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Research Article

The Rational Distance Problem for Equilateral Triangles

Roy Barbara and Antoine Karam*
Department of Mathematics, Faculty of Science II, Lebanese University, Fanar, Lebanon
***Corresponding author:** amkaram@ul.edu.lb

Abstract. We provide a complete characterization of all equilateral triangles T for which there exists a point in the plane of T, that is at rational distance from each vertex of T.

Keywords. Equilateral triangle; Rational distance problem; Bi-quadric number; Legendre's symbol; Non-degenerated triangle; Primitive integral triangle

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1. Introduction

Let (P) denote the problem of existence of a point in the plane of a given triangle T, that is at rational distance from all the vertices of T. Answer to (P) is positive if T has a rational side and the square of all sides are rational (see [1]). In [3], a complete solution to (P) is given for all isosceles triangles with one rational side. In this article, we provide a complete solution to (P) for all equilateral triangles.

In all what follows, θ denotes an arbitrary positive real number and $T = [\theta]$ denotes the equilateral triangle with side-length θ. For convenience, we say that θ is "good" (or "suitable") if answer to (P) is positive for the triangle $T = [\theta]$. Clearly, the property "θ is good" is invariant by any rational re-scaling of θ.

It turns out that the *good* θ must have algebraic degree 1, 2 or 4, and they form a subclass of the *positive* bi-quadric numbers, that is, the positive roots of equations of the form $x^4+ux^2+v=0$, $u,v\in\mathbb{Q}$. The general form of such numbers is

$$\sqrt{\alpha\pm\sqrt{\beta}},\quad \alpha,\beta\in\mathbb{Q},\ \beta\geq 0,\ \alpha\pm\sqrt{\beta}\geq 0$$

that includes positive numbers of the form

$$\alpha,\sqrt{\alpha},\alpha\pm\sqrt{\beta}\sqrt{\alpha}\pm\sqrt{\beta},\quad \alpha,\beta\in\mathbb{Q},\ \alpha,\beta\geq 0.$$

Notations and Conventions

(x,y) and (x,y,z) denote the gcd.$\left(\frac{x}{p}\right)$ denotes Legendre's symbol. A triangle with side-lengths a, b, c is denoted by $T=[a,b,c]$. A triangle is non-degenerated if it has positive area. A radical is non-degenerated if it is irrational.

2. The results

Theorem 2.1. *If θ is good, then, θ is bi-quadric. More precisely, $\theta^2=\alpha\pm\sqrt{\beta}$ for some $\alpha,\beta\in\mathbb{Q}$, $\beta\geq 0$ and α positive.*

Theorem 2.2. *Suppose $\theta\notin\mathbb{Q}$ and $\theta^2\in\mathbb{Q}$. Then, θ is good $\Leftrightarrow$ θ has the form $\theta=\lambda\sqrt{p_1\dots p_r}$, where $\lambda\in\mathbb{Q}$, $\lambda>0$, $r\geq 1$, $p_1,\dots,p_r$ are distinct odd primes, p_i is either 3 or of the form $6k+1$.*

Theorem 2.3. *Suppose $\theta^2=\alpha\pm\sqrt{\beta}$, $\alpha,\beta\in\mathbb{Q}$, $\alpha,\beta>0$, $\sqrt{\beta}\notin\mathbb{Q}$. Then, θ is good $\Leftrightarrow$ up to a rational re-scaling of θ, θ is described as follows:*

$$2\theta^2=(a^2+b^2+c^2)\pm 4\Delta\sqrt{3},$$

where $[a,b,c]$ is a non-degenerated primitive integral triangle with area Δ such that $4\Delta\sqrt{3}\notin\mathbb{Q}$.

Remark. Δ is given by Hero's formula, $\Delta=\sqrt{s(s-a)(s-b)(s-c)}$, $s=\frac{1}{2}(a+b+c)$. Equivalently, $4\Delta\sqrt{3}=\sqrt{3(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$, and the condition $4\Delta\sqrt{3}\notin\mathbb{Q}$ means that this latter radical is non-degenerated.

3. Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1. Suppose θ good. Let M be a point in the plane of triangle $T=[\theta]$, whose distances from the vertices of T are all rational. The following fundamental relation is well-known (see [4]):

$$3(a^4+b^4+c^4+\theta^4)=(a^2+b^2+c^2+\theta^2)^2. \qquad (\bullet)$$

Expanding $(\bullet)$ yields a relation as $\theta^4-u\theta^2+v=0$, where $u,v\in\mathbb{Q}$ and $u=a^2+b^2+c^2>0$. Solving for θ^2 yields $\theta^2=\alpha\pm\sqrt{\beta}$, with $\alpha,\beta\in\mathbb{Q}$ and $\alpha=\frac{1}{2}u>0$. □

Lemma 3.1. *Let $q>1$ be a square-free integer. Then, we have: The equation $x^2+3y^2=qz^2$ has a solution in integers x, y, z with $z\neq 0$ if and only if any prime factor of q is either 3 or of the form $6k+1$.*

Proof. Suppose first that q has only prime factors as 3 or $6k+1$. Since the quadratic form x^2+3y^2, $x,y\in\mathbb{Z}$, represents 3 and every prime $p=6k+1$, and since the set $\{x^2+3y^2,\ x,y\in\mathbb{Z}\}$ is closed by multiplication, we conclude that the equation $x^2+3y^2=q\cdot z^2$ has a solution in integers x,y,z with $z=1$.

Conversely, suppose that $x^2+3y^2=q\cdot z^2$ has a solution in integers x,y,z, $z\neq 0$. Pick such a solution with $|z|$ minimum. Clearly, $(x,y)=1$. We claim that q is *odd* and has no prime factor $6k-1$. For the purpose of contradiction, we consider two cases:

Case 1: q is even. Set $q=2w$, w odd. From $x^2+3y^2=2wz^2$, we see that $x\equiv y(\mathrm{mod}\,2)$. As $(x,y)=1$, x and y must be odd, so $x^2+3y^2\equiv 4(\mathrm{mod}\,8)$. Now, $4/2wz^2$ yields wz^2 even. But w is odd, hence z is even, so $2wz^2\equiv 0\,(\mathrm{mod}\,8)$. We get a contradiction.

Case 2: $q=p\cdot w$ for some prime $p=6k-1$. $x^2+3y^2=pwz^2$ yields $x^2+3y^2\equiv 0(\mathrm{mod}\,p)$. As $(x,y)=1$, p cannot divide y. Hence for some $t\in\mathbb{Z}$, $yt\equiv 1(\mathrm{mod}\,p)$. Therefore, $x^2t^2+3y^2t^2\equiv x^2t^2+3\equiv 0(\mathrm{mod}\,p)$, so $-3\equiv(xt)^2(\mathrm{mod}\,p)$. Hence $\left(\frac{-3}{p}\right)=+1$ contradicting $p=6k-1$. □

Lemma 3.2. *Let $\theta=\lambda\sqrt{q}$, $\lambda\in\mathbb{Q}$, $\lambda>0$, $q>1$ square-free integer. We have: θ is good $\Leftrightarrow$. There are $a,b,e,r,s\in\mathbb{Q}$, $e\neq 0$, such that*

$$a^2+3b^2=q\,, \tag{3.1}$$

$$(a+e)^2+3(b+e)^2=qr^2\,, \tag{3.2}$$

$$(a-e)^2+3(b+e)^2=qs^2\,. \tag{3.3}$$

Proof. By re-scaling, we take $\theta=2\sqrt{q}$. Let $T=ABC=[\theta]$. Choose a $x-y$ axis to get the coordinates $A(0,\sqrt{3q})$, $B(-\sqrt{q},0)$, $C(\sqrt{q},0)$.

• Suppose first that θ is good:

There is a point $M=M(x,y)$ in the plane of T such that $MA,MB,MC\in\mathbb{Q}$. Clearly, $M\neq A,B,C$. Set $w=\frac{MA}{q}$, $r=\frac{MB}{wq}$, $s=\frac{MC}{wq}$. Then, $w,r,s\in\mathbb{Q}-\{0\}$.

The Pythagoras relations are:

$$\overline{MA}^2=x^2+(y-\sqrt{3q})^2=w^2q^2\,, \tag{3.1$'$}$$

$$\overline{MB}^2=(x+\sqrt{q})^2+y^2=w^2q^2r^2\,, \tag{3.2$'$}$$

$$\overline{MC}^2=(x-\sqrt{q})^2+y^2=w^2q^2s^2\,. \tag{3.3$'$}$$

Subtracting (3.2′) and (3.3′) yields $x=\frac{1}{4}w^2q(r^2-s^2)\cdot\sqrt{q}$, that is,

$$x=\alpha\sqrt{q},\quad \alpha\in\mathbb{Q}. \tag{3.4}$$

Then (3.2′) gives $y^2\in\mathbb{Q}$, and then (3.1′) gives $2y\sqrt{3q}\in\mathbb{Q}$, hence, $y=\gamma\sqrt{3q}$, $\gamma\in\mathbb{Q}$.

For convenience, we put $\gamma=\beta+1$, obtaining

$$y=(\beta+1)\sqrt{3q},\quad \beta\in\mathbb{Q}, \tag{3.5}$$

Due to (3.4) and (3.5), equations (3.1′), (3.2′), (3.3′) become after dividing by q:

$$\alpha^2+3\beta^2=qw^2\,,$$

$$(\alpha+1)^2+3(\beta+1)^2 = qw^2r^2,$$
$$(\alpha-1)^2+3(\beta+1)^2 = qw^2s^2.$$

Set $a=\frac{\alpha}{w}$, $b=\frac{\beta}{w}$, $e=\frac{1}{w}$. Dividing by w^2, we get precisely relations (3.1), (3.2), (3.3).

• Conversely suppose that relations (3.1), (3.2), (3.3) hold with some $a,b,e,r,s\in\mathbb{Q}$, $e\neq 0$. Define point $M=M(x,y)$ in the plane of T by

$$x=\frac{a}{e}\sqrt{q},\quad y=\left(\frac{b}{e}+1\right)\sqrt{3q}.$$

We may write:

$$\overline{MA}^2 = x^2+(y-\sqrt{3q})^2 = q\frac{a^2}{e^2}+3q\frac{b^2}{e^2} = \frac{q}{e^2}(a^2+3b^2) = \frac{q}{e^2}\cdot q = \left(\frac{q}{e}\right)^2,$$
$$\overline{MB}^2 = \left(\left(\frac{a+e}{e}\right)\sqrt{q}\right)^2+\left(\left(\frac{b+e}{e}\right)\sqrt{3q}\right)^2 = \frac{q}{e^2}\left((a+e)^2+3(b+e)^2\right) = \frac{q}{e^2}\cdot qr^2 = \left(\frac{qr}{e}\right)^2,$$
$$\overline{MC}^2 = \left(\left(\frac{a-e}{e}\right)\sqrt{q}\right)^2+\left(\left(\frac{b+e}{e}\right)\sqrt{3q}\right)^2 = \frac{q}{e^2}\left((a-e)^2+3(b+e)^2\right) = \frac{q}{e^2}\cdot qs^2 = \left(\frac{qs}{e}\right)^2.$$

Therefore, MA, MB, MC are all rational. □

Proof of Theorem 2.2. Let θ such that $\theta\notin\mathbb{Q}$ and $\theta^2\in\mathbb{Q}$: θ can be written as $\theta=\lambda\sqrt{q}$, $\lambda\in\mathbb{Q}$, $\lambda>0$, $q>1$ square-free integer.

• Suppose first that q is even or has a prime factor $6k-1$. By Lemma 3.1, $a^2+3b^2=q$, $a,b\in\mathbb{Q}$, is impossible.

Hence, relation (3.1) in Lemma 3.2 fails, so θ is not good.

• Suppose now that q has only prime factors as 3 or $6k+1$. We show that θ is good using the characterization of Lemma 3.2:

By Lemma 3.1, for some $a,b\in\mathbb{Q}$, we have $a^2+3b^2=q$. Set $e=-\frac{q}{4b}=\frac{-(a^2+3b^2)}{4b}$, $r=\frac{a-b}{2b}$, $s=\frac{a+b}{2b}$.
We have

$$\begin{aligned}(a+e)^2+3(b+e)^2 &= (a^2+3b^2)+4e^2+2e(a+3b)\\ &= q+\frac{q^2}{4b^2}-\frac{q}{2b}(a+3b)\\ &= \frac{q}{4b^2}(4b^2+q-2b(a+3b))\\ &= \frac{q}{4b^2}(4b^2+a^2+3b^2-2ab-6b^2)\\ &= \frac{q}{4b^2}(a^2+b^2-2ab)\\ &= q\frac{(a-b)^2}{4b^2}\\ &= q\cdot r^2\end{aligned}$$

and

$$(a-e)^2+3(b+e)^2 = (a^2+3b^2)+4e^2-2e(a-3b)$$

$$
\begin{aligned}
&= q + \frac{q^2}{4b^2} + \frac{q}{2b}(a-3b) \\
&= \frac{q}{4b^2}(4b^2 + q + 2b(a-3b)) \\
&= \frac{q}{4b^2}(4b^2 + a^2 + 3b^2 + 2ab - 6b^2) \\
&= \frac{q}{4b^2}(a^2 + b^2 + 2ab) \\
&= q\frac{(a+b)^2}{4b^2} \\
&= q \cdot s^2 .
\end{aligned}
$$

□

4. Proof of Theorem 2.3

Lemma 4.1. *Let x, y, z, t be positive real numbers such that*

$$3(x^4 + y^4 + z^4 + t^4) = (x^2 + y^2 + z^2 + t^2)^2 . \qquad (\circledcirc)$$

Then, any three of x, y, z, t satisfy the triangle inequality.

Proof. Since x, y, z, t play symmetric roles, it suffices to show that x, y, z satisfy the triangle inequality. Write $(\circledcirc)$ as

$$t^4 - (x^2 + y^2 + z^2)t^2 + (x^4 + y^4 + z^4 - x^2y^2 - y^2z^2 - z^2x^2) = 0 .$$

The discriminant Δ of this trinomial in t^2 must be non-negative. But, $\Delta = 6(x^2y^2 + y^2z^2 + z^2x^2) - 3(x^4 + y^4 + z^4)$ that factors as $\Delta = 3(x + y + z)(-x + y + z)(x - y + z)(x + y - z)$.
Hence, $(-x + y + z)(x - y + z)(x + y - z) \geq 0$. The reader can easily check (using contraposition) that x, y, z must satisfy the triangle inequality. □

Lemma 4.2. *Let $T = ABC = [\theta]$. Let a, b, c be positive real numbers satisfying*

$$3(a^4 + b^4 + c^4 + \theta^4) = (a^2 + b^2 + c^2 + \theta^2)^2 .$$

Then, there is a point M in the plane of T such that $MA = a$, $MB = b$ and $MC = c$.

Proof. By Lemma 4.1, a, b and θ satisfy the triangle inequality. In particular, $a + b \geq \theta$. It follows that the circle $\mathcal{C}(A, a)$ intersects the circle $\mathcal{C}(B, b)$ at two points M_1 and M_2 ($M_1 = M_2$ if $a + b = \theta$). Set $c_1 = M_1C$ and $c_2 = M_2C$. By the fundamental relation ($\bullet$), we have $3(a^4+b^4+c_1^4+\theta^4) = (a^2+b^2+c_1^2+\theta^2)^2$ and $3(a^4+b^4+c_2^4+\theta^4) = (a^2+b^2+c_2^2+\theta^2)^2$. Therefore, c_1^2 and c_2^2 are the roots of the trinomial in T

$$T^2 - (a^2 + b^2 + \theta^2)T + (a^4 + b^4 + \theta^4 - a^2b^2 - b^2\theta^2 - \theta^2a^2) = 0 .$$

Since by hypothesis c^2 is also a root of this trinomial, we must have $c^2 = c_1^2$ or $c^2 = c_2^2$. Hence $c = c_1$ or $c = c_2$. Therefore, a, b and c are the distances from either point M_1 or M_2 to the vertices A, B and C of T. □

Proof of Theorem 2.3. Let $\theta > 0$ such that $\theta^2 = \alpha \pm \sqrt{\beta}$, $\alpha, \beta \in \mathbb{Q}$, $\alpha, \beta > 0$, $\sqrt{\beta} \notin \mathbb{Q}$.

• Suppose first that θ is good:
Let P be a point in the plane of $T = ABC = [\theta]$ such that $PA = a$, $PB = b$, $PC = c$ are all rational. We have

$$3(a^4+b^4+c^4+\theta^4) = (a^2+b^2+c^2+\theta^2)^2. \qquad (*)$$

By Lemma 4.1, a, b and c satisfy the triangle inequality. Relation $(*)$ yields

$$\theta^4 - U\theta^2 + V = 0 \text{ with } U = a^2+b^2+c^2 \text{ and } V = a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2 \ \ (U,V \in \mathbb{Q}).$$

Solving for θ^2, we get

$$2\theta^2 = (a^2+b^2+c^2) \pm \sqrt{3(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}. \qquad (\star)$$

Since θ^2 has algebraic degree 2, then, the radical in $(\star)$ is non-degenerated. In particular, the triangle $[a,b,c]$ is non degenerated. Select a sufficiently large positive integer N such that Na, Nb, Nc are all integers and set $D = (Na, Nb, Nc)$. If we multiply relation $(\star)$ by $\frac{N^2}{D^2}$, this results in replacing in $(\star)$ θ by $\frac{N}{D}\cdot\theta$ and a, b, c by the integers $\frac{Na}{D}$, $\frac{Nb}{D}$, $\frac{Nc}{D}$, respectively. As an outcome, we obtain *essentially* the same relation $(\star)$ where θ has been re-scaled by the rational $\frac{N}{D}$, and where the new symbols a, b, c represent relatively prime *positive* integers, satisfying the triangle inequality.

• Conversely, suppose that for some positive rational λ, $\theta_0 = \lambda\cdot\theta$ is described precisely as in Theorem 2.3. Eliminating the radical

$$4\Delta\sqrt{3} = \sqrt{6(a^2b^2+b^2c^2+c^2a^2) - 3(a^4+b^4+c^4)}$$

in the relation $2\theta_0^2 = (a^2+b^2+c^2) \pm 4\Delta\sqrt{3}$ leads to

$$3(a^4+b^4+c^4+\theta_0^4) = (a^2+b^2+c^2+\theta_0^2)^2.$$

By Lemma 4.2 there is a point M in the plane of $T = [\theta_0]$ that is at distances a, b, c from the vertices of T. Since a, b, c are integers, then, θ_0 is good. Therefore, $\theta = \lambda^{-1}\theta_0$ is also good. □

5. Exercises

(1) Check which are "good" among the radicals: $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{10}$.

(2) Show that the positive real number $\theta = \sqrt{25+12\sqrt{3}}$ is "good".

(3) Suppose that $2\theta^2 = \alpha + \sqrt{\beta}$, $\alpha,\beta \in \mathbb{Q}$, $\alpha,\beta > 0$, $\sqrt{\beta} \notin \mathbb{Q}$, and $\alpha^2 < \beta$. Show that θ is not good.

(4) Produce solution-points to problem (P) for the triangle $T = [\sqrt{3}]$.

(5) Let $\theta = \alpha + \beta\sqrt[4]{q} > 0$, $\alpha,\beta \in \mathbb{Q}$, $\beta \neq 0$, $q > 1$ square-free integer. Show that θ is not good.

(6) Suppose that $2\theta^2 = \alpha \pm \sqrt{\beta} > 0$, $\alpha,\beta \in \mathbb{Q}$, $\alpha,\beta > 0$, $\sqrt{\beta} \notin \mathbb{Q}$. Write the fraction α in *lowest terms* as $\alpha = \frac{m}{n}$ $(m,n$ positive integers) and suppose that mn has the form $mn = 4^l(8k+7)$, k, l non-negative integers. Then, prove that θ is not good.

6. Conclusion

We have a complete answer to problem (P) for equilateral triangles $T = [\theta]$:

If θ is transcendental or has algebraic degree ≥ 5, then, θ is not good. If θ has algebraic degree 3, or if both θ and θ^2 have algebraic degree 4, then, θ is not good. If θ is irrational and $\theta^2 \in \mathbb{Q}$, so θ has the form $\lambda\sqrt{q}$, where $\lambda \in \mathbb{Q}$, $\lambda > 0$ and $q > 1$ is a square-free integer, then, θ is good if and only if q is *odd* and has no prime factor $6k-1$. Finally, if $\theta^2 = \alpha \pm \sqrt{\beta}$, $\alpha, \beta \in \mathbb{Q}$, $\beta > 0$, $\sqrt{\beta} \notin \mathbb{Q}$, then, θ is not good if $\alpha \leq 0$ or if $\alpha^2 < \beta$, while if $\alpha > 0$ and $\alpha^2 > \beta$, θ is good if and only if θ satisfies the geometric property described in Theorem 2.3.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] T.G. Berry, Points at rational distance from the vertices of a triangle, *Acta Arithmetica* **LXII** (4) (1992), 391 – 398.

[2] R. Barbara, The rational distance problem for polygons, *Mathematical Gazette* **97** (538) (2013), note 97.11.

[3] R. Barbara and A. Karam, The rational distance problem for isosceles triangles with one rational side, *Communications in Mathematics and Applications* **4** (2) (2013), 169 – 179.

[4] Wikipedia, *Equilateral Triangle*, `https://en.wikipedia.org/wiki/Equilateral_triangle`.

[5] Wolfram Mathworld, *Equilateral Triangle*, `http://mathworld.wolfram.com/EquilateralTraiangle.html`.