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Research Article

Some Results of the Normal Intersection Graph of a Group

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Abstract. Let G be a group. We denote the normal intersection graph of subgroups of G by $\Delta(G)$, and define it as an undirected graph with no loops and multiple edges, whose vertex set is the set of all non-trivial subgroups of G and two distinct vertices H and K are adjacent if and only if $H \cap K$ is normal in G. In this paper, we characterize all of groups G whose the normal intersection graph of G is planar and we investigate some other properties of this graph.

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1. Introduction

In recent years, many researchers and mathematicians have defined and studied the graphs on several algebraic structures, like groups, rings, vector spaces, modules and obtained interesting results, for instance see [2, 3, 5, 7, 12].

Let G be a group. We define the *normal intersection graph* of G, as an undirected graph with no loops and multiple edges, whose vertex set is the set of all non-trivial subgroups of G and two distinct vertices H and K are adjacent if and only if $H \cap K$ is normal in G and denote it by $\Delta(G)$.

Let $X=(V,E)$ be a simple graph with the vertex set V and edge set E. We say that X is a *planar graph* if it can be drawn in the plane without their edges crossing. Planarity of some graphs associated to groups, has been studied by some authors, see [1, 8, 11]. For the standard terminology concerning graph theory and group theory, we the reader refer to [6] and [9, 10], respectively. The maximum possible distance in X is called the *diameter* of X and denoted by $\operatorname{diam}(X)$. The *girth* of X, denoted by $\operatorname{girth}(X)$, is the length of the shortest cycle of X, ($\operatorname{girth}(X)=\infty$ if X has no cycle). Two graphs K_n and $K_{m,n}$ denote the *complete graph* of order n and the *complete bipartite graph* with part sizes m and n. A graph whose edge set is empty is called a *null graph* or *totally disconnected graph*. Let F be a graph, a graph G is F*-free* if G contains no induced subgraph isomorphic to F. A set D of vertices of X is a *dominating set* if every vertex in $V(X)\setminus D$ is adjacent to at least one vertex in D. The minimum cardinality among the dominating sets of X is called the *domination number* of X and is denoted by $\gamma(X)$.

For a group G, by $\Pi(G)$, we mean the set of all prime divisors of $|G|$ and by $Syl_p(G)$, the set of all Sylow p-subgroups of G for $p\in\Pi(G)$. The number of the Sylow p-subgroups of a group G is denoted by $n_p(G)$ or simply by n_p. We denote the normal subgroup N and the characteristic subgroup H in G briefly by $N\lhd G$ and H char G. For any natural numbers n, as usual, we write $\mathbb{Z}_n$, D_{2n}, A_n and S_n to denote the cyclic group of order n, the dihedral group of order $2n$, the alternating group and the symmetric group of degree n, respectively. We will denote the multiplicative order of a non-zero element $x\in\mathbb{Z}_n$ by $ord_n(x)$.

In this paper, we study the planarity of the normal intersection graph of G and determine groups that the normal intersection graph of subgroups is planar. In the rest of the paper, we study connectivity, diameter and girth of this graph and conclude that in the normal intersection graph of abelian groups and finite non-abelian groups, the following condition are equivalent:

(i) $\Delta(G)$ is K_3-free.

(ii) $\Delta(G)$ is forest.

(iii) $\Delta(G)$ is bipartite.

(iv) $G\simeq\mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$.

In Section 2 some preliminary results are mentioned. In Sections 3 and 4, we study the planarity of the normal intersection graph of subgroups of all abelian groups and finite non-abelian groups.

2. Preliminary Results

We devote this section to study some basic properties about planarity of the normal intersection graph of subgroups of a given group. At first, we express the well-known Kuratowski's Theorem.

Theorem 2.1 ([6, 6.13]). *A graph is planar if and only if it contains no subgraph that is a subdivision of either K_5 or $K_{3,3}$.*

Remark 2.2. (i) [10, 5.20] Let H and K be subgroups of a group G. If H char K and $K \lhd G$, then $H \lhd G$.

(ii) [10, 4.6] If G is a finite p-group, then every maximal subgroup of G is normal and has index p.

(iii) [10, 4.8] If G is a finite p-group of order p^n and r_s is the number of subgroups of G having order p^s $(1 < s < n)$, then r_s is congruent to 1 $(\operatorname{mod} p)$.

(iv) [10, P.77] If $|G| = p^n$, where p is a prime, then for $k = 0, 1, \dots, n$, G contains a normal subgroup of order p^k.

(v) [9, 5.3.7 (Dedekind, Baer)] All the subgroups of a group G are normal if and only if G is abelian or the direct product of a quaternion group of order 8, an elementary abelian 2-group and an abelian group with all its elements of odd order.

(vi) [10, P.107 (Burnside's Theorem)] If p and q are primes, then every group of order $p^m q^n$ is solvable.

Lemma 2.3. *Let G be a group. Then $\Delta(G)$ is non-planar if one of the following holds:*

(i) *There exists a normal subgroup N of G such that $\Delta(G/N)$ is non-planar.*

(ii) *There exists a characteristic subgroup M of G such that $\Delta(M)$ is non-planar.*

Proof. **(i):** Assume that H_1/N and H_2/N be two subgroups of G/N. If $(H_1/N) \cap (H_2/N)$ is normal in G/N, then clearly $H_1 \cap H_2$ is normal in G. Thus if $\Delta(G/N)$ is non-planar, by the Kuratowski's Theorem $\Delta(G/N)$ contains a subdivision of K_5 or $K_{3,3}$ as a subgraph, so does $\Delta(G)$. Hence using Kuratowski's Theorem, $\Delta(G)$ is non-planar.

(ii): According to Remark 2.2(i), it is easy to see that if M is a characteristic subgroup of G, then $\Delta(M)$ is a subgraph of $\Delta(G)$. Thus if $\Delta(M)$ is non-planar, like the previous, by theorem Kuratowski's non-planarity of $\Delta(M)$ implies that $\Delta(G)$ is non-planar. □

Lemma 2.4. *Let $G \simeq G_1 \times \cdots \times G_n$. If there exist $1 \le i \le n$ such that $\Delta(G_i)$ is non-planar. Then $\Delta(G)$ is non-planar.*

Proof. Assume that $N \lhd G_i$, thus $N \lhd G$, hence $\Delta(G_i)$ is a subgraph of $\Delta(G)$. Thus, non-planarity of $\Delta(G_i)$ implies that $\Delta(G)$ is non-planar. □

Remark 2.5. Assume that G_1 and G_2 are two groups. If $\Delta(G_1) \simeq \Delta(G_2)$, then $G_1 \simeq G_2$ is not true in general. For example, we consider two non-isomorphic groups $\mathbb{Z}_{p^5}$ and Q_8, both groups have 5 non-trivial subgroups which are normal and hence $\Delta(\mathbb{Z}_{p^5}) \simeq \Delta(Q_8) \simeq K_5$.

3. The Planarity of Abelian Groups

In this section, we will classify the abelian groups whose normal intersection graph of subgroups are planar. At first, we investigate the groups whose normal intersection graphs are complete. We have the following theorem:

Theorem 3.1. *Let G be a group. Then $\Delta(G)$ is a complete graph if and only if G is an abelian group or $G \simeq Q_8 \times H \times K$, where H is a (necessarily abelian) group of exponent 2 and K is an abelian group in which every element has an odd order.*

Proof. If $\Delta(G)$ is a complete, then G is adjacent to all non-trivial subgroups of G in $\Delta(G)$. So every subgroup of G is a normal subgroup. Hence by Remark 2.2(v), G is an abelian group or $G \simeq Q_8 \times H \times K$. The other side is obvious. □

Theorem 3.2. *Let G be an abelian group. Then $\Delta(G)$ is a planar graph if and only if G is isomorphic to one of the following groups:*

(i) $\mathbb{Z}_{p^\alpha}$, *where p is a prime and $1 \le \alpha \le 4$.*

(ii) $\mathbb{Z}_{pq}$, *where p and q are distinct primes.*

(iii) $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Assume that G is an infinite abelian group. Then $\Delta(G)$ is an infinite complete graph and hence $\Delta(G)$ is non-planar.
Now, assume that G is a finite abelian group of $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where every p_i is a prime and α_i is a positive integer for $i = 1, \dots, k$. We have the two following cases.
Case 1. If G is cyclic, then the number of distinct subgroups of G is the number of distinct positive divisors of $|G|$. Thus, by Theorem 3.1, we have $\Delta(G) = K_t$, where $t = (\alpha_1 + 1)(\alpha_2 + 1)\dots(\alpha_k + 1) - 1$. So $\Delta(G)$ is planar if and only if $t \le 4$. This implies that the following statements hold:

- If $k = 1$, then $\Delta(G)$ is planar if and only if $\alpha_1 \le 4$, so $G \simeq \mathbb{Z}_{p^\alpha}$, where $\alpha \le 4$.
- If $k = 2$, then $\Delta(G)$ is planar only if $\alpha_1 = \alpha_2 = 1$. This implies $G \simeq \mathbb{Z}_{p_1 p_2}$.
- If $k > 2$, then it is easy to see that $\Delta(G)$ is non-planar.

Case 2. If G is non-cyclic, then G contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Note that this subgroup contains $p+1$ subgroups of order p such that pairwise have trivial intersection. Thus, these $p+1$ subgroups with $\mathbb{Z}_p \times \mathbb{Z}_p$ form a complete graph K_{p+2} as a subgraph of $\Delta(G)$. If $p > 2$, by Kuratowski's Theorem $\Delta(G)$ is non-planar. If $p = 2$, two cases occur: if $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. It is easy to see $\Delta(G) = K_4$, so $\Delta(G)$ is planar. But if $|G| > 4$, then $\Delta(G)$ contains a complete subgraph with more than 5 vertices, thus $\Delta(G)$ is non-planar. The converse is obvious. □

4. The Planarity of Finite Non-abelian Groups

In this section we assume that G is a finite non-abelian group. We first show that if G is a finite group whose order has more than two distinct prime factors, then $\Delta(G)$ is non-planar and we conclude that the normal intersection graph of non-solvable groups is non-planar. Then we investigate the planarity of the normal intersection graph of solvable groups.

Theorem 4.1. *Let G be a finite group whose order has more than two distinct prime factors, then $\Delta(G)$ is non-planar.*

Proof. Let G be a group of order $p^{\alpha}q^{\beta}r^{\gamma}m$, where p,q,r are distinct primes which are coprime to positive integer m. Now, we consider two cases:

Case 1. Assume that G has only one subgroups of each orders p, q and r, we call them P, Q and R, respectively. Therefore, P, Q, R, PQ, PR are normal subgroups of G. So by Kuratowski's Theorem, $\Delta(G)$ is non-planar.

Case 2. Assume that G has more than one subgroup of at least one of the orders p, q, r like p, then according to Remark 2.2(iii), G have at least $p+1$ subgroups of order p. These $p+1$ subgroups with Q and R form a complete graph of order at least $p+3$ in $\Delta(G)$ and the proof is completed. □

Theorem 4.2. *Let G be a finite non-solvable group. Then $\Delta(G)$ is non-planar.*

Proof. By Remark 2.2(vi), if G is a non-solvable group, then $\Pi(G) \geq 3$ and by the previous theorem $\Delta(G)$ is non-planar. □

Lemma 4.3. *Let G be a finite non-abelian p-group, where p is a prime. Then $\Delta(G)$ is non-planar.*

Proof. We have $|G| = p^n$ $(n > 2)$. By Remark 2.2(ii), every maximal subgroup of G is normal and has index p. We know that, if the number maximal subgroups of G is 1, then G is cyclic which is a contradiction. Thus according to Remark 2.2(iii), G contains at least $p+1$ maximal subgroups, but according to Remark 2.2(iv), G has at least one normal subgroup of order p^{n-2}, say N. Thus maximal subgroups of G and N and G form a complete graph of order at least $p+3$ in $\Delta(G)$, and the proof is completed. □

Theorem 4.4. *Let G be a finite non-abelian nilpotent group. Then $\Delta(G)$ is non-planar.*

Proof. By [10, 5.39] G is the direct product of its Sylow subgroups and since G is a non-abelian group, it has at least one non-abelian Sylow subgroup. By Lemma 4.3, normal intersection graph of its non-abelian Sylow subgroup is non-planar, so by Lemma 2.4, $\Delta(G)$ is non-planar. □

Now we investigate the planarity of non-nilpotent solvable groups and show that $\Delta(G)$ is always non-planar except for S_3.

Lemma 4.5. *Let G be a non-abelian group of order pq, where p and q are distinct primes and $p < q$. Then $\Delta(G)$ is planar if and only if $G \simeq \mathbb{Z}_3 \rtimes \mathbb{Z}_2 \simeq S_3$.*

Proof. By [4, p. 48], up to isomorphism, the only groups of order pq are $\mathbb{Z}_{pq}$, if $p \nmid q-1$ and $\mathbb{Z}_q \rtimes \mathbb{Z}_p$, if $p \mid q-1$. Since G is non-abelian, $G \simeq \mathbb{Z}_q \rtimes \mathbb{Z}_p$. So, G consists q Sylow p-subgroups of order p, such that pairwise have trivial intersection and a unique subgroup of order q which is normal in G. These $q+1$ subgroups form K_{q+1} as a subgraph $\Delta(G)$. Since $p < q$, so $q \geq 3$. If $q > 3$, then $\Delta(G)$ is non-planar. But, when $(p,q) = (2,3)$, then $G \simeq \mathbb{Z}_3 \rtimes \mathbb{Z}_2 \simeq S_3$. Group S_3 has 5 non-trivial subgroups and since subgroups of order 2 are not normal subgroups of S_3, these are not adjacent to S_3. By Kuratowaski's Theorem, the proof is completed. □

Consider the semi-direct product $\mathbb{Z}_q \rtimes_t \mathbb{Z}_{p^\alpha} = \langle a, b \mid a^q = b^{p^\alpha} = 1, bab^{-1} = a^i, ord_q(i) = p^t \rangle$, where p and q are distinct primes with $p^t \mid q-1$, $t \geq 0$. Then every semi-direct product $\mathbb{Z}_q \rtimes_t \mathbb{Z}_{p^\alpha}$ is one of these types. Note that in the future, when $t = 1$, we suppress the subscript.

Lemma 4.6. *Let G be a non-nilpotent group of order p^2q, where p and q are distinct primes. Then $\Delta(G)$ is non-planar.*

Proof. Burnside provides a classification of groups of order p^2q in [4, pp. 76-80]. By this classification, we have the following cases:

Case 1. $p < q$. In this case, we have three subcases:

Subcase 1. $p \nmid q-1$. By Sylow's Theorem, we have $n_p = 1$. Let $P \in Syl_p(G)$ and $Q \in Syl_q(G)$, then $G \simeq P \rtimes_\varphi Q$, where $\varphi : Q \longrightarrow Aut(P)$ is a homomorphism. It is easy to see that φ is the trivial homomorphism, so $G \simeq \mathbb{Z}_{p^2q}$. There are no non-abelian groups in this case.

Subcase 2. $p \mid q-1$, $p^2 \nmid q-1$. In this case, we have two non-abelian groups. The first is $G_1 = \mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}$, where $n_q \in \{1, p, p^2\}$. If $n_q = p$, then $q \mid p-1$ which is impossible and if $n_q = p^2$, then $q \mid (p+1)(p-1)$, which implies that $q \mid p+1$ or $q \mid p-1$ which is true only when $(p,q) = (2,3)$, we check the case separately. Now, consider $n_q = 1$. Since G_1 is a non-nilpotent, G_1 has q Sylow p-subgroups of order p^2, say $P_i, (1 \leq i \leq q)$. Thus G_1 has q subgroups of order p, say P'_i. P'_i together with a subgroup of order q form a complete graph K_{q+1} as a subgraph of $\Delta(G_1)$. Since $q > 3$, $\Delta(G_1)$ is non-planar.

The second group in this subcase is $G_2 = \langle a, b, c \mid a^q = b^p = c^p = 1, bab^{-1} = a^i, ca = ac, cb = bc, ord_q(i) = p \rangle$. Clearly, $c \in Z(G_2)$, so $p \mid |Z(G_2)|$, whereas G_2 is non-abelian, so $|Z(G_2)| = p$, hence $G_2/Z(G_2) \simeq \mathbb{Z}_q \rtimes \mathbb{Z}_p$. According to the previous lemma, the normal intersection graph of $\mathbb{Z}_q \rtimes \mathbb{Z}_p$ is non-planar unless $(p,q) = (2,3)$. Thus for $(p,q) \neq (2,3)$, by Lemma 2.3(i), $\Delta(G_2)$ is non-planar.

Subcase 3. $p^2 \mid q-1$. We automatically have both groups G_1 and G_2 from Subcase 2. As well as the group $G_3 = \mathbb{Z}_q \rtimes_2 \mathbb{Z}_{p^2} = \langle a, b \mid a^q = b^{p^2} = 1, bab^{-1} = a^i, ord_q(i) = p^2 \rangle$. In Subcase 2, we have already showed that $\Delta(G_1)$ and $\Delta(G_2)$ are non-planar. We now check the planarity of $\Delta(G_3)$. G_3 has a unique subgroup $K = \langle a, b^p \rangle \simeq Z_q \rtimes Z_p$ of order pq, which is a characteristic subgroup in G_3. By Lemma 4.5, if $(p,q) \neq (2,3)$, then $\Delta(K)$ is non-planar and by Lemma 2.4, $\Delta(G_3)$ is non-planar.

Case 2. $p > q$. We have the following subcases:

Subcase 1. $q \nmid p^2 - 1$. There are no non-abelian groups in this subcase.

Subcase 2. $q \mid (p-1)$. In this case there are two groups:

The first one is $G_4 = \langle a, b \mid a^{p^2} = b^q = 1, bab^{-1} = a^i, ord_{p^2}(i) = q \rangle$. By Sylow's Theorem, G_4 has a unique subgroup P of order p^2 and has a unique subgroup P' of order p, G_4 contains p^2 Sylow q-subgroups of order q, say Q_i, $1 \leq i \leq p^2$, $Q'_i s$ pairwise have trivial intersection. Thus, these p^2 subgroups together with P and P' form K_{p^2+2} as a subgraph of $\Delta(G_4)$. Note that $p \geq 3$, therefore $\Delta(G_4)$ is not planar.

Next, we have the family of groups $\langle a,b,c|a^p = b^p = c^q = 1, cac^{-1} = a^i, cbc^{-1} = b^{i^t}, ab = ba, ord_p(i) = q\rangle$. There are $(q+3)/2$ isomorphism types in this family (one for $t = 0$ and one for each pair $\{x, x^{-1}\}$ in $\mathbb{F}_p \setminus \{0\}$). We will refer to all of these groups as $G_5(t)$ of order p^2q. Here $H_1 = \langle a\rangle, H_2 = \langle b\rangle$ are subgroups of order p in $G_5(t)$. So, by Remark 2.2(iii), it has at least $p+1$ subgroups of order p and it has one subgroup Q of order q, such that $p+2$ subgroups pairwise have trivial intersection. Thus, $\Delta(G_5(t))$ contains a complete graph K_{p+2} as a subgraph. Therefore $\Delta(G_5(t))$ is non-planar.

Subcase 3. $q|p+1$. Only the non-abelian group here is $G_6 = (Z_p \times Z_p) \rtimes Z_q = \langle a,b,c|a^p = b^p = c^q = 1, ab = ba, cac^{-1} = a^ib^j, cbc^{-1} = a^kb^l\rangle$, where $M = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$ has order q in $GL_2(p)$. The same as $\Delta(G_5(t))$ can be demonstrated that $\Delta(G_6)$ is non-planar. Now, we consider the case that $(p,q) = (2,3)$. Note that if $(p,q) = (2,3)$, then cases 1 and 2 are not mutually exclusive. There are three non-abelian groups of order 12 up to isomorphism: $\mathbb{Z}_3 \rtimes \mathbb{Z}_4, D_{12}$ and A_4. In $\mathbb{Z}_3 \rtimes \mathbb{Z}_4 = \langle a,b|a^3 = b^4 = 1, bab^{-1} = a^i, ord_3(i) = 2\rangle$, $H_1 = \langle a\rangle$ is normal and $H_2 = \langle b^2\rangle$ and $H_3 = \langle a,b^2\rangle$ are unique subgroups of orders 2 and 6, respectively. So they are normal. $H_4 = \langle b\rangle, H_5 = \langle ab\rangle$ and $H_6 = \langle a^2b\rangle$ are also subgroups of G. H_2 is inclusion in each of H_i, $i = 3,4,5,6$ except H_1. The intersection H_1 with H_i, $i = 2,4,5,6$ is trivial and $H_1 \subseteq H_3$. So, $\Delta(\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ contains K_5 as subgraph of $\Delta(G)$, thus it is non-planar, see Figure 1.

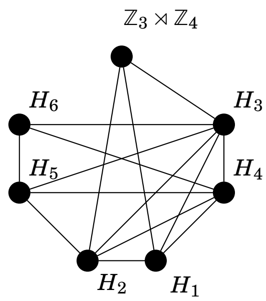

Figure 1. $\Delta(\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$

Group A_4 consist of 4 subgroups isomorphic to A_3 and 3 subgroups of order 2. These 7 subgroups pairwise have trivial intersection. Thus, $\Delta(A_4)$ is non-planar. In general, in D_{2n} for $n \geq 5$, $\langle a^ib\rangle$ are distinct minimal subgroups of order 2, where $0 \leq i < n-1$. Therefore, these n subgroups form K_n as a subgraph of $\Delta(D_{2n})$, so $\Delta(D_{2n})$ is non-planar. □

Lemma 4.7. *Let G be a non-nilpotent group of order $p^\alpha q$, where the p, q are distinct primes and $\alpha \geq 3$. Then $\Delta(G)$ is non-planar.*

Proof. For the proof of this lemma, we consider the following cases:

Case 1. $p > q$. Assume that P is a p-Sylow subgroup of G. By Sylow's Theorem, $n_p(G) = 1$, therefore $G \simeq P \rtimes \mathbb{Z}_q$. Since P char G by Lemma 2.3(ii), it suffices to consider only when $\Delta(P)$ is planar. That is, $P \simeq \mathbb{Z}_{p^3}$ and $P \simeq \mathbb{Z}_{p^4}$. So we have two groups, say, $G_1 \simeq \mathbb{Z}_{p^3} \rtimes \mathbb{Z}_q$ or $G_2 \simeq \mathbb{Z}_{p^4} \rtimes \mathbb{Z}_q$.

Here we prove non-planarity of $\Delta(G_1)$ and non-planarity of $\Delta(G_2)$ is similar to $\Delta(G_1)$. Since P char G_1, by Remark 2.2(i), 3 subgroups $\mathbb{Z}_{p^3}$, $p\mathbb{Z}_{p^3}$, $p^2\mathbb{Z}_{p^3}$ which are normal subgroups of P, are normal in G_1. Since G_1 is non-nilpotent, hence $n_q(G_1) \neq 1$. Therefore, G_1 has at least p Sylow q-subgroups, say Q_i, $i = 1,\dots,p$. Hence $V = \{\mathbb{Z}_{p^3}, p\mathbb{Z}_{p^3}, p^2\mathbb{Z}_{p^3}, Q_1,\dots,Q_p\}$ forms K_{p+3} as subgraph of $\Delta(G_1)$, Therefore by the Kuratowski's Theorem $\Delta(G_1)$ is non-planar.

Case 2. $p < q$. In this case, if $n_p(G) = 1$, then the same as Case 1, can be shown that $\Delta(G)$ is non-planar. If $n_p(G) \neq 1$, then $n_p(G) = q$. But this only leaves $p^\alpha q - q(p^\alpha - 1) = q$ elements, so $n_q(G) = 1$. Therefore $G = Q \rtimes P$ such that Q is a Sylow q-subgroup of G which it is normal in G and P is a Sylow p-subgroup of G. Now, we from G/Q. By Lemma 2.3(i), it suffices to consider only when $\Delta(G/Q)$ is planar, that is, $G/Q \simeq \mathbb{Z}_{p^3}$ or $G/Q \simeq \mathbb{Z}_{p^4}$, hence $G \simeq \mathbb{Z}_q \rtimes \mathbb{Z}_{p^3}$ or $G \simeq \mathbb{Z}_q \rtimes \mathbb{Z}_{p^4}$. In both cases, G has at least one subgroup of order p. If G has exactly one subgroup of order p, say N and N is a normal subgroup of G. So N, Q, N^2, NQ, G form K_5 as a subgraph of $\Delta(G)$, hence $\Delta(G)$ is non-planar. If we have more than one subgroup of order p, by Remark 2.2(iii), there are at least $p+1$ subgroups of order p say H_i. Also G has a normal subgroup K of order $p^{\alpha-1}q$. These $p+1$ subgroups with K and Q form K_{p+3} as subgraph of $\Delta(G)$ and since $p \geq 2$ proof is complete. □

Theorem 4.8. *Let G be a group of order $p^\alpha q^\beta$, where the p and q are distinct primes, $\alpha > 1$ and $\beta > 0$. Then $\Delta(G)$ is non-planar.*

Proof. We have shown by Theorems 3.2 and 4.4, if G is an abelian group or a finite non-abelian nilpotent group such that $\alpha > 1$ and $\beta > 0$, then $\Delta(G)$ is non-planar. So, assume that G is a solvable non-nilpotent group. If $\alpha, \beta \geq 3$, then $\Delta(G)$ contains the bipartite graph $K_{3,3}$ with partite sets $V_1 = \{A_1, A_2, A_3\}$ and $V_2 = \{B_1, B_2, B_3\}$, where A_i and B_i are subgroups of orders p^i and q^i for $i = 1,2,3$, respectively. So $\Delta(G)$ is non-planar. Also by Lemma 4.7, demonstrated that when $|G| = pq^\beta$, $\Delta(G)$ is non-planar. Now, it suffices to prove that it just when $|G| = p^2q^\beta$, $\beta > 1$. We consider the following cases:

Case 1. $p > q$. We know that $n_q \in \{1, p, p^2\}$. If $n_q = 1$, then $G = Q \rtimes P$ where Q is the q-Sylow subgroup of G, so Q is a normal and characteristic subgroup of G and P is a p-Sylow subgroup of G. By Lemma 2.3(ii), we shall prove non-planarity $\Delta(G)$ for when $\Delta(Q)$ is planar, that is, when G is isomorphic to $\mathbb{Z}_{q^3} \rtimes P$ or $\mathbb{Z}_{q^4} \rtimes P$. The proof of this case is similar to Lemma 4.7. If $n_q = p^2$, then number of elements only leaves $p^2q^\beta - p^2(q^\beta - 1) = p^2$ elements. So $n_p = 1$ and $G \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes Q$ or $G \simeq \mathbb{Z}_{p^2} \rtimes Q$. In the former event, $\mathbb{Z}_p \times \mathbb{Z}_p$ has $p+1$ subgroups of order p, say H_i where $i = 1,\dots,p+1$ and G contains p^2 Sylow q-subgroups , say Q_i where $i = 1,\dots,p^2$. Thus $\Delta(G)$ contains the bipartite graph K_{p+1,p^2} with partite sets $V_1 = \{H_i | i = 1,\dots,p+1\}$ and $V_2 = \{Q_i | i = 1,\dots,p^2\}$. Since $p \geq 3$, $\Delta(G)$ is non-planar. When $G \simeq \mathbb{Z}_{p^2} \rtimes Q$, whereas $\mathbb{Z}_{p^2}$ char G, so $p\mathbb{Z}_{p^2}$ is a normal subgroup of G. Assume that G has only one subgroup of order q, say N and it is normal in G. Thus $\{\mathbb{Z}_{p^2}, p\mathbb{Z}_{p^2}, N, \mathbb{Z}_{p^2}N, (p\mathbb{Z}_{p^2})N, G\}$ form a complete subgraph K_6 in $\Delta(G)$. Otherwise, if G has more than one subgroup of order q, by Remark 2.2(iii) G has at least $q+1$ subgroups of order q that with $\mathbb{Z}_{p^2}, p\mathbb{Z}_{p^2}$ form a complete subgraph K_{q+3} in $\Delta(G)$. Therefore,

$\Delta(G)$ is non-planar. Now, let $n_q = p$ and $n_p = 1$, then $G = P \rtimes Q$ and the proof is similar to an earlier case. Assume that $n_q = p$ and $n_p \neq 1$, so $n_p \mid q^\beta$, but by Sylow's Theorem, $n_p \neq q$, therefore $n_p \geq q^2$, we put p Sylow q-subgroups of G in a set and at least q^2 Sylow p-subgroups in the other set. It is clear that the two sets are partite sets of the bipartite graph, hence by the Kuratowski's Theorem $\Delta(G)$ is non-planar.

Case 2. $p < q$. In this case, $n_q \in \{1, p, p^2\}$, if $n_q = p$, then $q \mid p-1$, in contradiction to our assumption. Now, assume that $n_q = p^2$, so $q \mid p^2 - 1$ which implies that $q \mid p+1$ or $q \mid p-1$. But this is possible only in the case $(p,q) = (2,3)$. Let $(p,q) = (2,3)$, it is clear that $n_3 = 4$ and $n_2 = 1$ which imply that $G \simeq (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes Q$ or $G \simeq \mathbb{Z}_4 \rtimes Q$. In both cases, we in the previous case have seen that $\Delta(G)$ is non-planar. Now, assume that $(p,q) \neq (2,3)$, so $n_q = 1$. Since G is a non-nilpotent group, $n_p \neq 1$ and $G \simeq Q \rtimes P$. As, Q char G by Lemma 2.3(ii), It suffices to consider only when $\Delta(Q)$ is planar. In a similar manner to Lemma 4.7, we may prove that also in these cases $\Delta(G)$ is non-planar. The proof is completed. □

Corollary 4.9. *Let G be a finite group. Then $\Delta(G)$ is planar graph if and only if G is isomorphic to one of the following groups:*

(i) $\mathbb{Z}_{p^\alpha}$, *where p is a prime and $1 \leq \alpha \leq 4$.*

(ii) $\mathbb{Z}_{pq}$, *where p and q are distinct primes.*

(iii) $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(iv) S_3.

In continuation, we obtain some results related to the normal intersection graph.

Corollary 4.10. *Let G be an abelian group or a finite non-abelian group. Then* $\operatorname{girth}(\Delta(G)) = \{3, \infty\}$.

Proof. According to the proof of Theorem 3.2, if G is an abelian group, then $\operatorname{girth}(G) = 3$ except for $G \simeq \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$. If G is a finite non-abelian p-group, then by proof of Lemma 4.5, $\operatorname{girth}(G) = 3$. Now, let G be a non-abelian group of order $p_1^{\alpha_1} p_2^{\alpha_2} m$, where p_1, p_2 are distinct primes which are coprime to positive integer m and $\alpha_i > 0$. Consider two following cases:

Case 1. Assume that G has only one subgroups of each orders p_1 and p_2, we call them H_1 and H_2, respectively. Therefore, H_1, H_2 are normal subgroups of G. So $\{H_1, H_2, G\}$ forms a 3-cycle in $\Delta(G)$.

Case 2. Assume that G has more than one subgroup of at least one of the orders p_1 or p_2 like p_1, therefore according to Remark 2.2(iii), G has at least $p_1 + 1$ subgroups of order p_1. These $p_1 + 1$ subgroups form a complete subgraph K_{p_1+1} in $\Delta(G)$. So in this case, $\operatorname{girth}(\Delta(G)) = \{3\}$ and the proof is completed. □

The following corollary is an immediate consequence of the previous corollary.

Corollary 4.11. *Let G be an abelian group or a finite non-abelian group. Then the following are equivalent:*

(i) $\Delta(G)$ *is* K_3*-free.*

(ii) $\Delta(G)$ *is forest.*

(iii) $\Delta(G)$ *is bipartite.*

(iv) $G \simeq \mathbb{Z}_p$ *or* $\mathbb{Z}_{p^2}$.

Theorem 4.12. *Let $G \not\simeq \mathbb{Z}_p$ be a finite nilpotent group. Then $\Delta(G)$ is a connected graph with domination number $\gamma(\Delta(G)) = 1$ and* $\operatorname{diam}(\Delta(G)) \le 2$. *Moreover, if G is not a Dedekind group, then* $\operatorname{diam}(\Delta(G)) = 2$.

Proof. Assume that p is any prime dividing $|G|$, there is a minimal normal subgroup of size p, say N. Intersection of every subgroup of G with N is N or 1, so all of the subgroups of G adjacent to N. Therefore, $\Delta(G)$ is connected and $\gamma(\Delta(G)) = 1$ and hence $\operatorname{diam}(G) \le 2$. Now, if G is not a Dedekind group, then according to Theorem 3.1, $\operatorname{diam}(\Delta(G)) = 2$. □

Lemma 4.13. *Let G be a simple group. Then $\Delta(G)$ is a disconnected graph.*

Proof. Since G is simple and has no proper normal subgroups, $G \simeq \mathbb{Z}_p$ or there is not any adjacent between G and other non-trivial subgroups of G, so $\Delta(G)$ is a disconnected graph. □

Finally, with respect to results of sections 2 and 3, we have the following corollary.

Corollary 4.14. *Let G be an abelian group or a finite non-abelian group. Then $\Delta(G)$ is totally disconnected if and only if $G \simeq \mathbb{Z}_p$, where p is a prime.*

5. Conclusion

In this paper, we define the normal intersection graph of a group and we study the planarity of this graph. We also obtain connectivity, diameter and girth of this graph. For future research, one can investigate other properties of this graph and can obtain some properties of groups using the properties of this graph. Also, we can study one subgraph of this graph whose vertex set is the set of all non-trivial proper subgroups of G and two distinct vertices H and K are adjacent if and only if $H \cap K$ is non-trivial normal subgroups of G.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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