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Well Posedness of A Common Coupled Fixed Point Problem

Research Article

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Abstract. In this paper, we prove first some common coupled fixed point theorems for mappings $T: X \times X \to X$ and $g: X \to X$ satisfying a generalized contractive condition on a metric space. We provide examples of new concepts introduced herein. We also study the well posedness of a common coupled fixed point problem. Our results generalize several well known comparable results in the literature.

Keywords. Coincidence point; Point of coincidence; Contractive type mappings; Well posedness

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1. Introduction and Preliminaries

There are many generalizations of the Banach's contraction mapping principle in the literature (see for example [16], [17]). These generalizations were made either by using the contractive condition or by imposing some additional conditions on an ambient space. In 1968, Kannan [29] proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. This paper was a genesis for a multitude of fixed point papers over the next two decades. In 1976, Jungck [23] extended and generalized the celebrated Banach contraction principle exploiting the idea of commuting maps. Sessa [47] coined the term weakly commuting maps. Jungck [24] generalized the notion of weak commutativity by introducing compatible maps and then weakly compatible maps [25]. Since then, many interesting coincidence and common fixed point theorems of compatible and weakly compatible maps under various contractive conditions and assuming the continuity of at least one of the mappings, have been obtained by a number of authors. Ćirić [18] studied necessary conditions to obtain a fixed point result of asymptotically regular mappings on complete metric spaces.

Bhashkar and Lakshmikantham in [15] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered complete metric spaces. They also discussed an application of their result by investigating the existence and uniqueness of solution for a periodic boundary value problem. Afterwards, Lakshmikantham and Ćirić [34] proved coupled coincidence and coupled common fixed point theorems for nonlinear mappings $F : X \times X \to X$ and $g : X \to X$ in partially ordered complete metric spaces. Choudhury and Kundu [19] obtained coupled coincidence point results in partially ordered metric spaces for compatible mappings. Samet [44] proved coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces. Abbas et al. [1] proved coupled coincidence and coupled common fixed point results in cone metric spaces for w-compatible mappings (see also, [3–13, 21, 27, 28, 31, 32, 35, 40–42, 45, 48–50] and references therein).

The aim of this paper is to present a common coupled fixed point theorem for mappings $T : X \times X \to X$ and $g : X \to X$ which satisfy a generalized contractive condition. We also study the well-posedness of a common coupled fixed point problem involving the mappings T and g.

2. Common Coupled Fixed Point Theorem

Let X be a non-empty set.

Definition 2.1. Let $F : X \times X \to X$ and $g : X \to X$. An element $(x, y) \in X \times X$ is called:

(i) a coupled fixed point of mapping F if $x = F(x, y)$ and $y = F(y, x)$ ([15]),

(ii) a coupled coincidence point of F and g if $gx = F(x, y)$ and $gy = F(y, x)$, and (gx, gy) is called a coupled point of coincidence,

(iii) a common coupled fixed point of F and g if $x = gx = F(x, y)$ and $y = gy = F(y, x)$ ([34]).

Definition 2.2 ([25, 26])**.** Self mappings f and g on X is said to be weakly compatible if they commute at their coincidence point (i.e. $fgx = gfx$ whenever $fx = gx = y$). A point $y \in X$ is called point of coincidence of f and g.

Definition 2.3 ([2])**.** Mappings $F: X \times X \to X$ and $g: X \to X$ are called w-compatible if $gF(x,y) = F(gx,gy)$ whenever $gx = F(x,y)$ and $gy = F(y,x)$.

The following lemma is a variant of Proposition 1.4 in [2]. For sake of reader's convenience, we give its proof.

Lemma 2.1. *Suppose that $T: X \times X \to X$ and $f: X \to X$ have a unique coupled point of coincidence (p,q) in $X \times X$. If the pair (T,f) is w-compatible, then T and f have a unique common coupled fixed point.*

Proof. Since (p,q) is a coupled point of coincidence of f and T, so $(p,q) = (fu,fv) = (T(u,v),T(v,u))$. Also, f and T are w-compatible, we have

$$fp = fT(u,v) = T(fu,fv)) = T(p,q) \quad \text{and} \quad fq = fT(v,u) = T(fv,fu)) = T(q,p).$$

Thus (fp,fq) is a coupled point of coincidence of f and T. As, (p,q) is the only coupled point of coincidence of f and T, so $(p,q) = (fp,fq)$. Hence (p,q) is a common coupled fixed point of T and f. Moreover if $z = fz = T(z,w)$ and $w = fw = T(w,z)$, then (z,w) becomes a coupled point of coincidence of f and T. By uniqueness of (p,q), we have $(z,w) = (p,q)$. Thus, (p,q) is the unique common coupled fixed point of f and T. □

Let $T: X \times X \to X$ and $f: X \to X$ be two given mappings with $T(X \times X) \subset f(X)$, and $x_0, y_0 \in X$. Choose points x_1, y_1 in X such that $fx_1 = T(x_0,y_0)$ and $fy_1 = T(y_0,x_0)$. This can be done since $T(X \times X) \subset f(X)$. Continuing this process, having chosen $x_1,\dots,x_k$ and $y_1,\dots,y_k$, we choose x_{k+1}, y_{k+1} in X such that

$$fx_{k+1} = T(x_k,y_k) \text{ and } fy_{k+1} = T(y_k,x_k), \quad k = 0,1,2,\dots.$$

The above sequences $\{fx_n\}$ and $\{fy_n\}$ are called coupled (f,T)-sequences with initial point (x_0,y_0). If, $f = I_X$ (an identity map on X), then $\{x_n\}$ and $\{y_n\}$ are called coupled T-sequences with initial point (x_0,y_0).

Now, we introduce the following definition.

Definition 2.4. Let $T: X \times X \to X$ and $f: X \to X$ be mappings on a metric space (X,d), with $T(X \times X) \subset f(X)$, and $x_0, y_0 \in X$. A mapping T is said to be coupled asymptotically f-regular at point (x_0,y_0) if $d(fx_n, fx_{n+1}) \to 0$ and $d(fy_n, fy_{n+1}) \to 0$ as $n \to \infty$, where $\{fx_n\}$ and $\{fy_n\}$ are coupled (f,T)-sequences with initial point (x_0,y_0).

Example 2.1. Let $X = [0,+\infty)$ be endowed with the usual metric, $T: X \times X \to X$ and $f: X \to X$ be defined by

$$T(x,y) = \frac{x+y}{4}, \quad fx = x.$$

choose $x_0 = y_0 = 1$. We have $T(X \times X) \subset f(X)$. The sequences $\{x_n = \frac{1}{2^n}\}$ and $\{y_n = \frac{1}{2^n}\}$ are given by

$$fx_{n+1} = T(x_n,y_n) \text{ and } fy_{n+1} = T(y_n,x_n), \quad n = 0,1,2,\dots.$$

Then, $\{fx_n\}$ and $\{fy_n\}$ are coupled T-sequences with initial point (x_0, y_0). Again, as $n \to \infty$, we have

$$d(fx_n, fx_{n+1}) = d(fy_n, fy_{n+1}) = \frac{1}{2^{n+1}} \to 0,$$

that is, T is coupled asymptotically f-regular at the point (x_0, y_0).

Let $F_i : [0,\infty) \to [0,\infty)$ be functions such that $F_i(0) = 0$ and F_i is continuous at 0 $(i = 1,2)$. Our first result is the following:

Theorem 2.1. *Let (X, d) be a metric space, $T : X \times X \to X$ and $f : X \to X$ be mappings such that $T(X \times X) \subset f(X)$. Assume that the following condition holds:*

$$\begin{aligned} d(T(x,y), T(u,v)) \le\ & a_1 F_1[\min\{d(fx, T(x,y)), d(fu, T(u,v))\}] \\ & + a_2 F_2[d(fx, T(x,y)) d(fu, T(u,v))] + a_3[d(fx, fu) + d(fy, fv)] \\ & + a_4[d(fx, T(x,y)) + d(fu, T(u,v))] \\ & + a_5[d(fx, T(u,v)) + d(fu, T(x,y))] \end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$, where for $i = 1, \ldots, 5$, $a_i \ge 0$ such that for arbitrary fixed $k > 0$, $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$, we have $a_4 + a_5 \le \lambda_1$, $2a_3 + 2a_5 \le \lambda_2$ and $a_1, a_2 \le k$. If $f(X)$ is a complete subspace of X and T is coupled asymptotically f-regular at some point (x_0, y_0) in X, then T and f have a coupled point of coincidence.

Proof. Let (x_0, y_0) be arbitrary points in $X \times X$ and let $\{fx_n\}$ and $\{fy_n\}$ be coupled (f,T)-sequences with initial point (x_0, y_0). Since T is coupled asymptotically f-regular mapping at (x_0, y_0), therefore $d(fx_n, fx_{n+1}) \to 0$ and $d(fy_n, fy_{n+1}) \to 0$ as $n \to \infty$. Now for $m > n$, we have

$$\begin{aligned} d(fx_n, fx_m) &\le d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{m+1}) + d(fx_{m+1}, fx_m) \\ &= d(fx_n, fx_{n+1}) + d(T(x_n, y_n), T(x_m, y_m)) + d(fx_{m+1}, fx_m) \\ &\le d(fx_n, fx_{n+1}) + d(fx_{m+1}, fx_m) + a_1 F_1[\min\{d(fx_n, T(x_n, y_n)), d(fx_m, T(x_m, y_m))\}] \\ &\quad + a_2 F_2[d(fx_n, T(x_n y_n)) d(fx_m, T(x_m, y_m))] + a_3[d(fx_n, fx_m) + d(fy_n, fy_m)] \\ &\quad + a_4[d(fx_n, T(x_n, y_n)) + d(fx_m, T(x_m, y_m))] + a_5[d(fx_n, T(x_m, y_m)) + d(fx_m, T(x_n, y_n))] \\ &= d(fx_n, fx_{n+1}) + d(fx_{m+1}, fx_m) + a_1 F_1[\min\{d(fx_n, fx_{n+1}), d(fx_m, fx_{m+1})\}] \\ &\quad + a_2 F_2[d(fx_n, fx_{n+1}) d(fx_m, fx_{m+1})] + a_3[d(fx_n, fx_m) + d(fy_n, fy_m)] \\ &\quad + a_4[d(fx_n, fx_{n+1}) + d(fx_m, fx_{m+1})] + a_5[d(fx_n, fx_{m+1}) + d(fx_m, fx_{n+1})] \\ &\le d(fx_n, fx_{n+1}) + d(fx_{m+1}, fx_m) + a_1 F_1[\min\{d(fx_n, fx_{n+1}), d(fx_m, fx_{m+1})\}] \\ &\quad + a_2 F_2[d(fx_n, fx_{n+1}) d(fx_m, fx_{m+1})] + a_3[d(fx_n, fx_m) + d(fy_n, fy_m)] \\ &\quad + a_4[d(fx_n, fx_{n+1}) + d(fx_m, fx_{m+1})] \\ &\quad + a_5[d(fx_n, fx_m) + d(fx_m, fx_{m+1}) + d(fx_m, fx_n) + d(fx_n, fx_{n+1})] \\ &= (1 + a_4 + a_5)[d(fx_n, fx_{n+1}) + d(fx_{m+1}, fx_m)] \\ &\quad + a_1 F_1[\min\{d(fx_n, fx_{n+1}), d(fx_m, fx_{m+1})\}] + a_2 F_2[d(fx_n, fx_{n+1}) d(fx_m, fx_{m+1})] \end{aligned}$$

$$
\begin{aligned}
&+(a_3+2a_5)d(fx_n,fx_m)+a_3d(fy_n,fy_m)\\
&\le(1+\lambda_1)[d(fx_n,fx_{n+1})+d(fx_{m+1},fx_m)]+kF_1[\min\{d(fx_n,fx_{n+1}),d(fx_m,fx_{m+1})\}]\\
&+kF_2[d(x_n,fx_{n+1})d(fx_m,fx_{m+1})]+(a_3+2a_5)d(fx_n,fx_m)+a_3d(fy_n,fy_m)].
\end{aligned}
$$

Thus, we obtain that

$$
\begin{aligned}
d(fx_n,fx_m)\le{}&(1+\lambda_1)[d(fx_n,fx_{n+1})+d(fx_{m+1},fx_m)]+kF_1[\min\{d(fx_n,fx_{n+1}),d(fx_m,fx_{m+1})\}]\\
&+kF_2[d(fx_n,fx_{n+1})d(fx_m,fx_{m+1})]+(a_3+2a_5)d(fx_n,fx_m)+a_3d(fy_n,fy_m).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
d(fy_n,fy_m)\le{}&(1+\lambda_1)[d(fy_n,fy_{n+1})+d(fy_{m+1},fy_m)]+kF_1[\min\{d(fy_n,fy_{n+1}),d(fy_m,fy_{m+1})\}]\\
&+kF_2[d(fy_n,fy_{n+1})d(fy_m,fy_{m+1})]+(a_3+2a_5)d(fy_n,fy_m)+a_3d(fx_n,fx_m).
\end{aligned}
$$

Adding the above two inequalities, we get

$$
\begin{aligned}
d(fx_n,fx_m)+d(fy_n,fy_m)\le{}&(1+\lambda_1)[d(fy_n,fy_{n+1})+d(fy_{m+1},fy_m)]\\
&+(1+\lambda_1)[d(fx_n,fx_{n+1})+d(fx_{m+1},fx_m)]\\
&+kF_1[\min\{d(fx_n,fx_{n+1}),d(fx_m,fx_{m+1})\}]\\
&+kF_1[\min\{d(fy_n,fy_{n+1}),d(fy_m,fy_{m+1})\}]\\
&+kF_2[d(fx_n,fx_{n+1})d(fx_m,fx_{m+1})]\\
&+kF_2[d(fy_n,fy_{n+1})d(fy_m,fy_{m+1})]\\
&+(2a_3+2a_5)(d(fy_n,fy_m)+d(fx_n,fx_m)).
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
(1-\lambda_2)d(fx_n,fx_m)+d(fy_n,fy_m)\le{}&(1+\lambda_1)[d(fy_n,fy_{n+1})+d(fy_{m+1},fy_m)]\\
&+(1+\lambda_1)[d(fx_n,fx_{n+1})+d(fx_{m+1},fx_m)]\\
&+kF_1[\min\{d(fx_n,fx_{n+1}),d(fx_m,fx_{m+1})\}]\\
&+kF_1[\min\{d(fy_n,fy_{n+1}),d(fy_m,fy_{m+1})\}]\\
&+kF_2[d(fx_n,fx_{n+1})d(fx_m,fx_{m+1})]\\
&+kF_2[d(fy_n,fy_{n+1})d(fy_m,fy_{m+1})].
\end{aligned}
$$

As T is a coupled asymptotically f-regular and F_1 and F_2 are continuous at zero, so the right-hand side of the above inequality tends to zero, as $m,n\to\infty$. Thus,

$$\lim_{m,n\to\infty} d(fx_n,fx_m)=0,$$

and

$$\lim_{m,n\to\infty} d(fy_n,fy_m)=0.$$

It follows that $\{fx_n\}$ and $\{fx_n\}$ are Cauchy sequences in $f(X)$. Since $f(X)$ is a complete subspace of X, so there exist $u,p,v,q\in X$ such that

$$fx_n\to p=fu,\quad fy_n\to q=fv.$$

Now, we claim that (u,v) is a coupled coincidence point of f and T.

From (2.1), we obtain

$$
\begin{aligned}
d(fu,T(u,v)) = d(p,T(u,v)) &\le d(p,fx_{n+1}) + d(fx_{n+1},T(u,v)) \\
&= d(p,fx_{n+1}) + d(T(x_n,y_n),T(u,v)) \\
&\le d(p,fx_{n+1}) + a_1F_1[\min\{d(fx_n,T(x_n,y_n)),d(fu,T(u,v))\}] \\
&\quad + a_2F_2[d(fx_n,T(x_n,y_n))d(fu,T(u,v))] + a_3[d(fx_n,fu)+d(fy_n,fv)] \\
&\quad + a_4[d(fx_n,T(x_n,y_n))+d(fu,T(u,v))] + a_5[d(fx_n,T(u,v))+d(fu,T(x_n,y_n))] \\
&\le d(p,fx_{n+1}) + a_1F_1[\min\{d(fx_n,fx_{n+1}),d(p,T(u,v))\}] \\
&\quad + a_2F_2[d(fx_n,fx_{n+1})d(p,T(u,v))] + a_3[d(fx_n,p)+d(fy_n,q)] \\
&\quad + a_4[d(fx_n,fx_{n+1})+d(p,T(u,v))] + a_5[d(fx_n,p)+d(p,T(u,v))+d(p,fx_{n+1})] \\
&= (1+a_5)d(p,fx_{n+1}) + a_1F_1[\min\{d(fx_n,fx_{n+1}),d(p,T(u,v))\}] \\
&\quad + a_2F_2[d(fx_n,fx_{n+1})d(p,T(u,v))] + (a_3+a_5)d(fx_n,p) + a_3(fy_n,q) \\
&\quad + (a_4+a_5)d(p,T(u,v)) + a_5[d(fx_n,fx_{n+1})] \\
&\le (1+a_5)d(p,fx_{n+1}) + a_1F_1[\min\{d(fx_n,fx_{n+1}),d(p,T(u,v))\}] \\
&\quad + a_2F_2[d(fx_n,fx_{n+1})d(p,T(u,v))] + (a_3+a_5)d(fx_n,p) + a_3(fy_n,q) \\
&\quad + (a_4+a_5)d(p,T(u,v)) + a_5[d(fx_n,fx_{n+1})]
\end{aligned}
$$

which on taking limit as $n \to \infty$ gives that

$$d(p,T(u,v)) \le (a_4+a_5)d(p,T(u,v)),$$

which is true only if $d(p,T(u,v)) = 0$, and so $p = fu = T(u,v)$.

Similarly, one can get $q = fv = T(v,u)$. Hence (u,v) is a coupled coincidence point of f and T. Also, $(p,q) = (fu,fv)$ is a coupled point of coincidence of f and T. □

Lemma 2.2. *Let $(X,\,d)$ be a metric space, $T : X \times X \to X$ and $f : X \to X$ be two mappings such that $T(X \times X) \subset f(X)$. Assume that the following condition holds:*

$$
\begin{aligned}
d(T(x,y),T(u,v)) \le\ & a_1F_1[\min\{d(fx,T(x,y)),d(fu,T(u,v))\}] \\
& + a_2F_2[d(fx,T(x,y))d(fu,T(u,v))] + a_3[d(fx,fu)+d(fy,fv)] \\
& + a_4[d(fx,T(x,y))+d(fu,T(u,v))] \\
& + a_5[d(fx,T(u,v))+d(fu,T(x,y))]
\end{aligned}
$$

for all $x,y,u,v \in X$, where for $i = 1,\dots,5$, $a_i \ge 0$ such that for arbitrary fixed $k > 0$, $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$, we have $a_4 + a_5 \le \lambda_1$, $2a_3 + 2a_5 \le \lambda_2$ and $a_1, a_2 \le k$. Then, T and f have at most a unique coupled point of coincidence.

Proof. Assume that there exist points p,q,p^*,q^* in X such that $(p,q) = (fu,fv) = (T(u,v),T(v,u))$ and $(p^*,p^*) = (fu^*,fv^*) = (T(u^*,v^*),T(v^*,u^*))$, for some u,v,u^*,v^* in X.

Consider

$$\begin{aligned}
d(p,p^*) &= d(T(u,v),T(u^*,v^*)) \\
&\le a_1F_1[\min\{d(fu,T(u,v)),d(fu^*,T(u^*,v^*))\}]+a_2F_2[d(fu,T(u,v))d(fu^*,T(u^*,v^*))] \\
&\quad +a_3[d(fu,fu^*)+d(fv,fv^*)]+a_4[d(fu,T(u,v))+d(fu^*,T(u^*,v^*))] \\
&\quad +a_5[d(fu,T(u^*,v^*))+d(fu^*,T(u,v))] \\
&= a_3d(p,p^*)+a_5[d(p,p^*)+d(p^*,p)] \\
&= (a_3+2a_5)d(p,p^*)+a_3d(q,q^*),
\end{aligned}$$

Similarly, we get that

$$d(q,q^*)\le(a_3+2a_5)d(q,q^*)+a_3d(p,p^*).$$

Adding above two inequalities give

$$\begin{aligned}
d(p,p^*)+d(q,q^*) &\le (2a_3+2a_5)(d(p,p^*)+d(q,q^*)) \\
&\le \lambda_2(d(p,p^*)+d(q,q^*)),
\end{aligned}$$

which is true only if $d(p,p^*)+d(q,q^*)=0$. We deduce that $(p,q)=(p^*,p^*)$, that is, f and T have a unique coupled point of coincidence. □

From Theorem 2.1, and Lemmas 2.1 and 2.2, we obtain the following result.

Theorem 2.2. *Let $(X,\ d)$ be a metric space. Let $T:X\times X\to X$ and $f:X\to X$ be such that $T(X\times X)\subset f(X)$. Assume that T and f satisfy condition (2.1) for all $x,y\in X$. If $f(X)$ is a complete subspace of X and the pair (T,f) is w-compatible, then T and f have a unique common coupled fixed point provided that T is coupled asymptotically f-regular at some point (x_0,y_0) in $X\times X$.*

Proof. By Theorem 2.1 and Lemma 2.2, the mappings T and f have a unique coupled point of coincidence. Since the pair (T,f) is w-compatible, by Lemma 2.1, T and f have a unique common coupled fixed point. □

Now we give two examples to support our results.

Example 2.2. Let $X=[0,+\infty)$ be endowed with the usual metric. Define $f:X\to X$ and $T:X\times X\to X$ by

$$T(x+y)=\frac{x+y}{4}\quad\text{and}\quad fx=x.$$

Let $x_0=y_0=1$. We go back to Example 2.1 to say that $T(X\times X)\subset f(X)$ and T is a coupled asymptotically f-regular at the point (x_0,y_0). If $fx=F(x,y)$ and $fy=F(y,x)$, then $x=y=0$, so $T(fx,fy)=fT(x,y)$, that is the pair (T,f) is w-compatible. Also, the inequality (2.1) holds for all $x,y,u,v\in X$ with

$$a_1=a_2=a_4=a_5=0\quad\text{and}\quad a_3=\frac{1}{2}.$$

Thus, $f,T : X \to X$ satisfy all conditions of Theorem 2.2. Moreover, $(0,0)$ is the unique common coupled fixed point of f and T.

On the other hand, following [1, Example 2.9], we state:

Example 2.3. Let $X = [0,+\infty)$ be endowed with the usual metric. Define $f : X \to X$ and $T : X \times X \to X$ by

$$fx = \begin{cases} 3x & \text{if } x \in [0,1] \\ 2x & \text{if } x > 1 \end{cases} \quad \text{and} \quad T(x,y) = \begin{cases} \frac{x+y}{3} & \text{if } x \in [0,1],\ y \in \mathbb{R} \\ \frac{x+y}{4} & \text{if } x > 1,\ y \in \mathbb{R} \end{cases}$$

Let $x_0 = y_0 = 1$. It is clear that $T(X \times X) \subset f(X)$. Also, a simple calculation yields that T is coupled asymptotically f-regular at the point (x_0, y_0). If $fx = F(x,y)$ and $fy = F(y,x)$, then necessarily, $x = y = 0$, so $T(fx, fy) = fT(x,y)$, that is the pair (T,f) is w-compatible. Moreover, it is easy to verify that the inequality (2.1) holds for all $x,y,u,v \in X$ with

$$a_1 = a_2 = a_5 = 0, \quad a_3 = \frac{2}{5} \quad \text{and} \quad a_4 = \frac{1}{10}.$$

Thus, $f,T : X \to X$ satisfy all conditions of Theorem 2.2. Moreover, $(0,0)$ is the unique common coupled fixed point of f and T.

Now we state a common coupled fixed point theorem type of Abbas et al. [1] (which holds on cone metric spaces). The following corresponds to $a_1 = a_2 = 0$ in the inequality (2.1).

Corollary 2.1. *Let (X, d) be a metric space. Let $T : X \times X \to X$ and $f : X \to X$ be such that $T(X \times X) \subset f(X)$. Assume that the following condition holds:*

$$\begin{aligned} d(T(x,y),T(u,v)) \le{}& a_3[d(fx,fu)+d(fy,fv)] + a_4[d(fx,T(x,y))+d(fu,T(u,v))] \\ &+ a_5[d(fx,T(u,v))+d(fu,T(x,y))] \end{aligned} \tag{2.2}$$

for all $x,y,u,v \in X$, where for $i = 3,\dots,5$, $a_i \ge 0$ such that for arbitrary fixed $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$, we have $a_4 + a_5 \le \lambda_1$ and $2a_3 + 2a_5 \le \lambda_2$. Suppose that $f(X)$ is a complete subspace of X and T is coupled asymptotically $f-$regular at some point (x_0, y_0) in X. If the pair (T,f) is w-compatible, then T and f have a unique common coupled fixed point.

Remark 2.1. Corollary 2.1 generalizes Theorem 2.4 of Abbas et al. [1], that is, even with weaker hypotheses in Corollary 2.1 than those given in Theorem 2.4 in [1], we prove a common coupled fixed point result. In fact, if (2.2) holds with the assumption $a_3 + 2a_4 + 2a_5 < 1$ and $(T(X \times X) \subset f(X)$, we have got through Theorem 2.4 of [1] that $d(fx_n, fx_{n+1}) \to 0$ and $d(fy_n, fy_{n+1}) \to 0$ as $n \to \infty$, where $\{fx_n\}$ and $\{fy_n\}$ are coupled (f,T)-sequences with some initial point (x_0, y_0). Then, T is coupled asymptotically $f-$regular at the point (x_0, y_0). Also, there exist arbitrary fixed $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$ such that $a_4 + a_5 \le \lambda_1$ and $2a_3 + 2a_5 \le \lambda_2$. Thus, the hypotheses of Corollary 2.1 are verified.

As a consequence of Theorem 2.1, Lemma 2.2 and Theorem 2.2, we obtain the following result.

Corollary 2.2. *Let (X, d) be a metric space. Let $S, f : X \to X$ be such that $S(X) \subset f(X)$. Assume that the following condition holds:*

$$d(Sx,Su) \le a_1F_1[\min\{d(fx,Sx),d(fu,Su)\}] + a_2F_2[d(fx,Sx)d(fu,Su)] + a_3d(fx,fu) + a_4[d(fx,Sx)+d(fu,Su)] + a_5[d(fx,Su)+d(fu,Sx)] \tag{2.3}$$

for all $x,u \in X$, where for $i = 1,\dots,5$, $a_i \ge 0$ such that for arbitrary fixed $k > 0$, $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$, we have $a_4 + a_5 \le \lambda_1$, $a_3 + 2a_5 \le \lambda_2$ and $a_1, a_2 \le k$. Suppose that $f(X)$ is a complete subspace of X and S is asymptotically f-regular at some points x_0 and y_0 in X. If the pair (S, f) is weakly compatible, then S and f have a unique common fixed point.

Proof. Take $T : X \times X \to X$ defined by $T(x,y) = Sx$ for all $x, y \in X$. We will check that all the hypotheses of Theorem 2.2 are satisfied. For this, let $(x,y) \in X \times X$. Since $S(X) \subset f(X)$, there exists $u \in X$ such that $fu = Sx = T(x,y)$, so $T(X \times X) \subset f(X)$. Let $\{x_n\}$ and $\{y_n\}$ be such that

$$fx_{n+1} = T(x_n,y_n) \text{ and } fy_{n+1} = T(y_n,x_n), \quad n = 0,1,2,\dots,$$

that is, the sequence $\{fx_n\}$ and $\{fy_n\}$ are coupled $(f,T)-$sequences with initial point (x_0, y_0). We have $fx_{n+1} = Sx_n$ and $fy_{n+1} = Sy_n$, so $\{fx_n\}$ and $\{fy_n\}$ are Ssequences with initial points x_0 and y_0, respectively. Since S is asymptotically $f-$regular at the point x_0 and y_0, we get

$$d(fx_n, fx_{n+1}) \to 0, \quad d(fy_n, fy_{n+1}) \to 0,$$

so T is coupled asymptotically $f-$regular at the point (x_0, y_0). Finally, note that (T, f) is w-compatible. Indeed, for $(x,y) \in X \times X$ with $fx = T(x,y)$ and $fy = T(y,x)$ implies that $fx = Sx$ and $fy = Sy$. Since the pair (S, f) is weakly compatible, we get that

$$fT(x,y) := fSx = Sfx = T(fx, fy),$$

that is, the pair (T, f) is w-compatible. From (2.3), it is obvious that (2.1) holds for all $x, y, u, v \in X$.

Now, all the hypotheses of Theorem 2.2 are satisfied. Therefore f and T have a unique common coupled fixed point $(u,v) \in X \times X$, that is, $T(u,v) = fu = u$ and $T(v,u) = fv = v$. Hence

$$Su = T(u,u) = fu = u. \qquad \square$$

Remark 2.2. Corollary 2.2 generalizes Theorem 3.1 of Olaleru [39]. Our hypotheses are weaker than those of Olaleru [39] by using the arguments similar to those in Remark 2.1.

Remark 2.3. Corollary 2.2 remains true if we replace the real numbers a_i by real functions $a_i(x,u)$ for $x,u \in X$ and $i = 1,\dots,5$.

From Corollary 2.2, we have the following result of Ćirić [18], by taking $f = I_X$.

Corollary 2.3. *Let (X, d) be a metric space. Let $S : X \to X$ be such that the following condition holds:*

$$d(Sx,Su) \le a_1F_1[\min\{d(x,Sx),d(u,Su)\}] + a_2F_2[d(x,Sx)d(u,Su)] + a_3d(x,u) + a_4[d(x,Sx)+d(u,Su)] + a_5[d(x,Su)+d(u,Sx)] \tag{2.4}$$

for all $x,u \in X$, where for $i = 1,\dots,5$, $a_i \geq 0$ such that for arbitrary fixed $k > 0$, $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$, we have $a_4 + a_5 \leq \lambda_1$, $a_3 + 2a_5 \leq \lambda_2$ and $a_1, a_2 \leq k$. If S is asymptotically regular at some points x_0, y_0 in X, then S has a (unique) fixed point.

Remark 2.4. The fact that S is asymptotically regular at some $x_0 \in X$ corresponds to S is asymptotically I_X-regular at some $x_0 \in X$. Our results extend Theorem 1 in [17] and in turn extend and generalize results of Sharma and Yuel [46], Guay and Singh [22] and Babu, Sandhya and Kameswari [14] (of course when the constants $(a_i)_{i=1,\dots,5}$ are taken real functions $a_i(x,u)$).

Now, we have the following result on the continuity on the set of common coupled fixed points. Let $CCF(T,f)$ denote the set of all common coupled fixed points of T and f.

Theorem 2.3. *Let (X, d) be a metric space. Assume that $f : X \to X$ and $T : X \times X \to X$ satisfy condition (2.1) for all $x,y \in X$. If $CCF(T,f) \neq \emptyset$, then T is continuous at $(p,q) \in CCF(T,f)$ whenever f is continuous at p and q.*

Proof. Fix$(p,q) \in CCF(T,f)$. Let (z_n,w_n) be any sequence in $X \times X$ converging to (p,q) (that is, $d(z_n,p) \to 0$ and $d(w_n,q) \to 0$). Then by taking $(u,v) := (z_n,w_n)$ and $(x,y) := (p,q)$ in (2.1), we get

$$\begin{aligned}
d(T(p,q),T(z_n,w_n)) \leq\, & a_1F_1[\min\{d(fp,T(p,q)),d(fz_z,T(z_n,w_n))\}] \\
& + a_2F_2[d(fp,T(p,q))d(fz_n,T(z_n,w_n))] + a_3[d(fp,fz_n) + d(fq,fw_n)] \\
& + a_4[d(fp,T(p,q)) + d(fz_n,T(z_n,w_n))] + a_5[d(fp,T(z_n,w_n)) + d(fz_n,T(p,q))]
\end{aligned}$$

which, in view of $T(p,q) = fp = p$ implies that

$$\begin{aligned}
d(T(p,q),T(z_n,w_n)) &\leq a_3d(fp,fz_n) + a_4[d(fz_n,T(z_n,w_n))] + a_5[d(fp,T(z_n,w_n)) + d(fz_n,T(p,q))] \\
&\leq a_3[d(fp,fz_n) + d(fq,fw_n)] + a_4[d(fz_n,fp)] + a_4[d(T(p,q),T(z_n,w_n))] \\
&\quad + a_5[d(T(p,q),T(z_n,w_n)) + d(fz_n,fp)].
\end{aligned}$$

Now, by letting $n \to \infty$, we get

$$\begin{aligned}
\limsup_{n\to\infty} d(T(p,q),T(z_n,w_n)) &\leq (a_4 + a_5)\limsup_{n\to\infty} d(T(p,q),T(z_n,w_n)) \\
&\leq \lambda_1 \limsup_{n\to\infty} d(T(p,q),T(z_n,w_n)),
\end{aligned}$$

whenever f is continuous at p and q. The last inequality is true only if $\limsup_{n\to\infty} d(T(p,q),T(z_n,w_n)) = 0$. We get that $T(z_n,w_n) \to T(p,q)$ as $n \to \infty$. □

3. Well-Posedness

The notion of well-posedness of a fixed point problem has evoked much interest to several mathematicians, for example, De Blassi and Myjak [20], Reich and Zaslavski [43], Lahiri and Das [33] and Popa [37, 38]. Recently, Karapinar [30] studied well-posed problem for a cyclic weak $\phi-$contraction mapping on a complete metric space (see also, [36]). Here, we study the well-posedness of a common coupled fixed point problem.

Definition 3.1. A common coupled fixed point problem of maps $f : X \to W$ and $T : X \times X \to X$, $CCFP(f,T,X)$, is called well-posed if $CCF(f,T)$ (the set of common coupled fixed points of f and T) is singleton in $X \times X$ and for sequence $\{x_n\}$ and $\{y_n\}$ in X with $(x^*, y^*) \in CCF(S,T)$ such that $\lim_{n\to\infty} d(fx_n, x_n) = \lim_{n\to\infty}(T(x_n, y_n), x_n) = 0$ and $\lim_{n\to\infty} d(fy_n, y_n) = \lim_{n\to\infty}(T(y_n, x_n), y_n) = 0$ implies that $(x^*, y^*) = \lim_{n\to\infty}(x_n, y_n)$.

Theorem 3.1. *Suppose that T and f be two maps as in Theorem 2.1 and lemma 2.2. Then, the common coupled fixed point problem of f and T is well-posed.*

Proof. From Theorem 2.1 and Lemma 2.2, the mappings f and T have a unique common coupled fixed point, say $(u,v) \in X \times X$. Let $\{x_n, y_n\}$ be a sequence in $X \times X$ such that $\lim_{n\to\infty} d(fx_n, x_n) = \lim_{n\to\infty}(Tx_n, x_n) = 0$ and $\lim_{n\to\infty} d(fy_n, y_n) = \lim_{n\to\infty}(T(y_n, x_n), y_n) = 0$. With any loss of generality, we may suppose that $(u,v) \neq (x_n, y_n)$ for every non-negative integer n. From (2.1), we have

$$\begin{aligned}
d(u,x_n) = d(T(u,v),x_n) &\le d(T(x_n,y_n),T(u,v)) + d(T(x_n,y_n),x_n)\\
&\le d(T(x_n,y_n),x_n) + a_1F_1[\min\{d(fx_n,T(x_n,y_n)), d(fu,T(u,v))\}]\\
&\quad + a_2F_2[d(fx_n,T(x_n,y_n))d(fu,T(u,v))]\\
&\quad + a_3[d(fx_n,fu) + d(fy_n,fv)] + a_4[d(fx_n,T(x_n,y_n)) + d(fu,T(u,v))]\\
&\quad + a_5[d(fx_n,T(u,v)) + d(fu,T(x_n,y_n))]\\
&= d(T(x_n,y_n),x_n) + a_3[d(fx_n,u) + d(fy_n,v)] + a_4[d(fx_n,T(x_n,y_n))]\\
&\quad + a_5[d(fx_n,u) + d(u,T(x_n,y_n))]\\
&\le d(T(x_n,y_n),x_n) + a_3d(fx_n,x_n) + a_3d(x_n,u) + a_3d(fy_n,y_n) + a_3d(y_n,v)\\
&\quad + a_4[d(fx_n,x_n)] + a_4[d(x_n,T(x_n,y_n))]\\
&\quad + a_5[d(fx_n,x_n) + d(x_n,u) + d(u,x_n) + d(x_n,T(x_n,y_n))].
\end{aligned}$$

Letting $n \to \infty$, we get

$$\limsup_{n\to\infty} d(u,x_n) \le (a_3 + 2a_5)\limsup_{n\to\infty} d(u,x_n) + a_3 \limsup_{n\to\infty} d(v,y_n).$$

Similarly, we have

$$\limsup_{n\to\infty} d(v,y_n) \le (a_3 + 2a_5)\limsup_{n\to\infty} d(u,x_n) + a_3 \limsup_{n\to\infty} d(v,y_n).$$

A summation of the above two inequalities yields that

$$[\limsup_{n\to\infty} d(u,x_n) + \limsup_{n\to\infty} d(v,y_n)] \le (2a_3 + 2a_5)[\limsup_{n\to\infty} d(u,x_n) + \limsup_{n\to\infty} d(v,y_n)],$$

which holds only if $\limsup_{n\to\infty} d(u,x_n) = \limsup_{n\to\infty} d(v,y_n) = 0$. Hence $x_n \to u$ and $y_n \to v$ as $n \to \infty$. □

4. Conclusion

We used the concept of coupled asymptotical f-regularly at some coupled point to establish some common coupled fixed point theorems for mappings $T : X \times X \to X$ and $g : X \to X$. We also studied the well posedness of a common coupled fixed point problem. Our presented results are generalizations of many known results in literature.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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