Research Article

Projectively Vanishing Nearly Cosymplectic Manifold

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Abstract. The present paper focuses on the study of the geometric properties of projective curvature tensor on the nearly cosymplectic manifold. In particular, the flatness properties of projective tensor have been studied, so related to these properties we defined three special classes of nearly cosymplectic manifold.

1. Introduction

A relatively long time ago, number of researchers were studied one of the most important subjects of differential geometry, whose application is used in the synthesis of the differential geometrical structure, it is called *almost contact manifold*. In 1953, Chern [9] proved that in each point of $n+1$ manifold there is a contact form η such that $\eta \wedge (d\eta)^n \neq 0$. In 1960, Sasaki [23] defined set of three tensors $\{\Phi, \xi, \eta\}$ that are called almost contact structure tensors and are satisfy some of properties.

Like the almost Hermitian manifold, appeared the study of classification of almost contact manifold into different components in an attempt to determine its specifications and features accurately. Chinea and Morrero [11], Chinea [10] and Kirichenko [17] defined some of different almost contact metric manifolds.

Almost contact manifold strongly related with the almost Hemitian manifold. The famous classical example of almost contact manifold is non even sphere S^{2n-1} which is considered as a hypersphere in C^n.

One of the interesting class of almost contact structure is nearly cosymplectic structure. Blair [6], Blair and Showers [7] studied some properties of nearly cosymplectic structure and they are considered analog the concept nearly Kahler structure in Hermition geometry. There are many researchers studied this class, for example, Banaru [4], Endo [12], [13], [14]. Kirichenko and Kusova [19] studied the geometry of nearly cosymplectic manifold in the G-adjoined structure space. In particular, they found its structure equations and components of Riemannian curvature tensor.

The geometric properties of projective tensor on some kinds of almost Hermitian manifold have been studied by many researchers. For more dealails we refer to [1], [2] and [3].

In this paper, we studied some of the geometrical properties of the projective curvature tensor on nearly cosymplectic manifold.

This next section contains recollection to many concepts and facts related to the content of the present paper. In particular, shown how to construct the structure equations of nearly cosymplectic manifold.

2. Preliminaries

Definition 2.1 ([5])**.** Let M be a smooth manifold of dimension $2n + 1$ greater than 3, and η is differential 1-form called *contact form*, ξ be a vector field called *the characteristic*, Φ endomorphism of $X(M)$ called the *structure endomorphisim*, then the triple (η, ξ, Φ) is called *Almost contact structure* if the following conditions hold:

(i) $\eta(\xi) = 1$;

(ii) $\Phi(\xi) = 0$;

(iii) $\eta \circ \Phi = 0$;

(iv) $\Phi^2 = -\operatorname{id} + \eta \otimes \xi$.

In addition, if there is a Riemannian structure $g = \langle \cdot, \cdot \rangle$ on M such that

$$\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), \quad X, Y \in X(M),$$

then the quadruple (η, ξ, Φ, g) is called an almost contact metric structure. In this case, the manifold M with this structure is called an almost contact metric manifold.

Definition 2.2 ([8])**.** Almost contact manifold is called a *Nearly Cosymplectic Manifold* (NC-manifold) if the equality

$$\nabla_X(\Phi)Y + \nabla_Y(\Phi)X = 0, \quad X, Y \in X(M),$$

holds.

Definition 2.3 ([17])**.** Let (M, η, Φ, g) be almost contat metric manifold (AC-manifold). In the module $X(M)$ we can define two complementary projections m, ℓ, where $m = \eta \otimes \xi$ and $\ell = -\Phi^2$; thus $X(M) = L \oplus \aleph$, where $L = \operatorname{Im}\Phi = \ker\eta$ and $\aleph = \operatorname{Im}m = \ker\Phi$ (where ℓ and m are the

projections onto the submodules L and $\aleph$, respectively).

Definition 2.4 ([17])**.** In the module $X^c(M)$ define two endomorphisms σ and $\bar{\sigma}$ as $\sigma = \frac{1}{2}(\mathrm{id} - \sqrt{-1}\Phi)$ and $\bar{\sigma} = -\frac{1}{2}(\mathrm{id} + \sqrt{-1}\Phi)$. We can define two projections by the form:

$$\Pi = \sigma \circ \ell = -\frac{1}{2}(\Phi^2 - \sqrt{-1}\Phi) \quad \text{and} \quad \bar{\Pi} = \bar{\sigma} \circ \ell = \frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi),$$

where $\sigma \circ \Phi = \Phi \circ \sigma = i\sigma$ and $\bar{\sigma} \circ \Phi = \Phi \circ \bar{\sigma} = -i\bar{\sigma}$. Therefore, if we denote $\mathrm{Im}\,\Pi = D_{\Phi}^{\sqrt{-1}}$ and $\mathrm{Im}\,\bar{\Pi} = D_{\Phi}^{-\sqrt{-1}}$, then

$$X^c(M) = D_{\Phi}^{\sqrt{-1}} \oplus D_{\Phi}^{-\sqrt{-1}} \oplus D_{\Phi}^0,$$

where $D_{\Phi}^{\sqrt{-1}}$, $D_{\Phi}^{-\sqrt{-1}}$ and D_{Φ}^0 are proper submodules of endomorphism Φ with proper values $\sqrt{-1}$, $-\sqrt{-1}$ and 0, respectively.

Definition 2.5 ([20])**.** The mappings $\sigma_p : L_p \to D_{\Phi}^{\sqrt{-1}}$ and $\bar{\sigma}_p : L_p \to D_{\Phi}^{-\sqrt{-1}}$ are isomorphism and anti-isomorphism respectively. Therefore, at each point $p \in M^{2n+1}$, there is a frame in $T_p(M)^c$ of the form $(p, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{\hat{1}}, \ldots, \varepsilon_{\hat{n}})$, where $\varepsilon_a = \sqrt{2}\sigma_p(e_p)$, $\varepsilon_{\hat{a}} = \sqrt{2}\bar{\sigma}(e_p)$, $\varepsilon_0 = \xi_p$, and e_a are orthonormal bases of L_p. The frame $(p, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{\hat{1}}, \ldots, \varepsilon_{\hat{n}})$ is called A-frame.

Lemma 2.1 ([18])**.** *The matrix components of tensors Φ_p and g_p in A-frame have the following forms, respectively:*

$$(\Phi_j^i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & 0 \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix}, \quad (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{pmatrix},$$

where I_n is the identity matrix of order n.

It is well known, that the set of such frames defines an G-structure on M with structure group $1 \times U(n)$, represented by matrices of the form: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$, where $A \in U(n)$. This structure is called an G-adjoined structure.

Theorem 2.1 ([19])**.** *The collection of the structure equations of NC-manifold in the G-adjoined structure space has the following forms:*

(i) $\ d\omega^a = \omega_b^a \wedge \omega^b + B^{abc}\omega_b \wedge \omega_c + \frac{3}{2}C^{ab}\omega_b \wedge \omega$;

(ii) $\ d\omega_a = -\omega_a^b \wedge \omega_b + B_{abc}\omega^b \wedge \omega^c + \frac{3}{2}C_{ab}\omega^b \wedge \omega$;

(iii) $\ d\omega = C^{bc}\omega_b \wedge \omega_c + C_{bc}\omega^b \wedge \omega^c$;

(iv) $\ d\omega_b^a = \omega_c^a \wedge \omega_b^c + \left[A_{bc}^{ad} - 2B^{adh}B_{hbc} + \frac{3}{2}C^{ad}C_{bc}\right]\omega^c \wedge \omega_d$,

$\qquad$ *where $B^{abc} = \frac{\sqrt{-1}}{2}\Phi_{\hat{b},\hat{c}}^a$, $C^{ab} = \sqrt{-1}\Phi_{0,\hat{b}}^a$, $C_{ab} = -\sqrt{-1}\Phi_{b,0}^{\hat{a}}$ and $B_{abc} = -\frac{\sqrt{-1}}{2}\Phi_{b,c}^{\hat{a}}$.*

The tensors B, C and A are called the first, second and third structure tensors, respectively.

The following theorem gives the components of Riemannian curvature tensor of *NC*-manifold in the *G*-adjoined structure space.

Theorem 2.2 ([19])**.** *In the G-adjoined structure space, the components of Riemannian curvature tensor of NC-manifold have the following forms:*

(i) $R_{\widehat{a}bcd} = 0$;

(ii) $R_{abcd} = -2B_{ab[cd]}$;

(iii) $R_{\widehat{a}\widehat{b}cd} = -2B^{abh}B_{hcd}$;

(iv) $R_{\widehat{a}0b0} = C^{ac}C_{bc}$;

(v) $R_{\widehat{a}bc\widehat{d}} = A^{ad}_{bc} - B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc}$.

The other components of Riemannian curvature tensor R can be obtained by the property of symmetry for R or equal to zero.

Definition 2.6 ([22])**.** A Ricci tensor is a tensor of type $(2,0)$ which is defined by:

$$r_{ij} = R^k_{ijk} = g^{kl}R_{kijl} \, .$$

Lemma 2.2 ([19])**.** *In the G-adjoined structure space, the components of the Ricci tensor of NC-manifold are given by the following forms:*

(i) $r_{ab} = 0$;

(ii) $r_{a\widehat{b}} = -A^{cb}_{ac} + 3B^{cbh}B_{hac} + \frac{2}{3}C^{bc}C_{ac}$;

(iii) $r_{a0} = 0$;

(iv) $r_{oo} = -2C^{cd}C_{cd}$.

The other components of Ricci tensor can be found by taking the conjugate operator to the above components or equal to zero.

Now, we are in a position to provide put the projective curvature tensor which is the focus of the present paper.

Definition 2.7 ([15])**.** A projective curvature tensor on M^{2n+1} is a tensor of type $(4,0)$ which is define by the forms

$$P_{ijkl} = R_{ijkl} - \frac{1}{2n}[r_{ik}g_{jl} - r_{jk}g_{il}],$$

where $P_{ijkl} = -P_{jikl} = -P_{ijlk} = P_{klij}$.

Definition 2.8 ([16])**.** Let M be a Riemannian manifold, t be a non-zero tensor field of the type (r,s) on M. A tensor t is said to be a recurrent if there is 1-form ρ on M such that $\nabla t = \rho \otimes t$, where ∇ is the Riemannian connection on M. The 1-form ρ is called a recurrence covector. An NC-manifold which allows a field of the recurrent tensor t is called t-recurrent.

Definition 2.9. An NC-manifold has Φ-invariant Ricci tensor, if $\Phi \circ r = r \circ \Phi$.

Lemma 2.3. *An NC-manifold has Φ-invariant Ricci tensor if and only if, in the G-adjoined structure space the following condition*

$$r^{\widehat{a}}_b = r_{ab} = 0$$

holds.

Lemma 2.4 ([17])**.** *In the G-adjoined structure space, an NC-manifold is a manifold of class*

(i) CR_1 *if and only if,* $R_{abcd} = R_{\widehat{a}bcd} = R_{\widehat{a}\widehat{b}cd} = 0$;

(ii) CR_2 *if and only if,* $R_{abcd} = R_{\widehat{a}bcd} = 0$;

(iii) CR_3 *if and only if,* $R_{\widehat{a}bcd} = 0.$

It easy to see that $CR_1 \subset CR_2 \subset CR_3$.

Definition 2.10. In the G-adjoined structure space, an NC-manifold is a manifold of class

(i) PR_1 if and only if, $P_{abcd} = P_{\widehat{a}bcd} = P_{\widehat{a}\widehat{b}cd} = 0$;

(ii) PR_2 if and only if, $P_{abcd} = P_{\widehat{a}bcd} = 0$;

(iii) PR_3 if and only if, $P_{\widehat{a}bcd} = 0$.

3. The Main Results

This section is devoted to study the geometric properties of projective tensor. In particular, the necessary and sufficient conditions for vanishing this tensor have been found.

Theorem 3.1. *In the G-adjoined structure space, the components of projective curvature tensor of NC-manifold are given by the following forms:*

(i) $P_{abcd} = -2B_{ab[cd]}$;

(ii) $P_{\widehat{a}\widehat{b}cd} = -2B^{abh}B_{hcd} - \frac{1}{2n}[r^a_c\delta^b_d - r^b_c\delta^a_d]$;

(iii) $P_{\widehat{a}bc\widehat{d}} = A^{ad}_{bc} - B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc} - \frac{1}{2n}r^a_c\delta^d_b$

(iv) $P_{\widehat{a}0b0} = C^{ac}C_{bc} - \frac{1}{2n}r^a_b$;

and the others are conjugate to the above components or equal to zero.

Proof. By using the Definition 2.7, Lemma 2.2 and Theorem 2.2, we compute the components P_{ijkl} for all possible indecies as follows:

(i) put $i = a$, $j = b$, $k = c$, $l = d$, we get

$$P_{abcd} = R_{abcd} - \frac{1}{2n}[r_{ac}g_{bd} - r_{bc}g_{ad}].$$

So, we have

$$P_{abcd} = -2B_{ab[cd]}.$$

(ii) put $i = \widehat{a}$, $j = \widehat{b}$, $k = c$, $l = d$, we obtain

$$P_{\widehat{a}\widehat{b}cd} = R_{\widehat{a}\widehat{b}cd} - \frac{1}{2n}[r_{\widehat{a}c}g_{\widehat{b}d} - r_{\widehat{b}c}g_{\widehat{a}d}],$$

$$P_{\widehat{a}\widehat{b}cd} = -2B^{abh}B_{hcd} - \frac{1}{2n}[r_{\widehat{a}c}g_{\widehat{b}d} - r_{\widehat{b}c}g_{\widehat{a}d}],$$

$$P_{\widehat{a}\widehat{b}cd} = -2B^{abh}B_{hcd} - \frac{1}{2n}[r^a_c\delta^b_d - r^b_c\delta^a_d].$$

By the same way we can found the other components.　□

Definition 3.1. An NC-manifold is called a projectively vanishing tensor, if the projective tensor vanishes.

Theorem 3.2. *If M is projectively vanishing NC-manifold of* $\dim n > 3$ *and* Φ-*invariant Ricci tensor, then M has vanishing Ricci tensor.*

Proof. Suppose that M is is projectively vanishing NC-manifold, then we have

$$-2B^{abh}B_{hcd} - \frac{1}{2n}[r_c^a \delta_d^b - r_c^b \delta_d^a] = 0. \tag{3.1}$$

Contracting (3.1) by indices (d, b), we get:

$$-2B^{adh}B_{hcd} - \frac{n-1}{2n}[r_c^a] = 0. \tag{3.2}$$

Symmetrizing and antisymmetrizing (3.2) by the indices (c, d), we have

$$r_c^a = 0.$$

Since M has Φ-invariant Ricci tensor, consequently we deduce that M has vanishing Ricci tensor. $\qquad\square$

Theorem 3.3. *If M is projectively vanishing NC-manifold and* Φ-*invariant Ricci tensor, then M of vanishing holomorphic sectional curvature tensor if and only if, M is vanishing Ricci tensor.*

Proof. Let M be a projectively vanishing NC-manifold and vanishing Ricci tensor.

By using Definition 3.1 and Theorem 3.1, we have

$$A_{bc}^{ad} - B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc} = 0. \tag{3.3}$$

Symmetrizing (3.3) by the indices (a, d), it follows that

$$A_{bc}^{ad} = 0.$$

Hence, M is a manifold of flat holomorphic sectional curvature tensor.

Conversely, let $A_{bc}^{ad} = 0$, by using the equation (3.3), we have

$$-B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc} - \frac{1}{2n}r_c^a \delta_b^d = 0. \tag{3.4}$$

Contracting (3.4) by indices (a, b), we get

$$-B^{adh}B_{hac} - \frac{5}{3}C^{ad}C_{ac} - \frac{1}{2n}r_c^d = 0. \tag{3.5}$$

Symmetrizing and antisymmetrizing (3.5) by the indices (a, d), we obtain

$$r_c^d = 0.$$

Since M has Φ-invariant Ricci tensor, consequently we get that M has vanishing Ricci tensor. $\quad\square$

Definition 3.2 ([21])**.** A Riemannian manifold is called an Einstein manifold, if the Ricci tensor satisfies the equation $r_{ij} = eg_{ij}$, where, e is an Einstein constant.

The following theorem gives the necessary and sufficient condition in which an NC-manifold is an Einstein manifold.

Theorem 3.4. *Suppose that M is a projectively vanishing NC-manifold and* Φ-*invariant Ricci tensor. Then the necessary and sufficient condition in which M is an Einstein manifold is* $A_{ac}^{ad} = \frac{5}{3}C^{ad}C_{ac} + C_0\delta_c^d$, *where* $C_0 = \frac{e}{2n}$.

Proof. Let M be projectively vanishing NC-manifold.

According to Definition 3.1 and Theorem 3.1, we have

$$A^{ad}_{bc} - B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc} - \frac{1}{2n}r^a_c\delta^d_b = 0. \tag{3.6}$$

Symmetrizing and antisymmetrizing (3.6) by the indices (h,d), we deduce

$$A^{ad}_{bc} - \frac{5}{3}C^{ad}C_{bc} - \frac{1}{2n}r^a_c\delta^d_b = 0. \tag{3.7}$$

Suppose that M is Einstein manifold. Using the Definition 3.2, so the equation (3.7) becomes

$$A^{ad}_{bc} - \frac{5}{3}C^{ad}C_{bc} - \frac{e}{2n}\delta^a_c\delta^d_b = 0. \tag{3.8}$$

Contracting (3.8) by indices (a,b), it follows that

$$A^{ad}_{ac} = \frac{5}{3}C^{ad}C_{ac} + C_0\delta^d_c. \tag{3.9}$$

Conversely, let $A^{ad}_{ac} = \frac{5}{3}C^{ad}C_{ac} + C_0\delta^d_c$.

By contracting the equation (3.7) by indices (a,b), we deduce

$$A^{ad}_{ac} - \frac{5}{3}C^{ad}C_{ac} - \frac{1}{2n}r^d_c = 0. \tag{3.10}$$

From the equations (3.9) and (3.10), we have

$$r^d_c = e\delta^d_c.$$

According to the Φ-invariant Ricci tensor, we get that M is Einstein manifold.   □

Theorem 3.5. *Any NC-manifold is a submanifold of the class PR_3.*

Proof. Let M be NC-manifold.

In this case we have $P_{\hat{a}bcd} = 0$.

This implies that M is a manifold of class PR_3.   □

Theorem 3.6. *Suppose that M is NC-manifold, then the following statements are equivalent:*

   (i) *M is a manifold of class PR_2;*

  (ii) *On the space of the G-adjoined structure space, the identities $R_{abcd} = 0$ hold;*

 (iii) *The first structure tensor is parallel in the first canonical connection.*

Proof. To prove (i)$\Rightarrow$(ii): Let M be a structure of class PR_2.

We know that $P_{abcd} = -2B_{ab[cd]}$ and $P_{\hat{a}bcd} = 0$.

Since $P_{abcd} = P_{\hat{a}bcd} = 0$, then, we have

$$-2B_{ab[cd]} = 0.$$

Consequently, we get

$$B_{abcd} = 0.$$

Now, we want to prove that (ii)$\Rightarrow$(iii). According to the fundamental theorem of tensor analysis we have

$$\nabla B_{abc} = dB_{abc} + B_{dbc}\omega^d_a + B_{adc}\omega^d_b + B_{abd}\omega^d_c = B_{abcd}\omega^d.$$

So, we obtain

$$\nabla B_{abc} = B_{abcd}\omega^d.$$

Therefore, it follows that

$$\nabla B_{abc} = 0,$$

which means the tensor B_{abc} is parallel in the connection if its covariant derivative is equal to zero.

The statement (iii)$\Rightarrow$(i) directly. $\qquad\square$

Theorem 3.7. *Let M be NC-manifold, then M is a manifold of class PR_1 if its first structure tensor identically vanishes.*

Proof. Let M be NC-manifold of class PR_1.
From the Definition 2.10, we have

$$-2B^{abh}B_{hcd} - \frac{1}{2n}[r^a_c\delta^b_d - r^b_c\delta^a_d] = 0. \tag{3.11}$$

Contracting (3.11) by the indices (a,c) and (b,d), we get

$$-2B^{abh}B_{hab} = 0.$$

Since B^{abh} and B_{hab} are antisymmetric tensors, then we get

$$\sum_{a,b,h}|B^{abh}|^2 = 0.$$

Consequently, we deduce

$$B^{abh} = 0. \qquad\square$$

Theorem 3.8. *Let M be NC-manifold of class PR_1 with Φ-invariant Ricci tensor, then M is a manifold of class CR_1 if and only if, M is a manifold of vanishing Ricci tensor.*

Proof. Let M be NC-manifold of class CR_1.
Since M of class PR_1, it follows that

$$P_{abcd} = P_{\hat{a}bcd} = P_{\hat{a}\hat{b}cd} = 0$$

or equivalent to

$$-2B_{ab[cd]} = 0 = -2B^{abh}B_{hcd} - \frac{1}{2n}[r^a_c\delta^b_d - r^b_c\delta^a_d] = 0.$$

Consequently, we get

$$\frac{1}{2n}[r^a_c\delta^b_d - r^b_c\delta^a_d] = 0. \tag{3.12}$$

Contracting (3.12) by indices (b,d), we have

$$\frac{n-1}{2n}r^a_c = o.$$

So, we deduce

$$r^a_c = o.$$

Since M has Φ-invariant Ricci tensor. Therefore, M is vanishing Ricci tensor.
Conversely, let M be NC-manifold of vanishing Ricci tensor.

Since M of class PR_1, then we have

$$P_{abcd} = P_{\widehat{a}bcd} = P_{\widehat{a}\widehat{b}cd} = 0$$

(i) $P_{abcd} = 0$, lead to $R_{abcd} = 0$

(ii) $P_{\widehat{a}bcd} = 0$, implies $R_{\widehat{a}bcd} = 0$

(iii) $P_{\widehat{a}\widehat{b}cd} = 0$.

So, according to (iii) and Theorem 3.1, we have

$$-2B^{abh}B_{hcd} - \frac{1}{2n}[r^a_c \delta^b_d - r^b_c \delta^a_d] = 0. \tag{3.13}$$

Since M is a manifold of vanishing Ricci tensor, so the equation (3.13) becomes:

$$-2B^{abh}B_{hcd} = 0.$$

Consequently, we get

$$R_{\widehat{a}\widehat{b}cd} = 0.$$

Frome (i), (ii) and (iii), we get that M is NC-manifold of class CR_1. □

Theorem 3.9. *Suppose that M is NC-manifold of class PR_1 with Φ-invariant tensor, then if the first structure tensor identically vanish then M is a manifold of vanishing ricci tensor.*

Proof. Let M be NC-manifold of class PR_1 with vanishing first structure tensor. According to Definition 2.10, we have

$$-2B_{ab[cd]} = -\frac{1}{2n}[r^a_c \delta^b_d - r^b_c \delta^a_d] = 0. \tag{3.14}$$

Symmetrizing and antisymmetrizing (3.14) by the indices (c, d), we get

$$-\frac{1}{2n}[r^a_c \delta^b_d - r^b_c \delta^a_d] = 0. \tag{3.15}$$

Contacting (3.15) by the indices (b, d), we obtain

$$\frac{n-1}{2n}r^a_c = 0.$$

Since M has Φ-invariant Ricci tensor. Therefore, we get that M is NC-manifold of vanishing Ricci tensor. □

Finally, the following theorem gives the relation between P-recurrent and r-recurrent NC-manifold.

Theorem 3.10. *Suppose that M is P-recurrent NC-manifold then M is r-recurrent NC-manifold.*

Proof. Let M be P-recurrent NC-manifold. According to the Definition 2.8, we have

$$P_{ijkl,h} = \rho_h P_{ijkl}. \tag{3.16}$$

In the G-adjoined structure space the equation (3.16) becomes:

$$P_{\widehat{a}bc\widehat{d},h} = \rho_h P_{\widehat{a}bc\widehat{d}}.$$

According to Theorem 3.1, we have

$$A^{ad}_{bc,k} - B^{adh}B_{hbc,k} - \frac{5}{3}C^{ad}C_{bc,k} - \frac{1}{2n}[r_{\widehat{a}c,k}g_{b\widehat{d}}]$$

$$= \rho_h[A^{ad}_{bc} - B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc} - \frac{1}{2n}[r_{\hat{a}c}g_{b\hat{d}}]. \tag{3.17}$$

Contracting (3.17) by the indices (a, b), we have

$$A^{ad}_{ac,k} - B^{adh}B_{hac,k} - \frac{5}{3}C^{ad}C_{ac,k} - \frac{1}{2n}[r_{\hat{a}c,k}g_{a\hat{d}}]$$

$$= \rho_h[A^{ad}_{bc} - B^{adh}B_{hac} - \frac{5}{3}C^{ad}C_{ac} - \frac{1}{2n}[r_{\hat{a}c}g_{a\hat{d}}]. \tag{3.18}$$

Symmetrizing and antisymmetrizing (3.18) by the indices (a, c), we get

$$r_{\hat{d}c,k} = \rho_h r_{\hat{d}c}.$$

Therefore, by Definition 2.8, M is r-recurrent NC-manifold. □

4. Conclusion

The present work deals with the study of the projective tensor when it acts on NC-manifold. We used the flatness property of this tensor to find application in theoretical physics. In particular, we found the necessary and sufficient condition that nearly cosymplectic manifold is an Einstein manifold. For future work, may study the generalized projective tensor on nearly cosymplectic manifold depending on the generalized Riemannian curvature and generalized Ricci tensors.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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