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New Common Coupled Coincidence Point Theorems for Generalized Weakly Contraction Mappings with Applications to Dynamic Programming

Research Article

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Abstract. In this paper, we define a new concept of generalized weakly contraction mapping for coupled common fixed points in the space of the bounded function. We also prove the existence and uniqueness theorems for common coupled fixed points. As an application of our result, we also study the problem of existence and uniqueness of solutions for a class of system of functional equations which appears in dynamic programming.

Keywords. Common coupled coincidence points; Coupled coincidence point; Weakly contraction mappings; Dynamic programming

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1. Introduction and Preliminaries

The study of common and coupled fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity [2, 24, 27]. In 2006, Mustafa and Sims [28] have generalized and studied the concept of a generalized metric space. Based on the notion of generalized metric spaces, Mustafa et al. [29] obtained some fixed point theorems for mappings satisfying different contractive conditions. Abbas and Rhoades [1] initiated the study of a common fixed point theorem in generalized metric spaces. Recently, Shatanawi [32], Sintunavarat and Kumam [19, 32, 35] generalized coupled fixed point theorems, coupled coincidence and coupled common fixed point theorems for weak contraction mappings in partially ordered metric spaces. The pivotal result of analysis on Banach principle contraction. Its significance lies in its vast applicability in a great number of branches of mathematics and other sciences.

Bhashkar and Lakshmikantham [19] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ (a nonempty set) and established some coupled fixed point theorems in partially ordered complete metric spaces. Later, Lakshmikantham and Ćirić [25] proved coupled coincidence and coupled common fixed point results for nonlinear mappings $F : X \times X \to X$ and $g : X \times X$ satisfying certain contractive conditions in partially ordered complete metric spaces. In 2010, Abbas et al. [3] proved coupled coincidence and coupled common fixed point results in cone metric spaces for w-compatible mappings.

The aim of this paper is to prove coupled coincidence point and coupled common fixed points results for generalized weakly mappings. Consider a nonempty S and by $B(S)$ we denote the set of all bounded real functions defined on S and generalizations of the above principle have been objects in a lot of papers appearing in the literature. Particularly, one of these generalizations is due to Rhoades [30] and uses weakly contractive mappings. Before presenting the definition of this class of mappings, we introduce the class $\mathscr{A}$ of functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ which are nondecreasing such that $\varphi(t) = 0$ if and only if $t = 0$.

Definition 1.1. Let (X, d) be a metric space and let $T : X \to X$ be a mapping. We say that T is weakly contractive if, for any $x, y \in X$

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \tag{1}$$

where $\varphi \in \mathscr{A}$.

The following fixed point theorem which appeared in [30] will be a crucial tool in our study.

Theorem 1.2. *If $T : X \to X$ is a weakly contractive mapping, where (X, d) is a complete metric space, then T has a unique fixed point.*

Introducing a new generalization of contraction principle, Dutta and Choudhury [22] proved the following theorem.

Theorem 1.3 ([22])**.** *Let (X,d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(Tx,Ty)) \le \psi(d(x,y)) - \varphi(d(x,y)), \tag{2}$$

where $\psi,\varphi : [0,\infty) \to [0,\infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Definition 1.4. A point x in X is a coincidence point (common fixed point) of f and T if $f(x) = T(x)(f(x) = T(x) = x)$.

Definition 1.5. Let f,g be two compatible mappings on X. If $f(x) = g(x)$ for some x in X, then $fg(x) = gf(x)$.

Theorem 1.6. *Let (X,d) be a metric space and let T be a weakly contractive mapping with respect to f. If the range of f contains the range of T and $f(X)$ is a complete subspace of X, then f and T have coincidence point in X.*

Lemma 1.7. *Let f,g be two compatible mappings on X. If $f(x) = g(x)$ for some x in X, then $fg(x) = gf(x)$.*

Beg and Abbas [15] proved the coincidence point result for two mappings which generalized weak contraction condition. They obtained common fixed point theorem for a pair of weakly compatible maps.

Definition 1.8. The mappings $F : X \times X \to X$ and $g : X \times X$ are called w-compatible if $g(F(x,y)) = F(gx,gy)$ whenever $g(x) = F(x,y)$ and $g(y) = F(y,x)$.

Definition 1.9 ([15])**.** Let X be a metric space. A mapping $T : X \to X$ is called *weakly contractive with respect to* $f : X \to X$ if for each $x,y \in X$

$$d(Tx,Ty) \le d(fx,fy) - \varphi(d(fx,fy)), \tag{3}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing such that φ is positive on $(0,\infty), \varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Lemma 1.10 ([23])**.** *Let X be a nonempty set and $g : X \to X$ be a mapping. Then, there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one.*

Theorem 1.11. *Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping satisfying $d(T(x,y),T(u,v)) \le \varphi(d(fx,fu))$, for any $x,y,u,v \in X$, where φ is a comparison function and $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then f and T have a coupled coincidence point.*

Lemma 1.12. *Let (X,d) be a metric space and let T be a weakly contractive mapping with respect to f. If T and f are weakly compatible and $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then f and T have common fixed point in X.*

The main purpose of this paper is to introduce the concept of generalized coupled coincidence points and coupled common fixed points, to prove a result about the existence and uniqueness of these points and apply the result to a problem which appears in dynamic programming.

2. Main Results

In this section, we consider a nonempty set S and by $B(S)$, we will denote the set of all bounded real functions defined on S. With respect to the ordinary addition of functions and scalar multiplication, $B(S)$ is a real vector space on $\mathbb{R}$. In $B(S)$, we consider the classical norm as follows:

$$\|h\| = \sup_{x\in S}|h(x)|, \quad \text{for } h \in B(S),$$

and it is well known that $(B(S),\|.\|)$ is a Banach space.

Notice that the distance in $B(S)$ is given by

$$d(u,v) = \sup\{|u(x)-v(x)| : x \in S\}, \quad \text{for } u,v \in B(S).$$

Definition 2.1 ([32])**.** Let (X,d) be a nonempty set.

(i) An element $(x,y) \in X \times X$ is said to be a *coupled fixed point* of the mapping $F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$;

(ii) An element $(x,y) \in X \times X$ is said to be a *coupled coincidence fixed point* of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x,y) = g(x)$ and $F(y,x) = g(y)$, and (gx,gy) is called *coupled point of coincidence;*

(iii) An element $(x,y) \in X \times X$ is said to be a *coupled common fixed point* of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x,y) = g(x) = x$ and $F(y,x) = g(y) = y$;

(iv) The mappings $F : X \times X \to X$ and $g : X \to X$ are called *w-compatible* if $g(F(x,y)) = F(gx,gy)$, whenever $g(x) = F(x,y)$ and $g(y) = F(y,x)$.

Note that if (x,y) is a coupled common fixed point of F then (y,x) is coupled common fixed point of F too.

Next, we define the new concept in the following:

Definition 2.2. Suppose that $F : B(S) \times B(S) \to B(S)$ and $g,\alpha : B(S) \to B(S)$ are two mappings. An element $(u,v) \in B(S)\times B(S)$ is called an *α-coupled coincidence point* of F and g if $F(u,v) = g(u)$ and $F(\alpha(u),\alpha(v)) = g(\alpha(v))$ (and $F(v,u) = g(v)$ and $F(\alpha(v),\alpha(u)) = g(\alpha(u))$).

Definition 2.3. Suppose that $F : B(S) \times B(S) \to B(S)$ and $g,\alpha : B(S) \to B(S)$ are two mappings. An element $(u,v) \in B(S) \times B(S)$ is called an *α-common coupled coincidence point* of F and g if $F(u,v) = g(u) = u$ and $F(\alpha(u),\alpha(v)) = g(\alpha(v)) = v$ (and $F(v,u) = g(v) = v$ and $F(\alpha(v),\alpha(u)) = g(\alpha(u)) = u$).

Before presenting our main result, we need to introduce the class of functions $\mathscr{B}$ given by those functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ which are nondecreasing and such that $I - \varphi \in \mathscr{A}$, where I denotes the identity mapping on $\mathbb{R}^+$ and $\mathscr{A}$ is the class of functions introduced in Definition 1.7.

We are ready to present the main result of the paper which gives us a sufficient condition for the existence and uniqueness of an α-coupled fixed point.

2.1 Coupled Coincidence Point Result

Theorem 2.4. *Suppose that $F : B(S) \times B(S) \to B(S)$ and $\alpha, g : B(S) \to B(S)$ are non-expansive mappings. Assume that F satisfies*

$$d(F(x,y),F(u,v)) \le \varphi(\max(d(gx,gu),d(gy,gv))), \tag{4}$$

for any $x,y,u,v \in B(S)$, where $\varphi \in \mathscr{B}$. and assume that F,g are weakly compatible Then F has an α-coupled coincidence point.

Proof. Consider the cartesian product $B(S) \times B(S)$ endowed with the distance

$$\overline{d}((x,y),(u,v)) = \max(d(x,u),d(y,v)), \tag{5}$$

for any $(x,y),(u,v) \in B(S) \times B(S)$. It is known that $(B(S) \times B(S), \overline{d})$ is a complete metric space.

Now, we consider the mapping $\overline{F} : B(S) \times B(S) \to B(S) \times B(S)$ defined by

$$\overline{F}(x,y) = (F(x,y),F(\alpha(x),\alpha(y))), \tag{6}$$

where $\alpha : B(S) \to B(S)$.

Next, we check that $\overline{F}$ satisfies assumptions of Theorem 1.2, i.e., $\overline{F}$ is a weakly contractive mapping (eq.(3)) on $B(S) \times B(S)$.

In fact, taking into account our assumption, for any $x,y,u,v \in B(S)$, we have

$$\begin{aligned}
\overline{d}(\overline{F}(x,y),\overline{F}(u,v)) &= \overline{d}(F(x,y),F(\alpha(x),\alpha(y)),(F(u,v),F(\alpha(u),\alpha(v)))) && \text{(by (6))}\\
&= \max(d(F(x,y),F(u,v)),d(F(\alpha(x),\alpha(y)),F(\alpha(u),\alpha(v)))) && \text{(by (5))}\\
&\le \max(\varphi(\max(d(gx,gu),d(gy,gv))),\\
&\qquad \varphi(\max(d(g(\alpha(x)),g(\alpha(u))),d(g(\alpha(y)),g(\alpha(v)))))) && \text{(by (4))}.
\end{aligned}$$

Since the mapping α, g are both non-expansive with

$$\begin{aligned}
\max(d(g(\alpha(x)),g(\alpha(u))),d(g(\alpha(y)),g(\alpha(v)))) &\le \max(d(gx,gu),d(gy,gv))\\
&\le \max(d(x,u),d(y,v)),
\end{aligned}$$

and since φ is nondecreasing, we infer

$$\begin{aligned}
\overline{d}(\overline{F}(x,y),\overline{F}(u,v)) &\le (\varphi(\max(d(gx,gu),d(gy,gv))))\\
&= \max(d(gx,gu),d(gy,gv)) - (\max(d(gx,gu),d(gy,gv))\\
&\qquad - \varphi(\max(d(gx,gu),d(gy,gv)))).
\end{aligned}$$

Now, taking into account that $\varphi\in\mathscr{B}$ and, consequently, $I-\varphi\in\mathscr{A}$, from the last expression we obtain that $\overline{F}$ is a weakly contractive mapping with respect to g. By using Lemma 1.10 and Theorem 1.11, there exists $(x_0,y_0)\in B(S)\times B(S)$ such that $\overline{F}(x_0,y_0)=(g(x_0),g(y_0))=(x_0,y_0)$. Since

$$\overline{F}(x_0,y_0)=(g(x_0),g(y_0)):=(F(x_0,y_0),F(\alpha(x_0),\alpha(y_0)).$$

This, imply that

$$g(x_0)=F(x_0,y_0)\quad\text{and}\quad g(y_0)=F(\alpha(x_0),\alpha(y_0)).$$

Therefore F has an *α-coupled coincidence point*. This completes the proof. □

2.2 Common Coupled Coincidence Point Result

Theorem 2.5. *Suppose that $F:B(S)\times B(S)\to B(S)$ and $\alpha,g:B(S)\to B(S)$ are non-expansive mappings. Assume that F satisfies*

$$d(F(x,y),F(\alpha(u),\alpha(v))\le\varphi(\max(d(gx,gu),d(gy,gv))),$$

for any $x,y,u,v\in B(S)$, where $\varphi\in\mathscr{B}$, and F,g are weakly compatible and $F(B(S),B(S))\subseteq g(B(S))$ and $g(B(S))$ is a complete subspace of $B(S)$. Then F has an α-common coupled coincidence point.

Proof. By Theorem 2.4, Lemma 1.12 and *w-compatible*, then g and F have a coupled common fixed point, this means that

$$F(x_0,y_0)=g(x_0)=x_0,$$

$$F(\alpha(x_0),\alpha(y_0))=g(y_0)=y_0.$$

Again, by Lemma 1.8, we have $(x_0,y_0)\in B(S)$ and define

$$(g(x_0),g(y_0)):=\overline{F}(x_0,y_0)$$

such that

$$\overline{F}(x_0,y_0)=(g(x_0),g(y_0))=(\widehat{x}_0,\widehat{y}_0)\ \text{(say)}\ \text{ for some }\ (\widehat{x}_0,\widehat{y}_0)\in B(S)\times B(S).$$

By w-compatible, we get

$$g(\overline{F}(x_0,y_0))=\overline{F}(g(x_0),g(y_0)),$$

where

$$g(y_0)=\overline{F}(x_0,y_0),$$

$$g(x_0)=\overline{F}(x_0,y_0),$$

$$g(\widehat{x}_0,\widehat{y}_0)=\overline{F}(\widehat{x}_0,\widehat{y}_0).$$

We need to show that

$$g(\widehat{x}_0,\widehat{y}_0)=(\widehat{x}_0,\widehat{y}_0).$$

Assume that $g(\widehat{x}_0,\widehat{y}_0)\neq(\widehat{x}_0,\widehat{y}_0)$.
Then, we observe that

$$d(g(\widehat{x}_0,\widehat{y}_0),(\widehat{x}_0,\widehat{y}_0))=d(\overline{F}(\widehat{x}_0,\widehat{y}_0),(\widehat{x}_0,\widehat{y}_0)).$$

Since $\overline{F}$ is weakly contractive with respect to g, we compute

$$\begin{aligned} d(g(\widehat{x}_0,\widehat{y}_0),(\widehat{x}_0,\widehat{y}_0)) &= d(g(\widehat{x}_0,\widehat{y}_0),g(x_0,y_0)) - \varphi(d(g(\widehat{x}_0,\widehat{y}_0),g(x_0,y_0))) \\ &< d(g(\widehat{x}_0,\widehat{y}_0),g(x_0,y_0)) - \varphi(d(g(\widehat{x}_0,\widehat{y}_0),g(x_0,y_0))) \\ &\le d(g(\widehat{x}_0,\widehat{y}_0),g(x_0,y_0)) \\ &= d(g(\widehat{x}_0,\widehat{y}_0),(x_0,y_0)). \end{aligned}$$

So, we obtain

$$d(g(\widehat{x}_0,\widehat{y}_0),(\widehat{x}_0,\widehat{y}_0)) < d(g(\widehat{x}_0,\widehat{y}_0),(\widehat{x}_0,\widehat{y}_0))$$

which is a contradiction to the assumption. Therefore,

$$g(\widehat{x}_0,\widehat{y}_0) = (\widehat{x}_0,\widehat{y}_0).$$

That is

$$\overline{F}(\widehat{x}_0,\widehat{y}_0) = g(\widehat{x}_0,\widehat{y}_0) = (\widehat{x}_0,\widehat{y}_0).$$

Since

$$\overline{F}(\widehat{x}_0,\widehat{y}_0) := (g(\widehat{x}_0),g(\widehat{y}_0)) := (F(\widehat{x}_0,\widehat{y}_0),F(\alpha(\widehat{x}_0),\alpha(\widehat{y}_0))$$

and

$$\overline{F}(\widehat{x}_0,\widehat{y}_0) := (g(\widehat{x}_0),g(\widehat{y}_0)) = (\widehat{x}_0,\widehat{y}_0) := (F(\widehat{x}_0,\widehat{y}_0),F(\alpha(\widehat{x}_0),\alpha(\widehat{y}_0)),$$

we obtain that

$$(\widehat{x}_0) = g(\widehat{x}_0) = F(\widehat{x}_0,\widehat{y}_0) \text{ and } (\widehat{y}_0) \;=\; g(\widehat{y}_0) = F(\alpha(\widehat{x}_0),\alpha(\widehat{y}_0)).$$

Since F is a weakly contractive mapping with respect to g, then, there exists $(x_0,y_0) \in B(S)\times B(S)$ such that $F(x_0,y_0) = (x_0,y_0)$. This means that $F(x_0,y_0) = g(x_0) = x_0$ and $F(\alpha(\widehat{x}_0),\alpha(\widehat{y}_0)) = g(y_0) = y_0$ and therefore, (x_0,y_0) is a α-common coupled coincidence point of F. This completes the proof. □

2.3 Uniqueness

Theorem 2.6. *Suppose that $F : B(S)\times B(S) \to B(S)$ with respect to $g : B(S) \to B(S)$, F is a weakly contractive mapping with respect to g and w-compatible, where $(B(S),d)$ is a complete metric space. Then F and g have a unique coupled common fixed point.*

Proof. By using [34, Theorem 2.2], we can conclude that F and g have a coupled coincidence point (x,y). Moreover, if (x_0,y_0) is another coupled coincidence point of F and g, then

$$g(x) = g(x_0) \quad \text{and} \quad g(y) = g(y_0). \tag{7}$$

As F and g are w-compatible

$$g(g(x)) = g(F(x,y)) = F(g(x),g(y)) \quad \text{and} \quad g(g(y)) = g(F(y,x)) = F(g(y),g(x)). \tag{8}$$

Denote $g(x) = a$ and $g(y) = b$. Then, from (8), we have

$$g(a) = F(a,b) \quad \text{and} \quad g(b) = F(b,a) \tag{9}$$

which implies that (a,b) is a coupled coincidence point of F and g. From (7), we have $g(a)=g(x)$ and $g(b)=g(y)$ that is

$$g(a)=a \quad \text{and} \quad g(b)=b. \tag{10}$$

By (9) and (10), we have

$$a=g(a)=F(a,b) \quad \text{and} \quad b=g(b)=F(b,a). \tag{11}$$

Therefore (a,b) is a coupled common fixed point of F and g.

To prove the uniqueness, assume that (c,d) is another coupled common fixed point of F and g and then (c,d) is also coupled coincidence point of F and g. From (7), we get $c=g(c)=g(a)=a$ and $d=g(d)=g(b)=b$. Therefore (a,b) is a unique coupled common fixed point of F and g. □

3. Some Applications to Dynamic Programming

3.1 Coupled Coincidence Point for a System in Dynamic Programming

The following types of systems of functional equations

$$\begin{cases} f(u(x))=\sup\limits_{y\in D}\{g(x,y)+F(x,y,u(T(x,y)),v(T(x,y))\} \\ f(v(x))=\sup\limits_{y\in D}\{g(x,y)+F(x,y,u(\alpha(T(x,y))),v(\alpha(T(x,y))))\} \end{cases} \tag{12}$$

appear in the study of dynamic programming (see [5]), where $x\in S$ and S is a state space, D is a decision space, $T:S\times D\to S, g:S\times D\to\mathbb{R}, \alpha:B(S)\to B(S), f:B(S)\to B(S)$ and $F:S\times D\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given mappings.

For further information about the functional equations appearing in dynamic programming, we refer the reader to [16, 18, 21].

The following result gives us a sufficient condition for the existence and uniqueness of solutions to problem (12).

Theorem 3.1. *Suppose the following assumptions hold:*

(i) $g:S\times D\to\mathbb{R}$ *and* $F(-,-,0,0):S\times D\to\mathbb{R}$ *are bounded functions and* $\alpha:B(S)\to B(S)$ *is non-expansive.*

(ii) *There exists a function* $\varphi\in B$ *such that, for any* $x\in S, y\in D$ *and* $t,s,t_1,s_1\in\mathbb{R}$ *one has the inequality,*

$$|F(x,y,t,s)-F(x,y,t_1,s_1)|\le\varphi(\max(|t-t_1|,|s-s_1|)).$$

Then, problem (12) *has a unique solution* $(u_0,v_0)\in B(S)\times B(S)$.

For the proof of Theorem 3.1, we need the following lemma.

Lemma 3.2. *Suppose that* $H,G:S\to\mathbb{R}$ *are two bounded functions. Then*

$$\left|\sup_{y\in S}H(y)-\sup_{y\in S}G(y)\right|\le\sup_{y\in S}|H(y)-G(y)|.$$

Proof. It is clear that the result is true when $\sup_{y\in S}\{H(y)\} = \sup_{y\in S}\{G(y)\}$.

Suppose that $\sup_{y\in S}\{H(y)\} > \sup_{y\in S}\{G(y)\}$ (same argument works under the assumption $\sup_{y\in S}\{H(y)\} < \sup_{y\in S}\{G(y)\}$).

For any $y_0 \in S$, we have

$$H(y_0) - \sup_{y\in S}\{G(y)\} \le H(y_0) - G(y_0) \le |H(y_0) - G(y_0)|$$

and, therefore,

$$\sup_{y\in S}\{H(y) - \sup_{y\in S}\{G(y)\}\} \le \sup_{y\in S}\{|H(y) - G(y)|\}.$$

Since $\sup_{y\in S}\{H(y) - a\} = \sup_{y\in S}\{H(y)\} - a$, for any $a \in \mathbb{R}$, it follows

$$\sup_{y\in S}\{H(y)\} - \sup_{y\in S}\{G(y)\} \le \sup_{y\in S}\{|H(y) - G(y)|\}$$

and this proves our claim.

Proof of Theorem 3.1. Consider the operator G defined on $B(S)\times B(S)$ by

$$G(u,v)(x) = \sup_{y\in D}\{g(x,y) + F(x,y,u(T(x,y)),v(T(x,y)))\}$$

for any $u,v \in B(S)$ and $x \in S$.

Taking into account our assumptions, we get

$$\begin{aligned}
|G(u,v)(x)| &\le \sup_{y\in D}|g(x,y) + F(x,y,u(T(x,y)),v(T(x,y)))| \\
&\le \sup_{y\in D}|g(x,y)| + \sup_{y\in D}|F(x,y,u(T(x,y)),v(T(x,y)))| \\
&\le \sup_{y\in D}|g(x,y)| + \sup_{y\in D}\{|F(x,y,u(T(x,y)),v(T(x,y))) - F(x,y,0,0)| + |F(x,y,0,0)|\} \\
&\le \sup_{y\in D}|g(x,y)| + \sup_{y\in D}\{\varphi(\max(|u(T(x,y))|,|v(T(x,y))|)\} + \sup_{y\in D}\{|F(x,y,0,0|\} \\
&\le \sup_{y\in D}|g(x,y)| + \sup_{y\in D}\{\varphi(\max(\|u\|,\|v\|)\} + \sup_{y\in D}\{|F(x,y,0,0)|\} < \infty.
\end{aligned}$$

Therefore, $G: B(S)\times B(S) \to B(S)$.

Now, we check that G satisfies assumptions of Theorem 2.4.

In fact, for any $u,v,u_1,v_1 \in B(S)$, we have

$$\begin{aligned}
d(G(u,v),G(u_1,v_1)) &= \sup_{x\in S}|G(u,v)(x) - G(u_1,v_1)(x)| \\
&= \sup_{x\in S}|\sup_{y\in D}\{g(x,y) + F(x,y,u(T(x,y)),v(T(x,y)))\} \\
&\quad - \sup_{y\in D}\{g(x,y) + F(x,y,u_1(T(x,y)),v_1(T(x,y)))\}| \\
&\le \sup_{x\in S}\{\sup_{y\in D}\|F(x,y,u(T(x,y)),v(T(x,y))) - F(x,y,u_1(T(x,y)),v_1(T(x,y)))\|\}
\end{aligned}$$

$$\leq \sup_{x\in S}\{\sup_{y\in D}\{\varphi(\max\{|u(T(x,y))-u_1(T(x,y))|,|v(T(x,y))-v_1(T(x,y))|\})\}\}$$

$$\leq \sup_{x\in S}\{\varphi(\max(\|u-u_1\|,\|v-v_1\|))\}$$

$$\leq \varphi(\max(d(u,u_1),d(v,v_1))).$$

where we have used assumption (ii), Lemma 3.2 and the fact that φ is a nondecreasing function.

Therefore, the contractive condition appearing in Theorem 2.4 is satisfied and, since α is non-expansive, Theorem 2.4 gives us the existence and uniqueness of a α-coupled coincidence point for the mapping G, namely, there exists a unique $(u_0,v_0)\in B(S)\times B(S)$ such that $G(u_0,v_0)=f(u_0)$ and $G(\alpha(u_0),\alpha(v_0))=f(v_0)$.

This means that, for any $x\in S$,

$$f(u_0(x))=\sup_{y\in D}\{g(x,y)+F(x,y,u_0(T(x,y)),v_0(T(x,y)))\},$$

$$f(v_0(x))=\sup_{y\in D}\{g(x,y)+F(x,y,u_0(\alpha(T(x,y))),v_0(\alpha(T(x,y))))\}.$$

Therefore, we obtain the desired result. □

3.2 Existence and Uniqueness of Common Solution of System of Functional Equation in Dynamic Programming

We aim to give the existence and uniqueness of common and bounded solution of functional equations from problem (12).

Suppose that the following conditions hold:

(C1) G and g are bounded,

(C2) For $x\in S, h\in B(S)$ and $b>0$, define,

$$Kh(x)=\sup_{y\in D}\{g(x,y)+G(x,y,h(T(x,y)),k(T(x,y))), \tag{13}$$

$$Jk(x)=\sup_{y\in D}\{g(x,y)+G(x,y,h(\alpha(T(x,y))),k(\alpha(T(x,y)))), \tag{14}$$

where J,K are self-maps of $B(S)$.
Moreover assume that there exists a comparison function φ such that for every $(x,y)\in S\times D, h,k\in B(S)$ and $t\in S$ we have

$$|F(x,y,h(t),k(t))-F(x,y,h_1(t),k_1(t))|\leq\varphi(\max(|h(t)-h_1(t)|,|k(t)-k_1(t)|)), \tag{15}$$

and

$$|F(x,y,\alpha h(t),\alpha k(t))-F(x,y,\alpha h_1(t),\alpha k_1(t))|\leq\varphi(\max(|\alpha h(t)-\alpha h_1(t)|,|\alpha k(t)-\alpha k_1(t)|)). \tag{16}$$

(C3) For any $h\in B(S)$, there exists $k\in B(S)$ such that for $x\in S$,

$$Kh(x)=Jk(x).$$

(C4) There exists $h\in B(S)$ such that

$$Kh(x)=Jh(x) \quad \text{implies that} \quad JKh(x)=KJh(x).$$

Theorem 3.3. *Assume that the conditions* (C1)-(C4) *are satisfied. If* $J(B(S))$ *is a closed convex subspace of* $B(S)$*, then the functional equations* (12) *have a unique, common and bounded solution.*

Proof. Note that $(B(S),d)$ is a complete metric space. By (C1), J, K are self-maps of $B(S)$. The condition (C3) implies that $K(B(S)) \subseteq J(B(S))$. It follows from (C4) that J and K commute at their coincidence points. Let $h_1,h_2,k_1,k_2 \in B(S)$. Choose $x \in S$ and $y_1, y_2 \in D$ such that

$$Kh_j < g(x,y_j) + F(x,y_j,h_j(x_j),k(x_j)) \tag{17}$$

and

$$Jk_j < g(x,y_j) + F(x,y_j,\alpha h_j(x_j),\alpha k(x_j)), \tag{18}$$

where $x_j = T(x,y_j)$, $j = 1,2$.

Further from (13) and (14), we have

$$Kh_1 \le g(x,y_2) + F(x,y_2,h_1(x_2),k_1(x_2)), \tag{19}$$

$$Kh_2 \le g(x,y_1) + F(x,y_1,h_2(x_1),k_2(x_1)). \tag{20}$$

and

$$Jk_1 \le g(x,y_2) + F(x,y_2,\alpha h_1(x_2),\alpha k_1(x_2)), \tag{21}$$

$$Jk_2 \le g(x,y_1) + F(x,y_1,\alpha h_2(x_1),\alpha k_2(x_1)). \tag{22}$$

From (17) and (19) together with (15) it follows

$$\begin{aligned} Kh_1(x) - Kh_2(x) &< F(x,y_1,h_1(x_1)) - F(x,y_1,h_2(x_2)) \\ &\le |F(x,y_1,h_1(x_1)) - F(x,y_1,h_2(x_2))| \\ &\le \varphi(\max(|h(t)-h_1(t_1)|,|k(t)-k_1(t_1)|)). \end{aligned} \tag{23}$$

By (17) and (20) together with (15), we have

$$\begin{aligned} Kh_2(x) - Kh_1(x) &< F(x,y_1,h_2(x_2)) - F(x,y_1,h_1(x_1)) \\ &\le |F(x,y_1,h_1(x_1)) - F(x,y_1,h_2(x_2))| \\ &\le \varphi(\max(|h(t)-h_1(t_1)|,|k(t)-k_1(t_1)|)) \end{aligned} \tag{24}$$

and then (18) and (21) together with (15) imply

$$\begin{aligned} Kh_1(x) - Kh_2(x) &< F(x,y_1,h_1(x_1)) - F(x,y_1,h_2(x_2)) \\ &\le |F(x,y_1,h_1(x_1)) - F(x,y_1,h_2(x_2))| \\ &\le \varphi(\max(|h_1(t_1)-h_2(t_2)|,|k_1(t_1)-k_2(t_2)|)). \end{aligned} \tag{25}$$

By using the same arguments of $Jk_j < g(x,y_j) + F(x,y_j,\alpha h_j(x_j),\alpha k(x_j))$, and from (18) (21) and (22), we get

$$Jk_1(x) - Jk_2(x) \le \varphi(\max(|\alpha h(t) - \alpha h_1(t_1)|,|\alpha k(t) - \alpha k_1(t_1)|)), \tag{26}$$

$$Jk_2(x) - Jk_1(x) \le \varphi(\max(|\alpha h_1(t_1) - \alpha h_2(t_2)|,|\alpha k_1(t_1) - \alpha k_2(t_2)|)). \tag{27}$$

From (24) and (25), we have

$$|Kh_1(x) - Kh_2(x)| \le \varphi(\max(|h(t) - h_1(t_1)|, |k(t) - k_1(t_1)|)) \tag{28}$$

and

$$|Jk_1(x) - Jk_2(x)| \le \varphi(\max(|\alpha h(t) - \alpha h_1(t_1)|, |\alpha k(t) - \alpha k_1(t_1)|)). \tag{29}$$

The inequality (28) and (29), we obtain

$$|Kh_1(x) - Kh_2(x)| \le \varphi(\max(|h(t) - h_1(t_1)|, |k(t) - k_1(t_1)|)),$$
$$d(Kh_1(x), Kh_2(x)) \le \varphi(\max(d(h(t), h_1(t_1))), d(k(t), k_1(t_1))) \tag{30}$$

and

$$|Jk_1(x) - Jk_2(x)| \le \varphi(\max(|\alpha h(t) - \alpha h_1(t_1)|, |\alpha k(t) - \alpha k_1(t_1)|)),$$
$$d(Jk_1(x), Jk_2(x)) \le \varphi(\max(d(\alpha h(t), \alpha h_1(t_1))), d(\alpha k(t), \alpha k_1(t_1))). \tag{31}$$

Therefore, by condition (C2) the pair (K, J) has common fixed point of h and k, that is $h(x)$ is unique, bounded and common solution of equation (12) □

4. Conclusions

In this paper, we studied and defied a new concept of generalized weakly contraction mapping for coupled common fixed points in the space of the bounded function. We also obtained the existence and uniqueness common coupled fixed point theorems. Moreover, as an application of our result, we also studied the problem of existence and uniqueness of solutions for a class of system of functional equations which appears in dynamic programming.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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