Communications in Mathematics and Applications
Vol. 9, No. 1, pp. 41–52, 2018
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

Special Issue:
Recent Advances in Fixed Point Theory for Set Valued Operators with Related Applications
Guest Editors: Muhammad Usman Ali, Mihai Postolache, Ishak Altun, Tayyab Kamran

Some Higher Order Algorithms for Solving Fixed Point Problems

Research Article

Asif Waheed[1], Syed Tauseef Mohyud Din[2], Muhammad Zeb[1] and Muhammad Usman[3]

[1] *Department of Mathematics, COMSATS Institute of Information Technology, Attock, Pakistan*
[2] *Center for Research, University of Islamabad, Islamabad, Pakistan*
[3] *School of Mathematical Sciences, Peking University, Beijing, China*
***Corresponding author:** drasif.waheed@ciit-attock.edu.pk

Abstract. In this paper, some higher order algorithms have been introduced for solving fixed point problems. These algorithms have been developed by Homotopy Perturbation Method. New algorithms are tested on diversified nonlinear problems. The results are very promising and useful. Comparison of numerical results along with existing proficient techniques explicitly reflects the very high level of accuracy of developed iterative schemes.

Keywords. Higher order; Algorithms; Fixed point problems; Homotopy perturbation method; Nonlinear equations; Efficiency index; Convergences order

MSC. 65F15; 65J15; 65H05

Received: October 6, 2017 **Accepted:** November 17, 2017

1. Introduction

This paper is devoted to the development of some higher order iterative techniques for solving nonlinear algebraic equations which are of utmost importance in mathematical physics and engineering sciences. Such equations arise frequently in number of scientific models including fluid mechanics, astrophysics, geo physics, solid state physics, plasma physics, chemical kinematics and optical fiber [1–19] and references therein. Finding the solution of following nonlinear algebraic equations:

$$f(x)=0,$$

is equivalent to fixed point problems $x=g(x)$, where $g(x)$ is a continuous function under certain conditions. We can define functions g with a fixed point at p in a number of ways, for example as $g(x)=x-f(x)$ or $g(x)=x+3f(x)$. Conversely, if the function g has fixed point at p, then the function defined by $g(x)=x-f(x)$ has a zero at p.

The through study of literature reveals that many researchers have investigated to find the root of nonlinear algebraic equations.

The most popular method to find the root of nonlinear algebraic equations is Newton's method [13]. Chun [3] improved Newton method by decomposition method. Abbasbandy [1] has also improved Newton Raphson method by using the Adomian decomposition method. Halley [6] developed a tremendous higher order iterative technique in the history for solving nonlinear equations more accurately and in fastest way which open the door for others to think in this direction. Ezquerro *et al.* [4, 5] developed Halley type iterative scheme free from second derivative. Noor and Noor [14–16] have suggested and analyzed many efficient techniques of third, fifth and sixth-order predictor-corrector Halley method for solving the nonlinear equations. Also, Kou *et al.* [12, 13] have suggested a class of fifth-order iterative methods. Moreover, several iterative type methods have been developed by using the Taylor series, decomposition and quadrature formulae [2, 18, 19] and the references therein.

Homotopy perturbation method was introduced by He [7] in 1999. Since then this technique has been successfully used by many researchers for solving initial and boundary value problems of diversified nature. New interpretation and new development of the homotopy perturbation methods have been given and well addressed by He [8–11]. Afterwards, Sehati *et al.* [17] has developed some new iterative schemes of higher order by using HPM for finding real and complex roots of nonlinear equations.

Inspired and motivated by the ongoing research in this area, we have introduced some higher order iterative schemes (Algorithm 1, Algorithm 2, Algorithm 3 and Algorithm 4) to find the solution of some fixed point problems. We have compared our results with well known Newton's method [13], Noor method [14–16], Chun method [3] and Sehati *et al.* [17] 5^{th}, 7^{th}, 10^{th} and 14^{th} order techniques. Some of the techniques used for the comparison purpose are given as follows:

Newton's Method (NM) ([13])**.**

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}.$$

Noor's Method ([14–16]).

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = -\frac{(y_n - x_n)^2}{2}\frac{f''(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{(y_n + z_n - x_n)^2}{2}\frac{f''(x_n)}{f'(x_n)}.$$

Chun's method ([3]).

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - 2\frac{f(y_n)}{f'(x_n)} + \frac{f(y_n)f'(y_n)}{f'^2(x_n)}.$$

It is worth mentioning that applied scheme is fully compatible with the complexity of such problems and obtained results are highly accurate. Moreover, it is also observed by considering *Computational Order of Convergence* (COC) and number or iterations that the proposed algorithms are very reliable and may be implemented on other physical problems also.

Definition 1. We choose an initial approximation x_0 and generate the sequence $\{x_k\}_{k=0}^{\infty}$, by letting $x_n = g(x_{n-1})$, for $n = 1$. If the sequence converges to x and g is continuous then

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} g(x_{n-1}) = g(\lim_{n\to\infty} x_{n-1}) = g(x),$$

and solution to $x = g(x)$ is obtained. This technique is called fixed point iteration.

Definition 2. A sequence of iterates $\{x_k\}_{k=0}^{\infty}$, is said to converge to the root α, if $\lim_{k\to\infty} |x_k - \alpha| = 0$ or $\lim_{k\to\infty} x_k = \alpha$.

Definition 3. Assume that sequence of iterates $\{x_k\}_{k=0}^{\infty}$, is converges to α and $e_k = x_k - \alpha$ for $k = 0$. If two positive constants $M \neq 0$ and $q > 0$ exist, and

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^q} = M.$$

Then the sequence is said to converge to α with order of convergence q. The number M is called asymptotic error constant.

2. Methodology

Consider the nonlinear algebraic equation

$$f(x) = 0, \tag{2.1}$$

which is equivalent to fixed point problem, i.e., $f(x) = 0 \Longleftrightarrow x = g(x)$, under certain conditions.

We assume that α is a simple root of (2.1) and λ an initial guess sufficiently close to it. Equation (2.1) can be rewrite as

$$f(\lambda) + (x-\lambda)f'(\lambda) + \frac{1}{2!}(x-\lambda)^2 f''(\lambda) \approx 0. \tag{2.2}$$

According to Homotopy perturbation method [1], construct the homotopy $H : R \times [0,1] \to R$ which satisfied

$$H(x,p) = p[f(x)] + (1-p)\left[f(\lambda) + (x-\lambda)f'(\lambda) + \frac{1}{2!}(x-\lambda)^2 f''(\lambda)\right], \tag{2.3}$$

where p is an embedding parameter. Hence, it is obvious that

$$H(x,0) = f(\lambda) + (x-\lambda)f'(\lambda) + \frac{1}{2!}(x-\lambda)^2 f''(\lambda) \approx 0 \tag{2.4}$$

and

$$H(x,1) = f(x) = 0, \tag{2.5}$$

and the changing process of p form 0 to 1, refers to $H(x,p)$ form $H(x,0)$ to $H(x,1)$. According to the Homotopy perturbation method consider the solution in series form as

$$x = x_0 + px_1 + p^2x_2 + p^3x_3 + \dots, \tag{2.6}$$

when $p \to 1$, (2.3) corresponding to (2.1) and (2.6) becomes the approximate solution of (2.1), that is

$$\tilde{x} = \lim_{p\to 1} x = x_0 + x_1 + x_2 + x_3 + \dots. \tag{2.7}$$

To apply Homotopy perturbation method to (2.1). The (2.3) can be rewrite as

$$\begin{aligned} H(x,p) = p&\left[f(x_0) + (x-x_0)f'(x_0) + \frac{1}{2!}(x-x_0)^2 f''(x_0) + \dots\right] \\ &+ (1-p)\left[f(\lambda) + (x-\lambda)f'(\lambda) + \frac{1}{2!}(x-\lambda)^2 f''(\lambda)\right]. \end{aligned} \tag{2.8}$$

Substituting eq. (2.6) into eq. (2.8), we get

$$\begin{aligned} H(x,p) = p&\left[f(x_0) + \left(px_1 + p^2x_2 + p^3x_3 + \dots\right)f'(x_0) + \frac{1}{2!}\left(px_1 + p^2x_2 + p^3x_3 + \dots\right)^2 f''(x_0) + \dots\right] \\ &+ (1-p)\Big[f(\lambda) + (x_0 + px_1 + p^2x_2 + p^3x_3 + \dots - \lambda)f'(\lambda) \\ &\qquad + \frac{1}{2!}(x_0 + px_1 + p^2x_2 + p^3x_3 + \dots - \lambda)^2 f''(\lambda)\Big] = 0. \end{aligned} \tag{2.9}$$

Comparing the coefficient of like powers of p, we have following system of equations

$$p^0 : f(\lambda) + (x_0-\lambda)f'(\lambda) + \frac{1}{2!}(x_0-\lambda)^2 f''(\lambda) = 0,$$

$$p^1 : f(x_0) + x_1 f'(\lambda) + x_1(x_0-\lambda)f''(\lambda) - f(\lambda) - (x_0-\lambda)f'(\lambda) - \frac{1}{2!}(x_0-\lambda)^2 f''(\lambda) = 0,$$

$$p^2 : x_1 f'(x_0) + x_2 f'(\lambda) + \frac{1}{2}x_1^2 f''(\lambda) + x_2(x_0-\lambda)f'''(\lambda) - x_1 f'(\lambda) - x_1(x_0-\lambda)f'''(\lambda) = 0,$$

$$\begin{aligned} p^3 : &f''(\lambda)x_0x_3 + f''(\lambda)x_1x_2 - f''(\lambda)x_3\lambda - f''(\lambda)x_0x_2 + f''(\lambda)x_2\lambda + f'(\lambda)x_3 + \frac{1}{2}f''(x_0)x_1^2 \\ &+ f'(x_0)x_2 - f'(\lambda)x_2 - \frac{1}{2}f''(\lambda)x_1^2 = 0. \end{aligned}$$

After solving the system we get

$$x_0 = \lambda + \frac{-f'(\lambda) + \sigma\sqrt{f'^2(\lambda) - 2f(\lambda)f''(\lambda)}}{f''(\lambda)},$$

where $\sigma = \pm 1$.

$$x_1 = \frac{-f(x_0)}{\sigma\sqrt{f'^2(\lambda) - 2f(\lambda)f''(\lambda)}},$$

$$x_2 = \frac{x_1\left\{f'(\lambda) - f'(x_0) + \left(x_0 - \lambda - \frac{1}{2}x_1\right)f''(\lambda)\right\}}{\sigma\sqrt{f'^2(\lambda) - 2f(\lambda)f''(\lambda)}},$$

$$x_3 = \frac{1}{2}\frac{\begin{pmatrix} -2f''(\lambda)x_1x_2 + 2f''(\lambda)x_0x_2 - 2f''(\lambda)\lambda x_2 - f''(x_0)x_1^2 - 2f'(x_0)x_2 \\ +2f'(\lambda)x_2 + f''(\lambda)x_1^2 \end{pmatrix}}{\sigma\sqrt{f'^2(\lambda) - 2f(\lambda)f''(\lambda)}}.$$

Substituting the values of x_0, x_1, x_2 and x_3 in (2.7), we get

$$\begin{aligned} \tilde{x} = \lambda &+ \frac{-f'(\lambda) + \sigma\sqrt{f'^2(\lambda) - 2f(\lambda)f''(\lambda)}}{f''(\lambda)} + \frac{-f(x_0)}{\sigma\sqrt{f'^2(\lambda) - 2f(\lambda)f''(\lambda)}} \\ &+ \frac{x_1\left\{f'(\lambda) - f'(x_0) + \left(x_0 - \lambda - \frac{1}{2}x_1\right)f''(\lambda)\right\}}{\sigma\sqrt{f'^2(\lambda) - 2f(\lambda)f''(\lambda)}} \\ &+ \frac{1}{2}\frac{\begin{pmatrix} -2f''(\lambda)x_1x_2 + 2f''(\lambda)x_0x_2 - 2f''(\lambda)\lambda x_2 - f''(x_0)x_1^2 - 2f'(x_0)x_2 \\ +2f'(\lambda)x_2 + f''(\lambda)x_1^2 \end{pmatrix}}{\sigma\sqrt{f'^2(\lambda) - 2f(\lambda)f''(\lambda)}} + \dots . \end{aligned} \tag{2.10}$$

Now by substituting $\sigma = 1$, these formulation allow us to suggest the following iterative methods for solving the nonlinear algebraic equation (2.1).

Algorithm 1. For a given x_0, calculate the approximate solution x_{n+1} by the iterative scheme:

$$y_n = x_n + \frac{-f'(x_n) + \sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}}{f''(x_n)},$$

$$z_n = \frac{-f(y_n)}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}},$$

$$w_n = \frac{z_n\left\{f'(x_n) - f'(y_n) + (y_n - x_n - \frac{1}{2}z_n)f''(x_n)\right\}}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}},$$

$$x_{n+1} = y_n + z_n + w_n + \frac{1}{2}\frac{\begin{pmatrix} -2f''(x_n)z_nw_n + 2f''(x_n)y_nw_n - 2f''(x_n)x_nw_n - f''(y_n)z_n^2 - 2f'(y_n)w_n \\ +2f'(x_n)w_n + f''(x_n)z_n^2 \end{pmatrix}}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}}.$$

Algorithm 2. For a given x_0, calculate the approximate solution x_{n+1} by the iterative scheme:

$$y_n = x_n + \frac{-f'(x_n) + \sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}}{f''(x_n)},$$

$$z_n = \frac{-f(y_n)}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}},$$

$$w_n = \frac{z_n\left\{f'(x_n) - f'(y_n) + (y_n - x_n - \frac{1}{2}z_n)f''(x_n)\right\}}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}},$$

$$h_n = y_n + z_n + w_n + \frac{1}{2}\frac{\begin{pmatrix} -2f''(x_n)z_n w_n + 2f''(x_n)y_n w_n - 2f''(x_n)x_n w_n - f''(y_n)z_n^2 - 2f'(y_n)w_n \\ +2f'(x_n)w_n + f''(x_n)z_n^2 \end{pmatrix}}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}},$$

$$x_{n+1} = h_n - \frac{f(h_n)}{f'(h_n)}.$$

Algorithm 3. For a given x_0, calculate the approximate solution x_{n+1} by the iterative scheme:

$$y_n = x_n + \frac{-f'(x_n) + \sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}}{f''(x_n)},$$

$$z_n = \frac{-f(y_n)}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}},$$

$$w_n = \frac{z_n\left\{f'(x_n) - f'(y_n) + (y_n - x_n - \frac{1}{2}z_n)f''(x_n)\right\}}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}},$$

$$h_n = y_n + z_n + w_n + \frac{1}{2}\frac{\begin{pmatrix} -2f''(x_n)z_n w_n + 2f''(x_n)y_n w_n - 2f''(x_n)x_n w_n - f''(y_n)z_n^2 - 2f'(y_n)w_n \\ +2f'(x_n)w_n + f''(x_n)z_n^2 \end{pmatrix}}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}},$$

$$x_{n+1} = h_n - \frac{2f(h_n)f'(h_n)}{2f'^2(h_n) - f(h_n)f''(h_n)}.$$

Algorithm 4. For a given x_0, calculate the approximate solution x_{n+1} by the iterative scheme:

$$y_n = x_n + \frac{-f'(x_n) + \sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}}{f''(x_n)},$$

$$z_n = \frac{-f(y_n)}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}},$$

$$w_n = \frac{z_n\left\{f'(x_n) - f'(y_n) + (y_n - x_n - \frac{1}{2}z_n)f''(x_n)\right\}}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}},$$

$$h_n = y_n + z_n + w_n + \frac{1}{2}\frac{\begin{pmatrix} -2f''(x_n)z_n w_n + 2f''(x_n)y_n w_n - 2f''(x_n)x_n w_n - f''(y_n)z_n^2 - 2f'(y_n)w_n \\ +2f'(x_n)w_n + f''(x_n)z_n^2 \end{pmatrix}}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}},$$

$$x_{n+1} = h_n - \frac{f(h_n)}{f'(h_n)} - \frac{1}{2}\frac{f^2(h_n)f''(h_n)}{f'^3(h_n)}.$$

3. Numerical Examples

In this section, we apply Algorithms 1-4 to solve the following nonlinear algebraic equations

$$f_1(x) = \sin^2(x) - x^2 + 1,$$

$$f_2(x) = e^x - 3x^2,$$

$$f_3(x) = e^{-x} + \cos(x),$$

$$f_4(x) = e^{-x^2+x+2} - 1.$$

And compare the result with

Newton's Method (NM) ([13]).

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Noor's Method ([14–16]).

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = -\frac{(y_n - x_n)^2}{2}\frac{f''(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{(y_n + z_n - x_n)^2}{2}\frac{f''(x_n)}{f'(x_n)}.$$

Chun's Method ([3]).

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - 2\frac{f(y_n)}{f'(x_n)} + \frac{f(y_n)f'(y_n)}{f'^2(x_n)},$$

and with some new iterative formulas developed by Sehati *et al.* [17], by taking 1500 digits precision. We used $\varepsilon = 10^{-25}$. The following stopping criteria were used in computer program:

(i) $|x_{n+1} - x_n| < \varepsilon$, and

(ii) $|f(x_{n+1})| < \varepsilon$.

4. Tables

Table 4.1 shows numerical results for $f_1(x) = \sin^2(x) - x^2 + 1$.

Table 4.1

Method	x_0	Iterations	x_n	$\lvert f(x_n)\rvert$	COC
Newton's Method [13]	2	6	−1.40449164821534122604	2.2600E-32	2.00
Noor Method [14–16]	2	4	−1.40449164821534122604	8.6400E-63	3.00
Chun Method [3]	2	4	−1.40449164821534122604	1.4500E-91	3.99
5^{th} order Method [17]	2	4	−1.40449164821534122604	7.1289E-33	5.05
7^{th} order Method [17]	2	4	−1.40449164821534122604	6.9011E-166	6.99
10^{th} order Method [17]	2	4	−1.40449164821534122604	1.2535E-234	9.99
14^{th} order Method [17]	2	3	−1.40449164821534122604	2.3351E-159	13.90
Algorithm 1	2	3	−1.40449164821534122604	6.7501E-64	9.17
Algorithm 2	2	2	−1.40449164821534122604	2.8880E-27	15.23
Algorithm 3	2	2	−1.40449164821534122604	1.8016E-60	23.46
Algorithm 4	2	2	−1.40449164821534122604	1.5839E-44	27.31

Table 4.2 shows numerical results for $f_2(x) = e^x - 3x^2$.

Table 4.2

Method	x_0	Iterations	x_n	$\lvert f(x_n)\rvert$	COC
Newton's Method [13]	0.5	6	0.91000757248870906066	1.3153E-29	2.00
Noor Method [14–16]	0.5	4	0.91000757248870906066	6.1201E-26	3.00
Chun Method [3]	0.5	4	0.91000757248870906066	1.6753E-29	3.99
5^{th} order Method [17]	0.5	3	−0.45896226753694851460	2.2472E-66	5.00
7^{th} order Method [17]	0.5	3	−0.45896226753694851460	1.2272E-163	6.99
10^{th} order Method [17]	0.5	2	−0.45896226753694851460	3.3091E-47	10.29
14^{th} order Method [17]	0.5	2	−0.45896226753694851460	1.7178E-86	14.29
Algorithm 1	0.5	2	−0.45896226753694851460	3.4556E-36	9.29
Algorithm 2	0.5	2	−0.45896226753694851460	6.5611E-136	18.28
Algorithm 3	0.5	2	−0.45896226753694851460	2.0723E-300	27.29
Algorithm 4	0.5	2	−0.45896226753694851460	1.1552E-291	27.28

Table 4.3 shows numerical results for $f_3(x) = e^{-x} + \cos(x)$.

Table 4.3

Method	x_0	Iterations	x_n	$\|f(x_n)\|$	COC
Newton's Method [13]	2.5	6	1.74613953040801241765	1.3724E-47	2.00
Noor Method [14–16]	2.5	4	1.74613953040801241765	3.9410E-30	3.00
Chun Method [3]	2.5	4	1.74613953040801241765	7.7501E-90	3.99
5^{th} order Method [17]	2.5	3	4.70332375945224380651	3.2258E-53	4.99
7^{th} order Method [17]	2.5	3	4.70332375945224380651	4.8969E-135	6.99
10^{th} order Method [17]	2.5	2	4.70332375945224380651	1.8709E-65	9.17
14^{th} order Method [17]	2.5	2	4.70332375945224380651	1.4130E-105	13.00
Algorithm 1	2.5	2	4.70332375945224380651	8.2551E-133	7.63
Algorithm 2	2.5	2	4.70332375945224380651	3.1749E-165	16.88
Algorithm 3	2.5	2	4.70332375945224380651	5.4067E-303	25.64
Algorithm 4	2.5	2	4.70332375945224380651	5.4787E-303	25.64

Table 4.4 shows numerical results for $f_4(x) = e^{-x^2+x+2} - 1$.

Table 4.4

Method	x_0	Iterations	x_n	$\|f(x_n)\|$	COC
Newton's Method [13]	0.5	1	Fail	—	
Noor Method [14–16]	0.5	1	Fail	—	
Chun Method [3]	0.5	1	Fail	—	
5^{th} order Method [17]	0.5	3	−1.00000000000000000000	1.6955E-117	4.99
7^{th} order Method [17]	0.5	3	−1.00000000000000000000	5.0264E-35	6.65
10^{th} order Method [17]	0.5	3	−1.00000000000000000000	1.5124E-103	9.97
14^{th} order Method [17]	0.5	2	−1.00000000000000000000	1.0709E-219	13.99
Algorithm 1	0.5	3	−1.00000000000000000000	1.8608E-152	8.79
Algorithm 2	0.5	3	−1.00000000000000000000	4.0495E-391	17.99
Algorithm 3	0.5	2	−1.00000000000000000000	3.6553E-51	23.15
Algorithm 4	0.5	2	−1.00000000000000000000	3.7313E-45	27.65

The test results in Table 4.1-4.4 show that for most of the function we tested. The proposed algorithms highly effective and have better performance compared with other methods.

Form Table 4.1, we see that Algorithms 1-4 have better approximation as compared with Sehati *et al.* [17] formulas, Newton method [13], Noor method [14–16] and Chun method [3]. In Table 4.2, we see that Algorithms 1-4 have different roots from Newton method, Noor method and Chun method with same initial approximation. In Table 4.3, with the same approximation Algorithms 1-4 have different roots from Newton method, Noor method and Chun method. In Table 4.4, Newton method, Noor method and Chum method fails to calculate the root but Algorithms 1-4 are successively applied to find the root and have better results as compare to new iterative schemes.

5. Figures

Figure 5.1 depicts graphical comparison between methods and number of iterations.

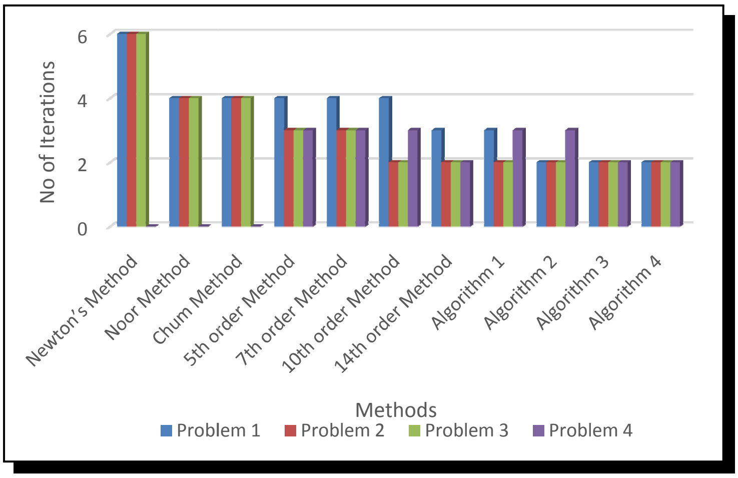

Figure 5.1

Figure 5.2 depicts the graphical comparison between methods and COC.

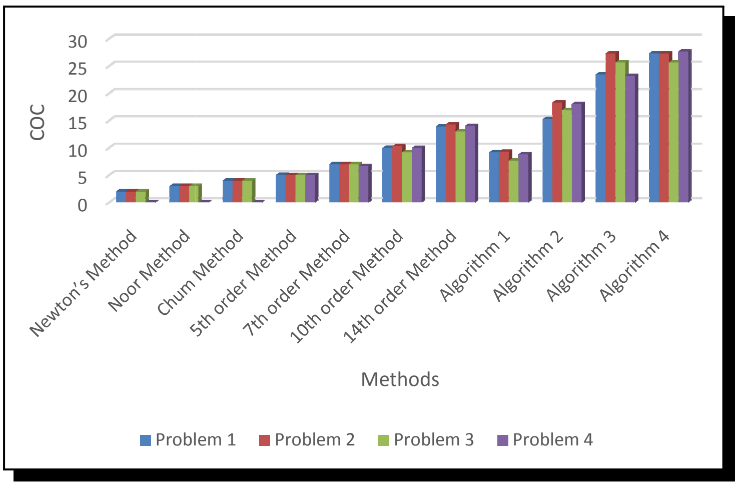

Figure 5.2

Figures 5.1 and 5.2 elaborate the efficiency, accuracy and reliability of the proposed algorithms to find the root of the problems 1-4. It is observed that the proposed algorithms have large *computational order of convergence* (COC) than the existing methods [3, 14–17] which evidence that our newly proposed methods are well-matched to investigate the roots. Moreover, for same initial guess our proposed algorithms are faster convergent because just in two iterations we achieved our required root. This figure witnesses that the Algorithms 1-4 are rapid convergent and more accurate and can be extend to find the root of nonlinear diversify problems.

6. Conclusions

New iterative schemes have been developed by using HPM. The comparison with other methods including Newton's method, Noor method, Chun method and Sehati *et al.* [17] shows the efficiency and reliability of the proposed techniques. The proposed algorithms are complicated and a lot of computation work, but with the aid of MAPLE 13, this deficiency have been removed. It is concluded that proposed algorithm can be extended to other nonlinear problems of physical nature too.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] S. Abbasbandy, Improving Newton–Raphson method for nonlinear equations by modified Adomian's decomposition method, *Appl. Math. Comput.* **145** (2) (2003), 887–893, doi:10.1016/S0096-3003(03)00282-0.

[2] S. Amat, S. Busquier and J.M. Gutiérrez, Geometric constructions of iterative functions to solve nonlinear equations, *J. Compu. Appl. Math.* **157** (1) (2003), 197–205, doi:10.1016/S0377-0427(03)00420-5.

[3] C. Chun, Iterative methods improving newton's method by the decomposition method, *Comput. Math. Applic.* **50** (10) (2005), 1559–1568, doi:10.1016/j.camwa.2005.08.022.

[4] J.A. Ezquerro and M.A. Hernández, A uniparametric Halley-type iteration with free second derivative, *Int. J. Pure Appl. Math.* **6** (2003), 103–114.

[5] J.A. Ezquerro and M.A. Hernandez, On Halley-type iterations with free second derivative, *J. Compu. Appl. Math.* **170** (2) (2004), 455–459, doi:10.1016/j.cam.2004.02.020.

[6] E. Halley, A new and general method of finding the roots of equations, *Phil. Trans. Royal Soci. London* **18** (1694), 136.

[7] J.H. He, Homotopy perturbation technique, *Comput. Meth. Appl. Mech. Engg.* **178** (3) (1999), 257–262, doi:10.1016/S0045-7825(99)00018-3.

[8] J.H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, *Int. J. Nonlin. Mech.* **35** (1) (2000), 37–43.

[9] J.H. He, Homotopy perturbation method: a new nonlinear analytical technique, *Appl. Math. Comput.* **135** (1) (2003), 73–79, doi:10.1016/S0096-3003(01)00312-5.

[10] J.H. He, A new iteration method for solving algebraic equations, *Appl. Math. Comput.* **135** (1) (2003), 81–84, doi:10.1016/S0096-3003(01)00313-7.

[11] J.H. He, Addendum: new interpretation of homotopy perturbation method, *Int. J. Moder. Phy. B* **20** (18) (2006), 2561–2568.

[12] J. Kou and Y. Li, The improvements of Chebyshev-Halley methods with fifth-order convergence, *Appl. Math. Comput.* **188** (1) (2007), 143–147, doi:10.1016/j.amc.2006.09.097.

[13] J. Kou, Y. Li and X. Wang, A family of fifth-order iterations composed of Newton and third-order methods, *Appl. Math. Comput.* **186** (2) (2007), 1258–1262.

[14] M.A. Noor, K.I. Noor, S.T. Mohyud-Din and A. Shabbir, An iterative method with cubic convergence for nonlinear equations, *Appl. Math. Comput.* 183 (2) (2006), 1249–1255.

[15] K.I. Noor and M.A. Noor, Predictor-corrector Halley method for nonlinear equations, *Appl. Math. Comput.* **188** (2) (2007), 1587–1591, doi:10.1016/j.amc.2006.11.023.

[16] M.A. Noor and K.I. Noor, Fifth-order iterative methods for solving nonlinear equations, *Appl. Math. Comput.* **188** (1) (2007), 406–410.

[17] M.M. Sehati, S.M. Karbassi, M. Heydari and G.B. Loghmani, Several new iterative methods for solving nonlinear algebraic equations incorporating homotopy perturbation method (HPM), *Int. J. Phy. Sci.* **7** (27) (2012), 5017–5025, doi:10.5897/IJPS12.279.

[18] J.F. Traub, Iterative methods for the solution of equations, *Amer. Math. Soci.* (1982).

[19] T.J. Ypma, Historical development of the Newton-Raphson method, *SIAM Review* **37** (4) (1995), 531–551, doi:10.1137/1037125.