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# Double Lacunary Statistical Convergence of Order $\alpha$ in Topological Groups via Ideal

Research Article

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**Abstract.** Recently, $\mathcal{I}$-lacunary double statistical convergence in topological groups is presented by Savas [31]. In this paper, we extend the concepts of $\mathcal{I}$-double statistical convergence and $\mathcal{I}$-double lacunary statistical convergence to the concepts of $\mathcal{I}$-double statistical convergence and $\mathcal{I}$-double lacunary statistical convergence of order $\alpha$, $0 < \alpha \leq 1$. We also investigate some inclusion relations between $\mathcal{I}$-double statistical of order $\alpha$ and $\mathcal{I}$-double lacunary double statistical convergence of order $\alpha$.

**Keywords.** Double lacunary; Ideal double lacunary statistical convergence; Topological groups

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## 1. Introduction

The notion of statistical convergence, which is an extension of the idea of usual convergence, was introduced by Fast [10] and also independently by Schoenberg [38] for real and complex sequences, but rapid developments were started after the papers of Šalát [21] and Fridy [12]. Nowadays, it has become one of the most active area of research in the field of summability. Maio and Kočinac [15] introduced the concept of statistical convergence in topological spaces and statistical Cauchy condition in uniform spaces and established the topological nature of this convergence. Quite recently, Savaş [30] studied $\mathcal{I}_\theta$-statistical convergence for sequences in topological groups where more references on this important summability method can be found. In many branches of science and engineering we often come across double sequences, i.e. sequences of matrices and certainly there are situations where either the idea of ordinary convergence does not work or the underlying space does not serve our purpose. Therefore to deal with such situations we have to introduce some new type of measures which can provide a better tool and suitable frame work. Cakalli and Savaş [5] studied the statistical convergence of double sequences to topological groups. Also lacunary statistical convergence of double sequences in topological groups was studied in [29]. Recently, Savas [36] introduced new notion, namely, $\mathcal{I}_\lambda$-double statistical convergence in topological groups and also some inclusion relations between $\mathcal{I}$-double statistical and $\mathcal{I}_\lambda$-double statistical convergence were investigated. Also, in [28] $\mathcal{I}_\lambda$-double statistical convergence of order $\alpha$ in topological groups was introduced by Savas. Mursaleen *et al*. [17] studied generalized statistical convergence and statistical core of double sequences.

Recall that a subset $E$ of the set $\mathbb{N}$ of natural numbers is said to have "natural density" $\delta(E)$ if

$$\delta(E) = \lim_n \frac{1}{n} |\{k \le n : k \in E\}|\,,$$

where the vertical bars denote the cardinality of the enclosed set. The number sequence $x = (x_k)$ is said to be *statistically convergent* to the number $\ell$ if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \le n : |x_k - \ell| \ge \varepsilon\}| = 0,$$

(see, Fridy [12]).

On the other hand in [1, 6] a different direction was given to the study of statistical convergence where the notion of statistical convergence of order $\alpha$, $0 < \alpha < 1$ was introduced by replacing $n$ by $n^\alpha$ in the denominator in the definition of statistical convergence. One can also see [7] for related works.

We present some definitions and preliminaries before continuing with this paper:

By a lacunary sequence, we mean an increasing sequence $\theta = (k_r)$ of positive integers such that $k_0 = 0$ and $h_r : k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper, the intervals determined by $\theta$ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio $(k_r)(k_{r-1})^{-1}$ will be abbreviated by $q_r$.

In [11], a new type of convergence called lacunary statistical convergence was defined as follows:

A sequence $(x_k)$ of real numbers is said to be lacunary statistically convergent to $L$ (or, $S_\theta$-convergent to $L$) if for any $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \ge \varepsilon\}| = 0\,,$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$. In [11] the relation between lacunary statistical convergence and statistical convergence was established among other things. The (relatively more general) concept of $\mathfrak{I}$-convergence was introduced by Kostyrko *et al.* [14] in a metric space. Later on, it was further studied by Dems [9] and Das *et al.* [8]. Several papers deal with this problem, see [13, 18, 23–27, 35].

## 2. Definitions and Notations

We recall some basic definitions which will be needed.

**Definition 2.1.** A family $\mathfrak{I} \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if the following conditions hold:

(a) $A, B \in \mathfrak{I}$ implies $A \cup B \in \mathfrak{I}$,

(b) $A \in \mathfrak{I}$, $B \subset A$ implies $B \in \mathfrak{I}$.

**Definition 2.2.** A non-empty family $F \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if the following conditions hold:

(a) $\phi \notin F$,

(b) $A, B \in F$ implies $A \cap B \in F$,

(c) $A \in F$, $A \subset B$ implies $B \in F$.

If $\mathfrak{I}$ is a proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathfrak{I}$), then the family of sets $F(\mathfrak{I}) = \{M \subset \mathbb{N} : \exists\, A \in \mathfrak{I} : M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

**Definition 2.3.** A proper ideal $\mathfrak{I}$ is said to be admissible if $\{n\} \in \mathfrak{I}$ for each $n \in \mathbb{N}$.

**Definition 2.4** (See [14])**.** Let $\mathfrak{I} \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$.
The sequence $(x_k)$ of elements of $\mathbb{R}$ is said to be $\mathfrak{I}$-convergent to $L \in \mathbb{R}$ if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in \mathfrak{I}$.

By $X$, we will denote an abelian topological Hausdorff group, written additively, which satisfies the first axiom of countability. For a subset $A$ of $X$, $s(A)$ will denote the set of all sequences $(x_n)$ such that $x_n$ is in $A$ for $n = 1, 2, \ldots, c(X)$ will denote the set of all convergent sequences. In [3], a sequence $(x_k)$ in $X$ is called to be statistically convergent to an element $L$ of $X$ if for each neighbourhood $U$ of $0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : x_k - L \notin U\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. The set of all statistically convergent sequences in $X$ is denoted by $st(X)$.

Also, Cakalli [4] defined lacunary statistical convergence in topological groups as follows: A sequence $(x_k)$ is said to be $S_\theta$-convergent to $L$ (or lacunary statistically convergent to $L$) if for each neighborhood $U$ of $0$, $\lim_{r\to\infty} (h_r)^{-1} |\{k \in I_r : x_k - L \notin U\}| = 0$. In this case, we write

$$S_\theta\text{-}\lim_{k\to\infty} x_k = L \quad \text{or} \quad x_k \to L(S_\theta).$$

## 3. Definitions and Notations

By the convergence of a double sequence we mean the convergence in Pringsheims sense ([20]). A double sequence $x = (x_{kl})$ of real numbers is said to be convergent in the Pringsheim's sense or $P$-convergent if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{kl} - L| < \varepsilon$, whenever $k, l \geq N$ and $L$ is called Pringsheim limit (denoted by $\text{P-}\lim x = L$).

Recently the studies of double sequences has a rapid growth. Mursaleen and Edely [16] extended the idea of statistical convergence of single sequences to double sequences of scalars and established relations between statistical convergence and strongly Cesáro summable double sequences. Also, the double lacunary statistical convergence was introduced by Patterson and Savas [19].

Mursaleen and Edely has presented main definition as follows:

**Definition 3.1** ([16]). A double sequences $x = (x_{k,l})$ is said to be P-statistically convergent to $L$ provided that for each $\varepsilon > 0$

$$\text{P-}\lim_{m,n} \frac{1}{mn}\{\text{number of } (k,l) : k < m \text{ and } l < n, |x_{kl} - L| \geq \varepsilon\} = 0.$$

In this case we write $st^2\text{-}\lim_{k,l} x_{k,l} = L$ and we denote the set of all statistical convergent double sequences by $st^2$.

It is clear that a convergent double sequence is also $st^2$-convergent but the inverse is not true, in general. Also, note that a $st^2$-convergence need not be bounded. For example, the sequence $x = (x_{k,l})$ defined by

$$x_{k,l} = \begin{cases} kl, & \text{if } k \text{ and } l \text{ are square} \\ 1, & \text{otherwise} \end{cases}$$

is $st^2$-convergent. Nevertheless, it neither convergent nor bounded.

The double sequence $\theta = \{(k_r, l_s)\}$ is called **double lacunary** if there exist two increasing of integers such that

$$k_0 = 0, h_r = k_r - k_{k-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

**Notations.** $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$, $\theta$ is determine by $I_r = \{(k) : k_{r-1} < k \leq k_r\}$, $I_s = \{(l) : l_{s-1} < l \leq l_s\}$, $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_s = \frac{l_s}{l_{s-1}}$ and $q_{r,s} = q_r \bar{q}_s$. We will denote the set of all double lacunary sequences by $\mathbf{N}_{\theta_{r,s}}$.

Let $K \subseteq \mathbf{N} \times \mathbf{N}$ has double lacunary density $\delta_2^{\theta}(K)$ if

$$\text{P-}\lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : (k,l) \in K\}| \text{ exists.}$$

In 2005, Patterson and Savas [19] studied double lacunary statistically convergence by giving the definition for complex sequences as follows:

**Definition 3.2.** Let $\theta$ be a double lacunary sequence; the double number sequence $x$ is $st_\theta^2$-*convergent* to $L$ provided that for every $\epsilon > 0$,

$$\text{P-}\lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : |x_{kl} - L| \geq \epsilon\}| = 0.$$

In this case write $st_\theta^2$-$\lim x = L$ or $x_{kl} \to L(S_\theta^2)$.

More investigation in this direction and more applications of double lacunary and double sequences can be found in ([2], [22], [32], [33], [34], [37]).

We have the following definition:

A double sequence $x = (x_{kl})$ of points in $X$ is said to be convergent to a point $L$ in $X$ in the Pringsheims sense if for every neighbourhood $U$ of 0 there exists $N \in \mathbb{N}$ such that $x_{kl} - L \in U$ whenever $k,l \geq N$. $L$ is called the Pringsheim limit of $x$.

Throughout, $\mathcal{I}_2$ will stand for a proper strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A double sequence $x = (x_{kl})$ of real number is said to be convergent to the number $L$ with respect to the ideal $\mathcal{I}$, if for each $\varepsilon > 0$

$$A(\varepsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N} : |x_{kl} - L| \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case we write $\mathcal{I}_2$-$\lim\limits_{k,l} x_{kl} = L$.

Quite recently, Cakalli and Savas[5] defined the statistical convergence of double sequences $x = (x_{k,l})$ of points in a topological group as follows.

In a topological group $X$, double sequence $x = (x_{kl})$ is called statistically convergent to a point $L$ of $X$ if for each neighbourhoud $U$ of 0 the set

$$\{(k,l), k \leq n \text{ and } l \leq m : x_{kl} - L \notin U\}$$

has double natural density zero. In this case we write $S^2$-$\lim\limits_{k,l} x_{kl} = L$ and we denote the set of all statistically convergent double sequences by $S^2(X)$.

The definition of double lacunary statistical convergence in topological groups was defined by Savas [29] as follows:

**Definition 3.3.** A sequence $x = (x_{kl})$ is said to be $S_\theta^2$-convergent to $L$ (or double lacunary statistically convergent to $L$) if for each neighborhood $U$ of 0,

$$\text{P-}\lim_{r,s\to\infty} (h_{rs})^{-1} |\{(k,l) \in I_{rs} : x_{kl} - L \notin U\}| = 0.$$

In this case, we write

$$S_\theta^2 - \lim_{k,l\to\infty} x_{kl} = L \quad \text{or} \quad x_{kl} \to L(S_\theta^2).$$

Let $K \subseteq \mathbb{N} \times \mathbb{N}$. If $(m,n) \in \mathbb{N} \times \mathbb{N}$, by $K_{m,n}$ here we denote the cardinality of the set

$$\{(k,l) \in K : 1 \leq k \leq m \text{ and } 1 \leq l \leq n\}.$$

Let $0 < \alpha \leq 1$ be a real number. Define $(k,l)$ in $K$ such that $k \leq n$ and $l \leq m$. Write

$$\underline{\delta_2^\alpha}(K) = \text{P-}\liminf_{m,n} \frac{K_{m,n}}{(mn)^\alpha} \quad \text{and} \quad \delta_2^\alpha(K) = \text{P-}\limsup_{m,n} \frac{K_{m,n}}{(mn)^\alpha}.$$

These are called the lower and upper double density of order $\alpha$ of the set $K$, respectively. If the limit P-$\lim\limits_{m,n} \frac{K_{m,n}}{(m,n)^{\alpha}}$ exists in Pringsheim's sense then we say that the double density of order $\alpha$ of the set $K$ exists and we denote it by $\delta_2^{\alpha}(K)$.

Now we are ready to present the main definitions of $\mathcal{I}$-double statistical convergence of order $\alpha$ and $\mathcal{I}$-double lacunary statistical convergence of order $\alpha$ in topological groups as follows:

**Definition 3.4.** A double sequence $x = (x_{kl})$ of points in a topological group $X$, is said to be $\mathcal{I}$-double statistically convergent of order $\alpha$ to $L$ or $S(\mathcal{I}_2)^{\alpha}$-convergent to $L$ if for each $\delta > 0$ and for each neighbourhood $U$ of 0,

$$\left\{(m,n) \in N \times N : \frac{1}{(mn)^{\alpha}} |\{k \le m \ \text{ and } \ l \le m : x_{kl} - L \notin U\}| \ge \delta\right\} \in \mathcal{I}_2.$$

In this case, we write $x_{kl} \to L(S(\mathcal{I}_2)^{\alpha})$. The set of all $S(\mathcal{I})^{\alpha}$-double statistically convergent sequences will be denoted by simply $S(\mathcal{I}_2)^{\alpha}(X)$.

**Remark 3.5.** For $\mathcal{I}_2 = \mathcal{I}_{2fin} = \{A \subset \mathbb{N} \times \mathbb{N}, A$ is a finite$\}$, $S(\mathcal{I}_2)^{\alpha}$-convergence coincides with double statistical convergence of order $\alpha$ in a topological group $X$ which was studied by Savaş [28]. For $\mathcal{I} = \mathcal{I}_{fin} = \{A \subseteq \mathbf{N} : A$ is a finite$\}$, and $\alpha = 1$, $\mathcal{I}$-double statistical convergence becomes statistical convergence in topological groups which is studied by Cakalli and Savas [5].

**Definition 3.6.** A sequence $x = (x_{kl})$ of points in a topological group $X$, is said to be $\mathcal{I}$-lacunary double statistically convergent of order $\alpha$ to $L$ or $S_{\theta}(\mathcal{I}_2)^{\alpha}$-lacunary convergent to $L$ if for each neighbourhood $U$ and any $\delta > 0$

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}^{\alpha}} |\{(k,l) \in I_{rs} : x_{kl} - L \notin U\}| \ge \delta\right\} \in \mathcal{I}_2.$$

In this case, we write

$$S_{\theta}(\mathcal{I}_2)^{\alpha}\text{-}\lim_{k,l \to \infty} x_{kl} = L \ \text{ or } \ x_{kl} \to L(S^{\theta}(\mathcal{I}_2)^{\alpha})$$

and define

$$S_{\theta}(\mathcal{I}_2)^{\alpha}(X) = \left\{x = (x_{kl}) : \text{ for some } L, S_{\theta}(\mathcal{I}_2)^{\alpha}\text{-}\lim_{k,l \to \infty} x_{kl} = L\right\}.$$

**Remark 3.7.** For $\mathcal{I}_2 = \mathcal{I}_{2fin} = \{A \subset \mathbb{N} \times \mathbb{N},\ A$ is finite$\}$, $\mathcal{I}$-lacunary double statistical convergence of order $\alpha$ becomes lacunary double statistical convergence of order $\alpha$ in topological groups. For $\mathcal{I} = \mathcal{I}_{fin} = \{A \subseteq \mathbf{N} : A$ is a finite$\}$, and $\alpha = 1$, $\mathcal{I}$-lacunary double statistical convergence of order $\alpha$ becomes lacunary double convergence in topological groups which is studied by Savas [29].

It is obvious that every $\mathcal{I}$-lacunary double statistically convergent sequence has only one limit, that is, if a double sequence is $\mathcal{I}_2^{\theta}$-statistically convergent to $L_1$ and $L_2$ then $L_1 = L_2$.

We now present the following theorems.

## 4. Inclusion Theorems

In this section, we prove the following theorems.

**Theorem 4.1.** *Let* $0 < \alpha \le \beta \le 1$. *Then* $S_\theta(\mathcal{I}_2)^\alpha(X) \subset S_\theta(\mathcal{I}_2)^\beta(X)$.

*Proof.* Let $0 < \alpha \le \beta \le 1$. Then

$$\frac{|\{(k,l) \in I_{rs} : x_{kl} - L \notin U\}|}{h_{rs}^\beta} \le \frac{|\{(k,l) \in I_{rs} : x_{kl} - L \notin U\}|}{h_{rs}^\alpha}$$

and so for any $\delta > 0$ and for each neighbourhood $U$ of $0$,

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{|\{(k,l) \in I_{rs} : x_{kl} - L \notin U\}|}{h_{rs}^\beta} \ge \delta\right\}$$
$$\subset \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{|\{(k,l) \in I_{rs} : x_{kl} - L \notin U\}|}{h_{rs}^\alpha} \ge \delta\right\}.$$

Hence if the set on the right hand side belongs to the ideal $\mathcal{I}$ then obviously the set on the left hand side also belongs to $\mathcal{I}$. This shows that $S_\theta(\mathcal{I}_2)^\alpha(X) \subset S_\theta(\mathcal{I}_2)^\beta(X)$. □

**Corollary 4.2.** *If a sequence is* $\mathcal{I}$*-lacunary double statistically convergent of order* $\alpha$ *to* $L$ *for some* $0 < \alpha \le 1$ *then it is* $\mathcal{I}$*-lacunary double statistically convergent to* $L$ *i.e.* $S_\theta(\mathcal{I}_2)^\alpha(X) \subset S_\theta(\mathcal{I}_2)(X)$.

Similarly, we can show that:

**Theorem 4.3.** *Let* $0 < \alpha \le \beta \le 1$. *Then*

(i) $S(\mathcal{I}_2)^\alpha(X) \subset S(\mathcal{I}_2)^\beta(X)$.

(ii) *In particular* $S(\mathcal{I}_2)^\alpha(X) \subset S(\mathcal{I}_2)(X)$.

We now prove some analogues for double sequences. For single sequences such results have been proved by Savas [30]. We now have

**Theorem 4.4.** *For any double lacunary sequence* $\theta = \{(k_r, l_s)\}$, $\mathcal{I}_2$*-statistical convergence of order* $\alpha$ *implies* $\mathcal{I}_2$*-lacunary statistical convergence of order* $\alpha$, *that is* $S(\mathcal{I}_2)^\alpha(X) \subset S_\theta(\mathcal{I}_2)^\alpha(X)$ *if* $\liminf q_r^\alpha > 1$ *and* $\liminf \bar{q}_s^\alpha > 1$.

*Proof.* Suppose that $\liminf q_r^\alpha > 1$, and $\liminf \bar{q}_s^\alpha > 1$, $\liminf q_r^\alpha = \gamma_1$ and $\liminf \bar{q}_s^\alpha = \gamma_2$ (say). Write $\beta_1 = (\gamma_1 - 1)/2$ and $\beta_2 = (\gamma_2 - 1)/2$. Then, there exist two positive integers $r_0$ and $s_0$ such that $q_r^\alpha \ge 1 + \beta_1$ for $r \ge r_0$ and $\bar{q}_s^\alpha \ge 1 + \beta_2$ for $s \ge s_0$. Thus for $r \ge r_0$, and $s \ge s_0$,

$$h_{rs}^\alpha (k_r^\alpha)^{-1} (l_s^\alpha)^{-1} \ge (1 - (1+\beta_1)^{-1}(1 - (1+\beta_2)^{-1}) = (\beta_1(1+\beta_1)^{-1})(\beta_2(1+\beta_2)^{-1}).$$

Take any $x = (x_{k,l}) \in S(\mathcal{I}_2)^\alpha(X)$, and $S(\mathcal{I}_2)^\alpha\text{-}\lim_{k,l\to\infty} x_{k,l} = L$ (say).

We have that $S_\theta(\mathcal{I}_2)^\alpha\text{-}\lim_{(k,l)\to\infty} x_{k,l} = L$. Let us take any neighborhood $U$ of $0$. Then for $r \ge r_0$, and $s \ge s_0$, we get

$$\begin{aligned}
&(k_r^\alpha)^{-1}(l_s^\alpha)^{-1} |\{k \le k_r \text{ and } l \le l_s : x_{kl} - L \notin U\}| \\
&\ge (k_r^\alpha)^{-1}(l_s^\alpha)^{-1} |\{(k,l) \in I_{rs} : x_{kl} - L \notin U\}| \\
&= h_{r,s}^\alpha (k_r^\alpha)^{-1}(l_s^\alpha)^{-1}(h_{rs}^\alpha)^{-1} |\{(k,l) \in I_{rs} : x_{kl} - L \notin U\}| \\
&\ge \beta_1(1+\beta_1)^{-1}\beta_2(1+\beta_2)^{-1}(h_{rs}^\alpha)^{-1} |\{(k,l) \in I_{rs} : x_{kl} - L \notin U\}|.
\end{aligned}$$

Then for any $\delta > 0$, we get

$$\begin{aligned}
&\{(r,s) \in \mathbb{N}\times\mathbb{N} : (h_{rs}^{\alpha})^{-1}|\{(k,l) \in I_{rs} : x_{kl} - L \notin U\}| \geq \delta\} \\
&\quad \subseteq \{(r,s) \in \mathbb{N}\times\mathbb{N} : (k_r^{\alpha})^{-1}(l_s^{\alpha})^{-1}|\{k \leq k_r, l \leq l_s : x_{kl} - L \notin U\}| \\
&\quad \geq \delta\beta_1(1+\beta_1)^{-1}\beta_2(1+\beta_2)^{-1}\} \in \mathcal{I}_2.
\end{aligned}$$

□

For the next result we assume that the double lacunary sequence $\theta$ satisfies the condition that for any set $C \in F(\mathcal{I}_2)$, $\cup\{(n,m) : k_{r-1} < n < k_r$ and $l_{s-1} < m < l_s, (r,s) \in C\} \in F(\mathcal{I}_2)$.

**Theorem 4.5.** *For any lacunary sequence $\theta = \{(k_r, l_s)\}$, satisfying the above condition $S_\theta(\mathcal{I}_2)^\alpha(X) \subseteq S(\mathcal{I}_2)^\alpha(X)$ if $\sup_{rs} \sum_{i,j=0}^{r-1,s-1} \frac{h_{i+1,j+1}^{\alpha}}{(k_{r-1}l_{s-1})^{\alpha}} = B$ (say) $< \infty$.*

*Proof.* Suppose that $S_\theta(\mathcal{I}_2)^\alpha\text{-}\lim_{k,l\to\infty} x_{k,l} = L$ (say). For any neighborhood $U$ of 0 and for $\delta, \delta_1 > 0$ define the sets

$$C = \left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{rs}^{\alpha}}|\{(k,l) \in I_{rs} : x_{kl} - L \notin U\}| < \delta\right\}$$

and

$$T = \left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{(n,m)^{\alpha}}|\{k \leq n \text{ and } l \leq m : x_{kl} - L \notin U\}| < \delta_1\right\}.$$

It is obvious from our assumption that $C \in F(\mathcal{I}_2)$, the filter associated with the ideal $\mathcal{I}_2$. Further observe that

$$A_{i,j} = \frac{1}{h_{ij}^{\alpha}}|\{(i,j) \in I_{ij} : x_{kl} - L \notin U\}| < \delta$$

for all $(i,j) \in C$. Let $n$ and $m$ be such that $k_{r-1} < m \leq k_r$ and $l_{s-1} < n \leq l_s$, hence we have the following

$$\begin{aligned}
&\frac{1}{(mn)^{\alpha}}|\{k \leq m \text{ and } l \leq n : x_{k,l} - L \notin U\}| \\
&\quad \leq \sup_{(i,j)\in C} A_{i,j} \cdot \sup_{r,s} \sum_{i,j=0}^{r-1,s-1} \frac{(k_{i+1}-k_i)^{\alpha}(l_{j+1}-l_j)^{\alpha}}{k_{r-1}^{\alpha} l_{s-1}^{\alpha}} < B\delta.
\end{aligned}$$

Choosing $\delta_1 = \frac{\delta}{B}$ and in view of the fact that $\cup\{(m,n) : k_{r-1} < n < k_r, l_{s-1} < m < l_s, (r,s) \in C\} \subset T$, where $C \in F(\mathcal{I}_2)$ it follows from our assumption on $\theta$ that the set $T$ also belongs to $F(\mathcal{I}_2)$ and this completes the proof of the theorem. □

**Theorem 4.6.** *If*

$$\liminf_{r,s\to\infty} \frac{h_{rs}^{\alpha}}{k_r l_s} > 0$$

*then $S(\mathcal{I}_2)(X) \subseteq S^{\theta}(\mathcal{I}_2)^{\alpha}(X)$.*

*Proof.* For neighbourhood $U$ of 0, we have

$$\{k \leq k_r \text{and } l \leq l_s : x_{kl} - L \notin U\} \supset \{(k,l) \in I_{rs} : x_{kl} - L \notin U\}.$$

Therefore,

$$\frac{1}{k_r l_s}\{k \le k_r \text{ and } l \le l_s : x_{kl} - L \notin U\} \ge \frac{1}{k_r l_s}\{(k,l)\in I_{rs} : x_{kl} - L \notin U\}$$
$$= \frac{h^{\alpha}_{rs}}{k_r l_s}\cdot\frac{1}{h^{\alpha}_{rs}}\{(k,l)\in I_{rs} : x_{kl} - L \notin U\}.$$

If $\liminf\limits_{r,s\to\infty}\frac{h^{\alpha}_{rs}}{k_r l_s} = \gamma$ then from definition $\left\{(r,s)\in N : \frac{h^{\alpha}_{rs}}{k_r l_s} < \frac{\gamma}{2}\right\}$is finite. For $\delta > 0$ and for each neighbourhood $U$ of 0,

$$\{(r,s)\in\mathbb{N}\times\mathbb{N} : (h^{\alpha}_{rs})^{-1}|\{(k,l)\in I_{rs} : x_{kl} - L \notin U\}| \ge \delta\}$$
$$\subset \left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{k_r l_s}|\{k \le k_r, l \le l_s : x_{kl} - L \notin U\}| \ge \frac{\gamma}{2}\delta\right\} \cup \left\{(r,s)\mathbb{N}\times\mathbb{N} : \frac{h^{\alpha}_{rs}}{k_r l_s} < \frac{\gamma}{2}\right\}.$$

The set on the right hand side belongs to $\mathcal{I}_2$ and this completed the proof. □

Finally, we prove the following theorem.

**Theorem 4.7.** *Let $\theta = \{(k_r, l_s)\}$ and $\theta^1 = \{(k_r, l_s)\}$ be two double lacunary sequences such that $I_{rs} \subset J_{rs}$ for all $(r,s)\in\mathbb{N}\times\mathbb{N}$, if*

$$\lim_{r,s\to\infty}\inf\frac{h^{\alpha}_{rs}}{\ell^{\alpha}_{rs}} > 0 \tag{4.1}$$

*then $S_{\theta^1}(\mathcal{I}_2)(X) \subseteq S_{\theta}(\mathcal{I}_2)(X)$.*

*Proof.* Suppose that $I_{rs} \subset J_{rs}$ for all $(r,s)\in\mathbb{N}\times\mathbb{N}$ and let (4.1) be satisfied. For neighbourhood $U$ of 0, we have

$$\{(k,l)\in J_{rs} : x_{kl} - L \notin U\} \supseteq \{(k,l)\in I_{rs} : x_{kl} - L \notin U\}.$$

Therefore, we can write

$$(\ell^{\alpha}_{rs})^{-1}|\{(k,l)\in J_{rs} : x_{kl} - L \notin U\}| \ge h^{\alpha}_{rs}(\ell^{\alpha}_{rs})^{-1}(h^{\alpha}_{rs})^{-1}|\{(k,l)\in I_{rs} : x_{kl} - L \notin U\}|$$

and so for all $(r,s)\in\mathbb{N}\times\mathbb{N}$, we have

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N} : (h^{\alpha}_{rs})^{-1}|\{(k,l)\in I_{rs} : x_{kl} - L \notin U\}| \ge \delta\right\}$$
$$\subseteq \left\{(r,s)\in\mathbb{N}\times\mathbb{N} : (\ell^{\alpha}_{rs})^{-1}|\{(k,l)\in J_{rs} : x_{kl} - L \notin U\}| \ge \delta h^{\alpha}_{rs}(l^{\alpha}_{rs})^{-1}\right\} \in \mathcal{I}_2$$

for all $(r,s)\in\mathbb{N}\times\mathbb{N}$ where $I_{r,s} = \{(k,l) : k_{r-1} < k \le k_r \text{ and } l_{s-1} < l \le l_s\}$, and $J_{r,s} = \{(p,q) : p_{r-1} < p \le p_r$ and $q_{s-1} < q \le q_s\}$. Hence $S^1_{\theta}(\mathcal{I}_2)^{\alpha}(X) \subseteq S_{\theta}(\mathcal{I}_2)^{\alpha}(X)$. □

## 5. Conclusion

Double lacunary sequence was studied by Savas and Patterson. It is natural question that whether this concept will be work for lacunary double statistical convergence of weighted $g$. In this paper, we gave some answers of this question and also we prove that $\mathcal{I}$-double statistical convergence a better tool than double statistical convergence.

### Competing Interests

The author declares that he has no competing interests.

## Authors' Contributions

The author wrote, read and approved the final manuscript.

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