1.4×10^{-3}	1.2×10^{-4}	4.7×10^{-8}
0.7	3.8×10^{-3}	9.8×10^{-4}	1.1×10^{-4}	3.4×10^{-8}
0.8	2.9×10^{-3}	6.1×10^{-4}	9.5×10^{-5}	2.2×10^{-8}
0.9	1.6×10^{-3}	2.8×10^{-4}	6.8×10^{-5}	1.0×10^{-8}
1.0	1.0×10^{-10}	3.5×10^{-8}	0	1.1×10^{-16}

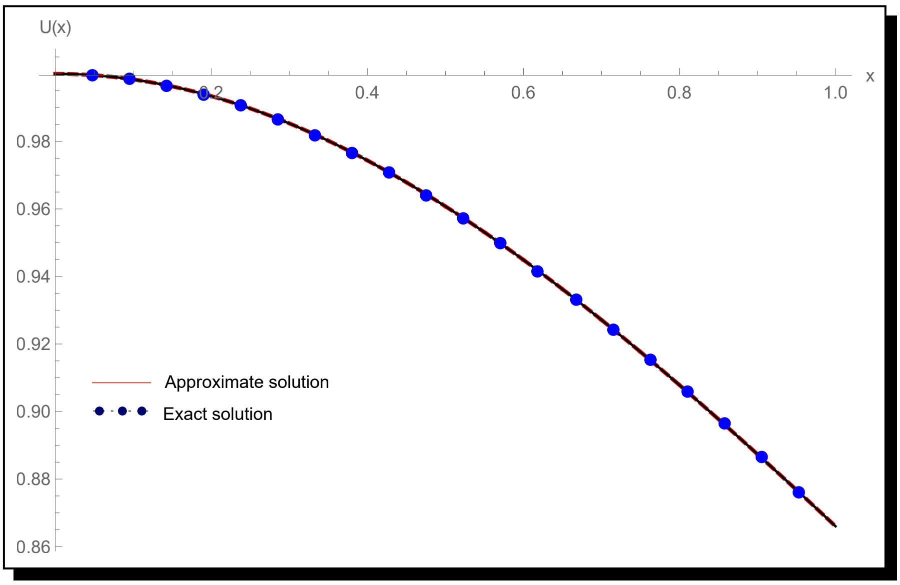
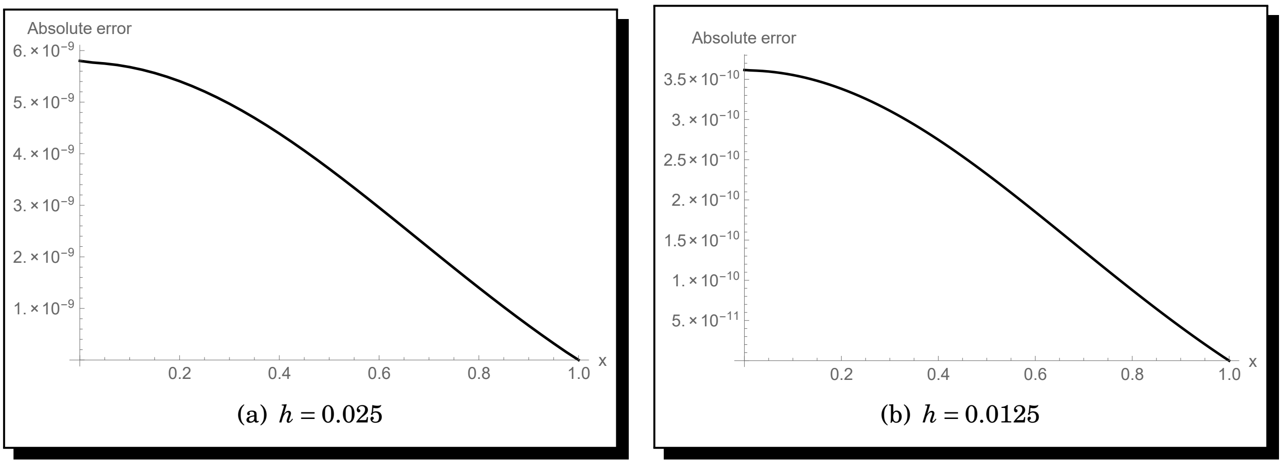


Figure 6.6. Comparison of approximate and exact solution for Problem 5 when $h = 0.05$



(a) $h = 0.025$ (b) $h = 0.0125$

Figure 6.7. Absolute numerical error for Problem 5 using different values of h

Conclusions

In this paper, a new cubic B-spline approximation method has been proposed for solving second order SBVP's. The authors conclude this work as

(1) The proposed scheme is based on cubic B-spline collocation method equipped with a new approximation for second order derivative.

(2) The presented algorithm is novel for second order singular boundary value problems.

(3) It produces fifth order accurate results.

(4) The proposed method generates a piecewise spline solution in the presence of the singularity which can be used to obtain a numerical solution at any point in the domain and not restricted to the values at the selected knots in contrast with existing finite-difference methods.

(5) Due to its simple and straightforward application, the approximate results computed by NCBSM are more accurate as compared to CPCBSM [3], CBSM [8], ECBM [16], QBSM [15], VIM [17], FDM [23], GIADM [28], ADM [29], SNM [29], RKM [30] and IDTM [32], .

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M. Abbas, A. Abd. Majid, A.I. Md. Ismail and A. Rashid, Numerical method using cubic B-spline for a strongly coupled reaction-diffusion system, *PloS-One* **9** (1) (2014), e83265, DOI: 10.1371/journal.pone.0083265.

[2] M. Abbas, A. Abd. Majid, A.I. Md. Ismail and A. Rashid, The application of cubic trigonometric B-spline to the numerical solution of the hyperbolic problems, *Applied Mathematics and Computation* **239** (2014), 74 – 88, DOI: 10.1016/j.amc.2014.04.031.

[3] M. Abukhaled, S.A. Khuri and A. Sayfy, A numerical approach for solving a class of singular boundary value problems arising in physiology, *International Journal of Numerical Analysis and Modeling* **8** (2) (2011), 353 – 363.

[4] E.L. Albasiny and W.D. Hoskins, Cubic spline solutions to two-point boundary value problems, *The Computer Journal* **12** (2) (1969), 151 – 153, DOI: 10.1093/comjnl/12.2.151.

[5] E. Babolian, A. Iftikhari and A. Saadatmandi, A Sinc-Galerkin technique for the numerical solution of a class of singular boundary value problems, *Computational and Applied Mathematics* **34** (1) (2015), 45 – 63, DOI: 10.1007/s40314-013-0103-x.

[6] W.G. Bickley, Piecewise cubic interpolation and two point boundary-value problems, *Computer Journal* **11** (1968), 202 – 208, DOI: 10.1093/comjnl/11.2.206.

[7] C.D. Boor, *A Practical Guide to Splines*, Springer-Verlag, New York (1978).

[8] N. Caglar and H. Caglar, B-spline solution of singular boundary value problems, *Applied Mathematics and Computation* **182** (2) (2006), 1509 – 1513, DOI: 10.1016/j.amc.2006.05.035.

[9] H. Caglar, N. Caglar and M. Ozer, B-spline solution of non-linear singular boundary value problems arising in physiology, *Chaos, Solitons & Fractals* **39** (3) (2009), 1232 – 1237, DOI: 10.1016/j.chaos.2007.06.007.

[10] Z. Cen, Numerical study for a class of singular two-point boundary value problems using Green's functions, *Applied Mathematics and Computation* **183** (1) (2006), 10 – 16, DOI: 10.1016/j.amc.2006.01.096.

[11] M.M. Chawala, R. Subramanian and H.L. Sathi, A fourth order method for a singular two-point boundary value problem, *BIT Numerical Mathematics* **28** (1) (1988), 88 – 97, DOI: 10.1007/BF01934697.

[12] U. Flesch, The distribution of heat sources in the human head: a theoretical consideration, *Journal of Theoretical Biology* **54** (2) (1975), 285 – 287, DOI: 10.1016/S0022-5193(75)80131-7.

[13] D.J. Fyfe, The use of cubic splines in the solution of two-point boundary value problems, *The Computer Journal* **12** (2) (1969), 188 – 192, DOI: 10.1093/comjnl/12.2.188.

[14] J.B. Garner and R. Shivaji, Diffusion problems with a mixed nonlinear boundary condition, *Journal of Mathematical Analysis and Applications* **148** (2) (1990), 422 – 430, DOI: 10.1016/0022-247X(90)90010-D.

[15] J. Goh, A. Abd. Majid and A.I. Md. Ismail, A quartic B-spline for second-order singular boundary value problems, *Computers & Mathematics with Applications* **64** (2) (2012), 115 – 120, DOI: 10.1016/j.camwa.2012.01.022.

[16] J. Goh, A. Abd. Majid and A.I. Md. Ismail, Extended cubic uniform B-spline for a class of singular boundary value problems, *Nuclear Physics* **2** (2011), 4 pages.

[17] A.S.V.R. Kanth and K. Aruna, He's variational iteration method for treating nonlinear singular boundary value problems, *Computers & Mathematics with Applications* **60** (3) (2010), 821 – 829, DOI: 10.1016/j.camwa.2010.05.029.

[18] S.A. Khuri and A. Sayfy, A novel approach for the solution of a class of singular boundary value problems arising in physiology, *Mathematical and Computer Modelling* **52** (3) (2010), 626 – 636, DOI: 10.1016/j.mcm.2010.04.009.

[19] F.G. Lang and X.P. Xu, A new cubic B-spline method for approximating the solution of a class of nonlinear second-order boundary value problem with two dependent variables, *ScienceAsia* **40** (6) (2014), 444 – 450, DOI: 10.2306/scienceasia1513-1874.2014.40.444.

[20] S.H. Lin, Oxygen diffusion in a spherical cell with nonlinear oxygen uptake kinetics, *Journal of Theoretical Biology* **60** (2) (1976), 449 – 457, DOI: 10.1016/0022-5193(76)90071-0.

[21] D.L.S. McElwain, A re-examination of oxygen diffusion in a spherical cell with Michaelis-Menten oxygen uptake kinetics, *Journal of Theoretical Biology* **71** (2) (1978), 255 – 263, DOI: 10.1016/0022-5193(78)90270-9.

[22] M. Mohsenyzadeh, K. Maleknejad and R. Ezzati, A numerical approach for the solution of a class of singular boundary value problems arising in physiology, *Advances in Difference Equations* **2015** (1) (2015), 231, DOI: 10.1186/s13662-015-0572-x.

[23] R.K. Pandey and A.K. Singh, On the convergence of a finite difference method for a class of singular boundary value problems arising in physiology, *Journal of Computational and Applied Mathematics* **166** (2) (2004), 553 – 564, DOI: 10.1016/j.cam.2003.09.053.

[24] K. Parand, M. Dehghan, A.R. Rezaei and S.M. Ghaderi, An approximation algorithm for the solution of the nonlinear Lane–Emden type equations arising in astrophysics using Hermite functions collocation method, *Computer Physics Communications* **181** (6) (2010), 1096 – 1108, DOI: 10.1016/j.cpc.2010.02.018.

[25] J. Rashidinia, R. Mohammadi and R. Jalilian, The numerical solution of non-linear singular boundary value problems arising in physiology, *Applied Mathematics and Computation* **185** (1) (2007), 360 – 367, DOI: 10.1016/j.amc.2006.06.104.

[26] P. Rentrop, A Taylor series method for the numerical solution of two-point boundary value problems, *Numerische Mathematik* **31** (4) (1978), 359 – 375, DOI: 10.1007/BF01404566.

[27] R.D. Russell and L.F. Shampine, Numerical methods for singular boundary value problems, *SIAM Journal on Numerical Analysis* **12** (1) (1975), 13 – 36, DOI: 10.1137/0712002.

[28] R. Singh and J. Kumar, An efficient numerical technique for the solution of nonlinear singular boundary value problems, *Computer Physics Communications* **185** (4) (2014), 1282 – 1289, DOI: 10.1016/j.cpc.2014.01.002.

[29] R. Singh, Abd. M. Wazwaz and J. Kumar, An efficient semi-numerical technique for solving nonlinear singular boundary value problems arising in various physical models, *International Journal of Computer Mathematics* **93** (8) (2016), 1330 – 1346, DOI: 10.1080/00207160.2015.1045888.

[30] Y. Wang, H. Yu, F. Tan and S. Li, Using an effective numerical method for solving a class of Lane-Emden equations, *Abstract and Applied Analysis*, **2004** (2004), Article ID 735831, 8 pages, DOI: 10.1155/2014/735831.

[31] Abd. M. Wazwaz, The variational iteration method for solving nonlinear singular boundary value problems arising in various physical models, *Communications in Nonlinear Science and Numerical Simulation* **16** (10) (2011), 3881 – 3886, DOI: 10.1016/j.cnsns.2011.02.026.

[32] L.J. Xie, C.L. Zhou and S. Xu, An effective numerical method to solve a class of nonlinear singular boundary value problems using improved differential transform method, *SpringerPlus* **5** (1) (2016), 1066, DOI: 10.1186/s40064-016-2753-9.