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Research Article

Geometric Means and Tracy-Singh Products for Positive Operators

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Abstract. We investigate relationship between metric/spectral/Sagae-Tanabe geometric means for several positive operators and Tracy-Singh products in terms of identities and inequalities. In particular, we obtain various generalizations of arithmetic-geometric-harmonic means inequality and its reverse. Moreover, we introduce the weighted Sagae-Tanabe spectral geometric mean for several positive operators and deduce its properties related to Tracy-Singh products.

Keywords. Metric (spectral) geometric mean; Sagae-Tanabe metric (spectral) geometric mean; Tensor product; Tracy-Singh product; Khatri-Rao product

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1. Introduction

It is well known that the tensor product (or Kronecker product) plays a fundamental role in linear algebra, functional analysis and related fields. Nowadays, theory of tensor products of operators is still developing, see [18] for instance. Recently, the authors of [12] introduced the Tracy-Singh product of operators which generalizes both tensor products of operators and Tracy-Singh products of complex matrices [17].

On the other hand, geometric means of positive definite matrices arise naturally in several areas of pure and applied mathematics. There are at least two types of geometric means.

The first one is the metric geometric mean, introduced by Ando [3]. Recall that the set of n-by-n positive definite matrices is a Riemannian manifold, in which the Riemannian metric between two matrices A and B is given by

$$\delta(A,B) = \|\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\|_2,$$

here, $\|\cdot\|_2$ denotes the Frobenius norm. The metric geometric mean $A\#B$ is indeed the unique metric midpoint of the geodesic line containing A and B (see, e.g., [9]). The second one is the spectral geometric mean $\natural$, introduced by Fiedler and Pták [6]. In fact, the square of $A\natural B$ is similar to the product AB, and in particular, its eigenvalues coincide with the positive square roots of the eigenvalues of AB. See more information about metric/spectral geometric means in [4, 10] and [5, Chapters 4 and 6].

These two kinds of geometric means can be extended to multiple matrices by iterative processes, see e.g. [8]. Another such iterative geometric mean is the Sagae-Tanabe geometric mean, introduced in [15]. One of the most interesting properties of geometric means is the arithmetic-geometric-harmonic means (AM-GM-HM) inequalities. Indeed, the AM-GM-HM inequality concerning the metric geometric mean was established in [4]. Another version concerning the Sagae-Tanabe geometric mean were discussed in [1, 15].

The Kronecker product of matrices turns out to be compatible with the metric geometric mean in the sense that

$$(A\#B) \otimes (C\#D) = (A \otimes C)\#(B \otimes D) \tag{1}$$

for positive semidefinite matrices A, B, C, D of appropriate sizes (see [4]). Of course, the similar result for the spectral geometric mean is also true ([8]). Moreover, Kilicman and Al-Zhour [8] discussed relations between Tracy-Singh products and metric geometric means, spectral geometric means, and Sagae-Tanabe geometric means of several positive definite matrices.

In this paper, we develop further theory for geometric means of Hilbert space operators. We investigate relationship between metric/spectral/Sagae-Tanabe geometric means and Tracy-Singh products in terms of operator identities and inequalities. In particular, we obtain various generalizations of the property (1) and the AM-GM-HM inequality and its reverse. Moreover, we introduce the weighted Sagae-Tanabe spectral geometric mean for several positive operators and deduce its properties related to Tracy-Singh products.

This paper is organized as follows. Section 2 consists of prerequisites on Tracy-Singh and Khatri-Rao products for Hilbert space operators. In Section 3, we establish certain identities and inequalities between metric geometric means and Tracy-Singh products of several positive operators. Sections 4 and 5 deal with spectral geometric means and Sagae-Tanabe metric geometric means, respectively. In Section 6, we define the weighted Sagae-Tanabe spectral geometric mean for several positive operators and prove certain results related to Tracy-Singh products.

2. Preliminaries on Tracy-Singh and Khatri-Rao Products for Operators

Throughout this paper, let $\mathbb{H}$ and $\mathbb{K}$ be complex separable Hilbert spaces. When X and Y are Hilbert spaces, denote by $\mathcal{B}(X,Y)$ the Banach space of bounded linear operators from X into Y, and abbreviate $\mathcal{B}(X,X)$ to $\mathcal{B}(X)$. For Hermitian operators $A,B \in \mathcal{B}(\mathbb{H})$, the notation $A \geqslant B$ means that $A-B$ is a positive operator, while $A>0$ indicates that A is positive and invertible.

The projection theorem for Hilbert spaces allows us to decompose

$$\mathbb{H} = \bigoplus_{i=1}^{m} \mathbb{H}_i, \quad \mathbb{K} = \bigoplus_{l=1}^{n} \mathbb{K}_k$$

where all $\mathbb{H}_i$ and $\mathbb{K}_k$ are Hilbert spaces. Each operator $A \in \mathcal{B}(\mathbb{H})$ and $B \in \mathcal{B}(\mathbb{K})$ thus can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,m} \text{ and } B = [B_{kl}]_{k,l=1}^{n,n}$$

where $A_{ij} \in \mathcal{B}(\mathbb{H}_j,\mathbb{H}_i)$ and $B_{kl} \in \mathcal{B}(\mathbb{K}_l,\mathbb{K}_k)$ for each $i,j=1,\dots,m$ and $k,l=1,\dots,n$.

Definition 1. Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathcal{B}(\mathbb{K})$ be operator matrices defined as above. The *Tracy-Singh product* of A and B is defined to be the operator matrix

$$A \boxtimes B = \left[[A_{ij} \otimes B_{kl}]_{kl} \right]_{ij}, \tag{2}$$

which is a bounded linear operator from $\bigoplus_{i,k=1}^{m,n} \mathbb{H}_i \otimes \mathbb{K}_k$ into itself.

Note that if $m=n=1$, the Tracy-Singh product $A \boxtimes B$ reduces to the tensor product $A \otimes B$.

Lemma 1 ([12, 13])**.** *Let $A,C \in \mathcal{B}(\mathbb{H})$ and $B,D \in \mathcal{B}(\mathbb{K})$ be compatible operator matrices.*

(i) *The Tracy-Singh product is compatible with the usual product in the sense that*

$$(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD.$$

(ii) *If $A,B>0$, then $(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha$ for any real number α.*

(iii) *If $A,B \geqslant 0$, then $A \boxtimes B \geqslant 0$.*

(iv) *If $A,B>0$, then $A \boxtimes B > 0$.*

(v) *If $A \geqslant C \geqslant 0$ and $B \geqslant D \geqslant 0$, then $A \boxtimes B \geqslant C \boxtimes D$.*

The notion of Khstri-Rao product for operators was introduced in [11].

Definition 2. Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{H})$ and $B = [B_{ij}]_{i,j=1}^{m,m} \in \mathcal{B}(\mathbb{K})$ be operator matrices. The *Khatri-Rao product* of A and B is defined to be

$$A \boxdot B = [A_{ij} \otimes B_{ij}]_{i,j=1}^{m,m} \tag{3}$$

as a bounded linear operator from $\bigoplus_{i=1}^{m} \mathbb{H}_i \otimes \mathbb{K}_i$ into itself.

Note that if $m=1$, then $A \boxdot B = A \otimes B$. We set $\boxplus_{i=1}^{1} A_i = A_1 = \boxdot_{i=1}^{1} A_i$. For $r \in \mathbb{N}-\{1\}$ and a finite number of operator matrices $A_i \in \mathcal{B}(\mathbb{H}_i)$ $(i=1,\dots,r)$, denote

$$\mathop{\boxtimes}_{i=1}^{r} A_i = ((A_1 \boxtimes A_2) \boxtimes \cdots \boxtimes A_{r-1}) \boxtimes A_r, \qquad \mathop{\boxdot}_{i=1}^{r} A_i = ((A_1 \boxdot A_2) \boxdot \cdots \boxdot A_{r-1}) \boxdot A_r.$$

Lemma 2 ([14]). *Let $r \in \mathbb{N} - \{1\}$. There exists an isometry Z such that*

$$\mathop{\boxdot}_{i=1}^{r} A_i = Z^* \left(\mathop{\boxtimes}_{i=1}^{r} A_i \right) Z \tag{4}$$

for any $A_i \in \mathcal{B}(\mathbb{H}_i)$, $i = 1, \dots, r$.

3. Metric Geometric Mean

In this section, we establish certain operator identities and inequalities involving metric geometric means and Tracy-Singh products. First of all, we recall some background about metric geometric means.

The metric geometric mean for matrices/operators was firstly defined by Ando [3]:

$$A \# B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}, \quad A, B > 0. \tag{5}$$

This formula comes from two natural requirements. First, it should coincide with the usual geometric mean for positive real numbers: $A \# B = (AB)^{1/2}$ provided that $AB = BA$. The second condition is the congruent invariance

$$T^*(A \# B)T = (T^*AT) \# (T^*BT)$$

for any invertible $T \in \mathcal{B}(\mathbb{H})$. Now, consider positive invertible operators A and B in $\mathcal{B}(\mathbb{H})$ and let $w \in [0,1]$. The w-weighted geometric mean of A and B is defined by

$$A \#_w B = A^{1/2}(A^{-1/2}BA^{-1/2})^w A^{1/2}.$$

For arbitrary positive operators A and B, we define the w-weighted geometric mean of A and B to be

$$A \#_w B = \lim_{\varepsilon \to 0^+} (A + \varepsilon I) \#_w (B + \varepsilon I).$$

Here, the limit is taken in the strong-operator topology. For briefly, we write $A \# B$ for $A \#_{1/2} B$.

Theorem 1. *Let A_1, A_2, B_1 and B_2 be positive operators in $\mathcal{B}(\mathbb{H})$ and $w \in [0,1]$. Then*

$$(A_1 \#_w A_2) \boxtimes (B_1 \#_w B_2) = (A_1 \boxtimes B_1) \#_w (A_2 \boxtimes B_2), \tag{6}$$

$$(A_1 \#_w A_2) \boxdot (B_1 \#_w B_2) \leqslant (A_1 \boxdot B_1) \#_w (A_2 \boxdot B_2). \tag{7}$$

Proof. First, consider the case $A_1, A_2, B_1, B_2 > 0$. By using Lemma 1, we get

$$\begin{aligned}
(A_1 \boxtimes B_1) \#_w (A_2 \boxtimes B_2) &= (A_1 \boxtimes B_1)^{1/2} \left[(A_1 \boxtimes B_1)^{-1/2} (A_2 \boxtimes B_2)(A_1 \boxtimes B_1)^{-1/2} \right]^w (A_1 \boxtimes B_1)^{1/2} \\
&= \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \left[\left(A_1^{-1/2} \boxtimes B_1^{-1/2} \right) (A_2 \boxtimes B_2) \left(A_1^{-1/2} \boxtimes B_1^{-1/2} \right) \right]^w \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \\
&= \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \left[\left(A_1^{-1/2} A_2 A_1^{-1/2} \right) \boxtimes \left(B_1^{-1/2} B_2 B_1^{-1/2} \right) \right]^w \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \\
&= \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \left[\left(A_1^{-1/2} A_2 A_1^{-1/2} \right)^w \boxtimes \left(B_1^{-1/2} B_2 B_1^{-1/2} \right)^w \right] \left(A_1^{1/2} \boxtimes B_1^{1/2} \right) \\
&= \left[A_1^{1/2} \left(A_1^{-1/2} A_2 A_1^{-1/2} \right)^w A_1^{1/2} \right] \boxtimes \left[B^{1/2} \left(B_1^{-1/2} B_2 B_1^{-1/2} \right)^w B_1^{1/2} \right] \\
&= (A_1 \#_w A_2) \boxtimes (B_1 \#_w B_2).
\end{aligned}$$

Ando's result [4] states that if Φ is a positive linear map, then for all $A, B \geqslant 0$,

$$\Phi(A \#_w B) \leqslant \Phi(A) \#_w \Phi(B). \tag{8}$$

By applying Lemma 2 and inequality (8), we have

$$\begin{aligned}(A_1\#_w A_2)\boxdot(B_1\#_w B_2) &= Z^*\,[(A_1\#_w A_2)\boxtimes(B_1\#_w B_2)]Z\\ &= Z^*\,[(A_1\boxtimes A_2)\#_w(A_2\boxtimes B_2)]Z\\ &\leqslant [Z^*(A_1\boxtimes B_1)Z]\#_w[Z^*(A_2\boxtimes B_2)Z]\\ &= (A_1\boxdot B_1)\#_w(A_2\boxdot B_2).\end{aligned}$$

For arbitrary $A_1,A_2,B_1,B_2\geqslant 0$, perturb each of them with εI and then take limit as $\varepsilon\to 0^+$. □

Corollary 1. *Let $r\in\mathbb{N}$ and $w\in[0,1]$. For each $1\leqslant i\leqslant r$, let $A_i,B_i\in\mathcal{B}(\mathbb{H})$ be positive operators. Then*

$$\mathop{\boxtimes}_{i=1}^{r}(A_i\#_w B_i)=\left(\mathop{\boxtimes}_{i=1}^{r}A_i\right)\#_w\left(\mathop{\boxtimes}_{i=1}^{r}B_i\right). \tag{9}$$

Proof. The proof is by induction on r. □

In [8], Kilicman and Al-Zhour invesigated weighted metric geometric means of any finite number of positive definite matrices. Now, we will extend this geometric mean to the case of finite number of positive operators.

Definition 3. Let $r\in\mathbb{N}-\{1\}$. For each $1\leqslant i\leqslant r$, let $A_i\in\mathcal{B}(\mathbb{H})$ be a positive operator. Let $\alpha_i\in[0,1]$ for each $1\leqslant i\leqslant r-1$ and denote $\alpha=(\alpha_1,\dots,\alpha_{r-1})$. We define

$$\mathcal{G}_{\alpha_1}(A_1,A_2)=A_2\ \#_{\alpha_1}\ A_1.$$

Now continue recurrently, setting

$$\mathcal{G}_{\alpha}(A_1,\dots,A_r)=\mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{\alpha}}(A_1,\dots,A_{r-1}),A_r)$$

where $\tilde{\alpha}=(\alpha_1,\dots,\alpha_{r-2})$. We call $\mathcal{G}_{\alpha}(A_1,\dots,A_r)$ the *iterative α-weighted metric geometric mean* of $A_1,\dots,A_r$.

The next two results asserts the compatibility between Tracy-Singh products and iterative weighted metric geometric means.

Theorem 2. *Let $r\in\mathbb{N}-\{1\}$. For each $1\leqslant i\leqslant r$, let $A_i,B_i\in\mathcal{B}(\mathbb{H})$ be positive operators. Let $\alpha_i\in[0,1]$ for each $1\leqslant i\leqslant r-1$. Then*

$$\mathcal{G}_{\alpha}(A_1\boxtimes B_1,\dots,A_r\boxtimes B_r)=\mathcal{G}_{\alpha}(A_1,\dots,A_r)\boxtimes\mathcal{G}_{\alpha}(B_1,\dots,B_r) \tag{10}$$

Proof. We use induction on r. By continuity, we may assume that $A_i,B_i>0$ for all $i=1,\dots,r$. When $r=2$, we have by Proposition 1 that

$$\begin{aligned}\mathcal{G}_{\alpha}(A_1\boxtimes B_1,A_2\boxtimes B_2) &= (A_2\boxtimes B_2)\#_{\alpha}(A_1\boxtimes B_1)\\ &= (A_2\#_{\alpha}A_1)\boxtimes(B_2\#_{\alpha}B_1)\\ &= \mathcal{G}_{\alpha}(A_1,A_2)\boxtimes\mathcal{G}_{\alpha}(B_1,B_2)\end{aligned}$$

where $\alpha\in[0,1]$. This gives the claim when $r=2$. Suppose that the property (10) holds for $r-1$ ($r\geqslant 3$). Let $\alpha=(\alpha_1,\dots,\alpha_{r-1})$ and $\tilde{\alpha}=(\alpha_1,\dots,\alpha_{r-2})$ where $\alpha_i\in[0,1]$ for any $1\leqslant i\leqslant r-1$.

Using Theorem 1, we have

$$\begin{aligned}\mathcal{G}_\alpha(A_1 \boxtimes B_1,\dots,A_r \boxtimes B_r) &= \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{\alpha}}(A_1 \boxtimes B_1,\dots,A_{r-1} \boxtimes B_{r-1}),A_r \boxtimes B_r)\\ &= \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{\alpha}}(A_1,\dots,A_{r-1}) \boxtimes \mathcal{G}_{\tilde{\alpha}}(B_1,\dots,B_{r-1}),A_r \boxtimes B_r)\\ &= \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{\alpha}}(A_1,\dots,A_{r-1}),A_r) \boxtimes \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{\alpha}}(B_1,\dots,B_{r-1}),B_r)\\ &= \mathcal{G}_\alpha(A_1,\dots,A_r) \boxtimes \mathcal{G}_\alpha(B_1,\dots,B_r).\end{aligned}$$ □

Corollary 2. *Let $r \in \mathbb{N}-\{1\}$ and $s \in \mathbb{N}$. For each $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$, let $A_{ij} \in \mathcal{B}(\mathbb{H})$ be a positive operator. Let $\alpha_i \in [0,1]$ for each $1 \leqslant i \leqslant r-1$. Then*

$$\mathcal{G}_\alpha\left(\boxtimes_{j=1}^{s} A_{1j},\dots,\boxtimes_{j=1}^{s} A_{rj}\right) = \boxtimes_{j=1}^{s} \mathcal{G}_\alpha(A_{1j},\dots,A_{rj}). \tag{11}$$

The *Thompson metric* [16] on the open convex cone of positive invertible operators is defined for each $A,B>0$ by

$$d(A,B) = \max\{\log M(A/B), \log M(B/A)\},$$

where $M(A/B) = \inf\{\lambda > 0 : A \leqslant \lambda B\}$. The *diameter* of $\{A_1,\dots,A_r\}$ with respect to the Thompson metric d is defined by

$$\triangle(A_1,\dots,A_r) = \max\{d(A_i,A_j) : 1 \leqslant i,j \leqslant r\}.$$

Lemma 3. *Let $r \in \mathbb{N}-\{1\}$. Let A_i for each $1 \leqslant i \leqslant r$ and B be positive invertible operators on $\mathbb{H}$. Let $\alpha_i \in [0,1]$ for each $1 \leqslant i \leqslant r-1$. Then*

$$d(\mathcal{G}_\alpha(A_1,\dots,A_r),B) \leqslant \triangle(A_1,\dots,A_r,B). \tag{12}$$

Proof. See [1, Proposition 3.1]. □

The next result is a generalization of inequality (12).

Proposition 1. *Let $r \in \mathbb{N}-\{1\}$ and $s \in \mathbb{N}$. For each $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$, let $A_{ij}, B_j \in \mathcal{B}(\mathbb{H})$ be positive invertible operators. Let $\alpha_i \in [0,1]$ for each $1 \leqslant i \leqslant r-1$. Then*

$$d\left(\boxtimes_{j=1}^{s} \mathcal{G}_\alpha(A_{1j},\dots,A_{rj}), \boxtimes_{j=1}^{s} B_j\right) \leqslant \triangle\left(\boxtimes_{j=1}^{s} A_{1j},\dots,\boxtimes_{j=1}^{s} A_{rj}, \boxtimes_{j=1}^{s} B_j\right). \tag{13}$$

Proof. This proposition follows from Lemma 3 and Corollary 2. □

4. Spectral Geometric Mean

Recall that for positive definite matrices A and B of the same size, its spectral geometric mean [6] is defined by

$$A \natural B = (A^{-1} \# B)^{\frac{1}{2}} A (A^{-1} \# B)^{\frac{1}{2}}.$$

Now, let A and B be positive invertible operators in $\mathcal{B}(\mathbb{H})$ and $w \in [0,1]$. The w-weighted spectral geometric mean of A and B is defined by

$$A \natural_w B = (A^{-1} \# B)^w A (A^{-1} \# B)^w.$$

For arbitrary positive operators A and B, we define the w-weighted spectral geometric mean of A and B to be

$$A\natural_w B = \lim_{\varepsilon\to 0^+}(A+\varepsilon I)\natural_w(B+\varepsilon I).$$

Theorem 3. *Let A_1,A_2,B_1 and B_2 be positive operators in $\mathcal{B}(\mathbb{H})$. Then*

$$(A_1\boxtimes B_1)\natural_w(A_2\boxtimes B_2)=(A_1\natural_w A_2)\boxtimes(B_1\natural_w B_2). \tag{14}$$

Proof. By continuity, we may assume that $A_1,A_2,B_1,B_2>0$. It follows from Lemma 1 and Proposition 1 that

$$\begin{aligned}(A_1\boxtimes B_1)\natural_w(A_2\boxtimes B_2)&=\left[(A_1\boxtimes B_1)^{-1}\#(A_2\boxtimes B_2)\right]^w(A_1\boxtimes B_1)\left[(A_1\boxtimes B_1)^{-1}\#(A_2\boxtimes B_2)\right]^w\\&=\left[(A_1^{-1}\boxtimes B_1^{-1})\#(A_2\boxtimes B_2)\right]^w(A_1\boxtimes B_1)\left[(A_1^{-1}\boxtimes B_1^{-1})\#(A_2\boxtimes B_2)\right]^w\\&=\left[(A_1^{-1}\#A_2)\boxtimes(B_1^{-1}\#B_2)\right]^w(A_1\boxtimes B_1)\left[(A_1^{-1}\#A_2)\boxtimes(B_1^{-1}\#B_2)\right]^w\\&=\left[(A_1^{-1}\#A_2)^w\boxtimes(B_1^{-1}\#B_2)^w\right](A_1\boxtimes B_1)\left[(A_1^{-1}\#A_2)^w\boxtimes(B_1^{-1}\#B_2)^w\right]\\&=\left[(A_1^{-1}\#A_2)^wA_1(A_1^{-1}\#A_2)^w\right]\boxtimes\left[(B_1^{-1}\#B_2)^wB_1(B_1^{-1}\#B_2)^w\right]\\&=(A_1\natural_w A_2)\boxtimes(B_1\natural_w B_2).\end{aligned}$$

□

Corollary 3. *Let $r\in\mathbb{N}-\{1\}$ and $w\in[0,1]$. For each $1\leqslant i\leqslant r$, let $A_i,B_i\in\mathcal{B}(\mathbb{H})$ be positive operators. Then*

$$\left(\boxtimes_{i=1}^{r}A_i\right)\natural_w\left(\boxtimes_{i=1}^{r}B_i\right)=\boxtimes_{i=1}^{r}(A_i\natural_w B_i). \tag{15}$$

Proof. The proof is by induction on r. We have that the property (15) holds for $r=2$ by Lemma 3. Suppose that the property (15) holds for $r-1$ ($r\geqslant 3$). By using Lemma 3, we get

$$\begin{aligned}\left(\boxtimes_{i=1}^{r}A_i\right)\natural_w\left(\boxtimes_{i=1}^{r}B_i\right)&=\left[\left(\boxtimes_{i=1}^{r-1}A_i\right)\boxtimes A_r\right]\natural_w\left[\left(\boxtimes_{i=1}^{r-1}B_i\right)\boxtimes B_r\right]\\&=\left[\left(\boxtimes_{i=1}^{r-1}A_i\right)\natural_w\left(\boxtimes_{i=1}^{r-1}B_i\right)\right]\boxtimes(A_r\natural_w B_r)\\&=\left(\boxtimes_{i=1}^{r-1}(A_i\natural_w B_i)\right)\boxtimes(A_r\natural_w B_r)\\&=\boxtimes_{i=1}^{r}(A_i\natural_w B_i).\end{aligned}$$

□

In [8], Kilicman and Al-Zhour studied weighted spectral geometric means of any finite number of positive definite matrices and proved several properties related to Tracy-Singh products. Now, we will extend this geometric mean to the case of any finite number of positive operators.

Definition 4. Let $r\in\mathbb{N}-\{1\}$. For each $1\leqslant i\leqslant r$, let $A_i\in\mathcal{B}(\mathbb{H})$ be positive operators. Let $\alpha_i\in[0,1]$ for each $1\leqslant i\leqslant r-1$. We define

$$\mathcal{G}^{sp}_{\alpha_1}(A_1,A_2)=A_1\natural_{\alpha_1}A_2.$$

Now continue recurrently, setting for each $r \geqslant 3$,

$$\mathcal{G}_{\alpha}^{sp}(A_1,\ldots,A_r) = \mathcal{G}_{\alpha_{r-1}}^{sp}\left(\mathcal{G}_{\tilde{\alpha}}^{sp}(A_1,\ldots,A_{r-1}),A_r\right)$$

where $\tilde{\alpha} = (\alpha_1,\ldots,\alpha_{r-2})$. We call $\mathcal{G}_{\alpha}^{sp}(A_1,\ldots,A_r)$ the *iterated α-weighted spectral geometric mean* of $A_1,\ldots,A_r$.

From Definition 4, we can rewrite (15) in Corollary 3 to be

$$\mathcal{G}_{\alpha}^{sp}\left(\boxtimes_{i=1}^{r} A_i, \boxtimes_{i=1}^{r} B_i\right) = \boxtimes_{i=1}^{r} \mathcal{G}_{\alpha}^{sp}(A_i,B_i)$$

where $\alpha = w$.

Corollary 4. *Let $r \in \mathbb{N} - \{1\}$. Let A_i and B_i be compatible positive operators in $\mathcal{B}(\mathbb{H})$ for each $i = 1,\ldots,r$. Then*

$$\mathcal{G}_{\alpha}^{sp}(A_1 \boxtimes B_1,\ldots,A_r \boxtimes B_r) = \mathcal{G}_{\alpha}^{sp}(A_1,\ldots,A_r) \boxtimes \mathcal{G}_{\alpha}^{sp}(B_1,\ldots,B_r). \tag{16}$$

Proof. The proof is by induction on r. By Theorem 3, we have that the property (16) is true for $r = 2$. Suppose that the property (16) is true for $r-1$. By Theorem 3, we obtain

$$\begin{aligned}
\mathcal{G}_{\alpha}^{sp}(A_1 \boxtimes B_1,\ldots,A_r \boxtimes B_r) &= \mathcal{G}_{\alpha_{r-1}}^{sp}\left(\mathcal{G}_{\tilde{\alpha}}^{sp}(A_1 \boxtimes B_1,\ldots,A_{r-1} \boxtimes B_{r-1}),A_r \boxtimes B_r\right) \\
&= \mathcal{G}_{\alpha_{r-1}}^{sp}\left(\mathcal{G}_{\tilde{\alpha}}^{sp}(A_1,\ldots,A_{r-1}) \boxtimes \mathcal{G}_{\tilde{\alpha}}^{sp}(B_1,\ldots,B_{r-1}),A_r \boxtimes B_r\right) \\
&= \mathcal{G}_{\alpha_{r-1}}^{sp}\left(\mathcal{G}_{\tilde{\alpha}}^{sp}(A_1,\ldots,A_{r-1}),A_r\right) \boxtimes \mathcal{G}_{\alpha_{r-1}}^{sp}\left(\mathcal{G}_{\tilde{\alpha}}^{sp}(B_1,\ldots,B_{r-1}),B_r\right) \\
&= \mathcal{G}_{\alpha}^{sp}(A_1,\ldots,A_r) \boxtimes \mathcal{G}_{\alpha}^{sp}(B_1,\ldots,B_r).
\end{aligned}$$

□

Corollary 5. *Let $r \in \mathbb{N} - \{1\}$ and $s \in \mathbb{N}$. Let $A_{ij} \in \mathcal{B}(\mathbb{H})$ be a positive invertible operator for each $i = 1,\ldots,r$, $j = 1,\ldots,s$. Then*

$$\mathcal{G}_{\alpha}^{sp}\left(\boxtimes_{j=1}^{s} A_{1j},\ldots,\boxtimes_{j=1}^{s} A_{rj}\right) = \boxtimes_{j=1}^{s} \mathcal{G}_{\alpha}^{sp}(A_{1j},\ldots,A_{rj}). \tag{17}$$

Proof. The proof is by induction on s. □

5. Sagae-Tanabe Metric Geometric Mean

Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i \in \mathcal{B}(\mathbb{H})$ be a positive invertible operator. Let $t = (t_1,\ldots,t_r)$ where $t_i \in [0,1]$ for each $1 \leqslant i \leqslant r$ and $\sum_{i=1}^{r} t_i = 1$. The *weighted arithmetic* and *harmonic means* of $A_1,\ldots,A_r$ are defined by

$$\mathcal{A}_t(A_1,\ldots,A_r) = \sum_{i=1}^{r} t_i A_i, \qquad \mathcal{H}_t(A_1,\ldots,A_r) = \left(\sum_{i=1}^{r} t_i A_i^{-1}\right)^{-1}.$$

Sagae and Tanabe [15] proposed weighted geometric means of severable positive definite matrices as follows.

Definition 5. Let A and B be positive invertible operators in $\mathcal{B}(\mathbb{H})$ and let $v = (v_1,v_2)$ where v_1, $v_2 \in [0,1]$ and $v_1 + v_2 = 1$. We define

$$\mathcal{G}_v(A,B) = A \#_{\alpha} B$$

where $\alpha = 1 - v_2$. Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i \in \mathcal{B}(\mathbb{H})$ be a positive invertible operator. Let $t = (t_1, \dots, t_r)$ where $t_i \in [0,1]$ for each $1 \leqslant i \leqslant r$ and $\sum_{i=1}^{r} t_i = 1$. For each $1 \leqslant i \leqslant r-1$, let

$$\alpha_i = 1 - \left(t_{i+1} \Big/ \sum_{j=1}^{i+1} t_j \right).$$

The *Sagae-Tanabe weighted geometric mean* of $A_1, \dots, A_r$ is defined by

$$\mathcal{G}_t(A_1, \dots, A_r) = \mathcal{G}_{\alpha_{r-1}}(\mathcal{G}_{\tilde{t}}(A_1, \dots, A_{r-1}), A_r)$$

where $\mathcal{G}_{\tilde{t}}(A_1, \dots, A_{r-1})$ is the Sagae-Tanabe weighted geometric mean of $A_1, \dots, A_{r-1}$ with weighted $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{r-1})$ where $\tilde{t}_i = t_i / \sum_{j=1}^{r-1} t_j$ for each $1 \leqslant i \leqslant r-1$. Note that

$$\mathcal{G}_t(A_1, \dots, A_r) = \mathcal{G}_\alpha(A_1, \dots, A_r)$$

where $\mathcal{G}_\alpha(A_1, \dots, A_r)$ is the weighted metric geometric mean of $A_1, \dots, A_r$ in Definition 3 with weight $\alpha = (\alpha_1, \dots, \alpha_{r-1})$.

Theorem 4. *Let $r \in \mathbb{N} - \{1\}$ and $s \in \mathbb{N}$. For each $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$, let $A_{ij} \in \mathcal{B}(\mathbb{H})$ be a positive invertible operator. Let $t = (t_1, \dots, t_r)$ where $t_i \in [0,1]$ for each $1 \leqslant i \leqslant r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\mathcal{G}_t\left(\boxtimes_{j=1}^{s} A_{1j}, \dots, \boxtimes_{j=1}^{s} A_{rj} \right) = \boxtimes_{j=1}^{s} \mathcal{G}_t(A_{1j}, \dots, A_{rj}). \tag{18}$$

Proof. Let $\alpha_i = 1 - (t_{i+1} / \sum_{j=1}^{i+1} t_j)$ for each $1 \leqslant i \leqslant r-1$ and denote $\alpha = (\alpha_1, \dots, \alpha_{r-1})$. By Definition 5, we have

$$\mathcal{G}_t(A_{11} \boxtimes A_{12}, \dots, A_{r1} \boxtimes A_{r2}) = \mathcal{G}_\alpha(A_{11} \boxtimes A_{12}, \dots, A_{rj} \boxtimes B_{rj}).$$

Applying Theorem 2, we obtain

$$\mathcal{G}_\alpha(A_{11} \boxtimes A_{12}, \dots, A_{rj} \boxtimes B_{rj}) = \mathcal{G}_\alpha(A_{11}, \dots, A_{r1}) \boxtimes \mathcal{G}_\alpha(A_{12}, \dots, A_{r2}).$$

This implies that

$$\mathcal{G}_t(A_{11} \boxtimes A_{12}, \dots, A_{r1} \boxtimes A_{r2}) = \mathcal{G}_t(A_{11}, \dots, A_{r1}) \boxtimes \mathcal{G}_t(A_{12}, \dots, A_{r2}).$$

We get the result by using induction on s. □

Lemma 4. *Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i \in \mathcal{B}(\mathbb{H})$ be a positive invertible operator and let $t = (t_1, \dots, t_r)$ where $t_i \in [0,1]$ for $i = 1, \dots, r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\mathcal{H}_t(A_1, \dots, A_r) \leqslant \mathcal{G}_t(A_1, \dots, A_r) \leqslant \mathcal{A}_t(A_1, \dots, A_r). \tag{19}$$

Proof. See [1, Proposition 2.4]. □

We extend [8, Theorem 4.6] to AM-GM-HM inequalities involving Tracy-Singh product of positive invertible operators as in the next two results.

Corollary 6. *Let $r \in \mathbb{N} - \{1\}$ and $s \in \mathbb{N}$. For each $1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s$, let $A_{ij} \in \mathcal{B}(\mathbb{H})$ be a positive invertible operator and let $t = (t_1, \dots, t_r)$ where $t_i \in [0,1]$ for $i = 1, \dots, r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\boxtimes_{j=1}^{s} \mathcal{H}_t(A_{1j}, \dots, A_{rj}) \leqslant \mathcal{G}_t\left(\boxtimes_{j=1}^{s} A_{1j}, \dots, \boxtimes_{j=1}^{s} A_{rj} \right) \leqslant \boxtimes_{j=1}^{s} \mathcal{A}_t(A_{1j}, \dots, A_{rj}). \tag{20}$$

Proof. Lemma 4 tells us that

$$\mathcal{H}_t(A_{1j},\dots,A_{rj}) \leqslant \mathcal{G}_t\left(A_{1j},\dots,A_{rj}\right) \leqslant \mathcal{A}_t(A_{1j},\dots,A_{rj})$$

for each $1 \leqslant j \leqslant s$. By using Lemma 1, we get

$$\mathop{\boxtimes}_{j=1}^{s} \mathcal{H}_t(A_{1j},\dots,A_{rj}) \leqslant \mathop{\boxtimes}_{j=1}^{s} \mathcal{G}_t\left(A_{1j},\dots,A_{rj}\right) \leqslant \mathop{\boxtimes}_{j=1}^{s} \mathcal{A}_t(A_{1j},\dots,A_{rj}).$$

Applying Theorem 4, we obtain

$$\mathop{\boxtimes}_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rj}) = \mathcal{G}_t\left(\mathop{\boxtimes}_{j=1}^{s} A_{1j},\dots,\mathop{\boxtimes}_{j=1}^{s} A_{rj}\right)$$

and the inequality (20) follows. □

Corollary 7. *Let $r \in \mathbb{N} - \{1\}$ and $s \in \mathbb{N}$. For each $1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s$, let $A_{ij} \in \mathcal{B}(\mathbb{H})$ be a positive invertible operator and let $t = (t_1,\dots,t_r)$ where $t_i \in [0,1]$ for $i = 1,\dots,r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\mathcal{H}_t\left(\mathop{\boxtimes}_{j=1}^{s} A_{1j},\dots,\mathop{\boxtimes}_{j=1}^{s} A_{rj}\right) \leqslant \mathop{\boxtimes}_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rj}) \leqslant \mathcal{A}_t\left(\mathop{\boxtimes}_{j=1}^{s} A_{1j},\dots,\mathop{\boxtimes}_{j=1}^{s} A_{rj}\right). \tag{21}$$

Proof. It follows directly from the AM-GM-HM inequality (19) and Theorem 4. □

We now turn to the AM-GM-HM inequality involving Khatri-Rao products.

Corollary 8. *Let $A_{ij} \in \mathcal{B}(\mathbb{H})$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ be compatible positive invertible operators and let $t = (t_1,\dots,t_r)$ where $t_i \in [0,1]$ for $i = 1,\dots,r$ and $\sum_{i=1}^{r} t_i = 1$. Then*

$$\mathop{\boxdot}_{j=1}^{s} \mathcal{H}_t(A_{1j},\dots,A_{rj}) \leqslant \mathop{\boxdot}_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rj}) \leqslant \mathcal{A}_t\left(\mathop{\boxdot}_{j=1}^{s} A_{1j},\dots,\mathop{\boxdot}_{j=1}^{s} A_{rj}\right). \tag{22}$$

Proof. We have by Lemmas 2 and 4 that

$$\begin{aligned}
\mathop{\boxdot}_{j=1}^{s} \mathcal{H}_t(A_{1j},\dots,A_{rj}) &= Z^*\left(\mathop{\boxtimes}_{j=1}^{s} \mathcal{H}_t(A_{1j},\dots,A_{rj})\right) Z \\
&\leqslant Z^*\left(\mathop{\boxtimes}_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rj})\right) Z \\
&= \mathop{\boxdot}_{j=1}^{s} \mathcal{G}_t(A_{1j},\dots,A_{rj}).
\end{aligned}$$

Using Lemma 2, we get

$$\begin{aligned}
Z^*\left[\mathcal{A}_t\left(\mathop{\boxtimes}_{j=1}^{s} A_{1j},\dots,\mathop{\boxtimes}_{j=1}^{s} A_{rj}\right)\right] Z &= Z^*\left[\sum_{i=1}^{r} t_i\left(\mathop{\boxtimes}_{j=1}^{s} A_{ij}\right)\right] Z \\
&= \sum_{i=1}^{r} t_i\left[Z^*\left(\mathop{\boxtimes}_{j=1}^{s} A_{ij}\right) Z\right] \\
&= \sum_{i=1}^{r} t_i\left(\mathop{\boxdot}_{j=1}^{s} A_{ij}\right) \\
&= \mathcal{A}_t\left(\mathop{\boxdot}_{j=1}^{s} A_{1j},\dots,\mathop{\boxdot}_{j=1}^{s} A_{rj}\right).
\end{aligned}$$

Applying Lemma 2 and Corollary 7, we obtain

$$\begin{aligned}\boxed{\cdot}_{j=1}^{s}\mathcal{G}_t(A_{1j},\dots,A_{rj}) &= Z^*\left(\boxtimes_{j=1}^{s}\mathcal{G}_t(A_{1j},\dots,A_{rj})\right)Z\\ &\leqslant Z^*\left[\mathcal{A}_t\left(\boxtimes_{j=1}^{s}A_{1j},\dots,\boxtimes_{j=1}^{s}A_{rj}\right)\right]Z\\ &= \mathcal{A}_t\left(\boxed{\cdot}_{j=1}^{s}A_{1j},\dots,\boxed{\cdot}_{j=1}^{s}A_{rj}\right).\end{aligned}$$ □

The next result is a generalization of Lemma 3.

Proposition 2. *Let A_{ij} and B_j $(1\leqslant i\leqslant r,\ 1\leqslant j\leqslant s,\ r\geqslant 2)$ be compatible positive invertible operators and let $t=(t_1,\dots,t_r)$ where $t_i\in[0,1]$ for $i=1,\dots,r$ and $\sum_{i=1}^{r}t_i=1$. Then*

$$d\left(\boxtimes_{j=1}^{s}\mathcal{G}_t(A_{1j},\dots,A_{rj}),\boxtimes_{j=1}^{s}B_j\right)\leqslant\triangle\left(\boxtimes_{j=1}^{s}A_{1j},\dots,\boxtimes_{j=1}^{s}A_{rj},\boxtimes_{j=1}^{s}B_j\right). \tag{23}$$

Proof. The desire result follows from Lemma 3 and Corollary 2. □

For $h,x\geqslant 1$, the *(generalized) Specht ratio* is defined by

$$S_h(x)=\frac{(h^x-1)h^{x(h^x-1)^{-1}}}{e\log h^x}\quad\text{for } h\neq 1 \text{ and } S_1(x)=1.$$

We denote $S_h(1)$ by S_h. See [1,7] for more information. The next result is a reverse version of AM-GM-HM inequality involving Tracy-Singh products via Specht ratio.

Proposition 3. *Let A_{ij} $(1\leqslant i\leqslant r,\ 1\leqslant j\leqslant s,\ r\geqslant 2)$ be compatible positive invertible operators and let $t=(t_1,\dots,t_r)$ where $t_i\in[0,1]$ for $i=1,\dots,r$ and $\sum_{i=1}^{r}t_i=1$. Then*

$$\mathcal{A}_t\left(\boxtimes_{j=1}^{s}A_{1j},\dots,\boxtimes_{j=1}^{s}A_{rj}\right)\leqslant S_h^{r-1}\cdot\left(\boxtimes_{j=1}^{s}\mathcal{G}_t(A_{1j},\dots,A_{rj})\right) \tag{24}$$

where $h=e^{\triangle(\boxtimes_{j=1}^{s}A_{1j},\dots,\boxtimes_{j=1}^{s}A_{rj})}$.

Proof. By using Lemma 3 and Corollary 2, we get the result. □

Lemma 5. *Let $A_i\in\mathcal{B}(\mathbb{H})$ $(1\leqslant i\leqslant r, r\geqslant 2)$ be positive invertible operators and t_i $(1\leqslant i\leqslant r)$ be real numbers such that $t_1>0, t_i<0$ $(2\leqslant i\leqslant r)$ and $\sum_{i=1}^{r}t_i=1$. Then*

$$\mathcal{A}_t(A_1,\dots,A_r)\leqslant\mathcal{G}_t(A_1,\dots,A_r). \tag{25}$$

If $\sum_{i=1}^{r}t_iA_i^{-1}>0$, then

$$\mathcal{G}_t(A_1,\dots,A_r)\leqslant\mathcal{H}_t(A_1,\dots,A_r). \tag{26}$$

Proof. The proof is similar to the case of matrices, given in [2, Theorem 2.1]. □

We now obtain reverse AM-GM-HM inequalities involving Tracy-Singh products as follows.

Theorem 5. *Let $A_{ij}\in\mathcal{B}(\mathbb{H})$ $(1\leqslant i\leqslant r,\ 1\leqslant j\leqslant s,\ r\geqslant 2)$ be compatible positive invertible operators and t_i $(1\leqslant i\leqslant r)$ be real numbers such that $t_1>0, t_i<0$ $(2\leqslant i\leqslant r)$ and $\sum_{i=1}^{r}t_i=1$.*

Then

$$\boxtimes_{j=1}^{s} \mathcal{A}_t(A_{1j},\ldots,A_{rj}) \leqslant \mathcal{G}_t\left(\boxtimes_{j=1}^{s} A_{1j},\ldots,\boxtimes_{j=1}^{s} A_{rj}\right). \tag{27}$$

If $\mathcal{H}_t(A_{1j},\ldots,A_{rj}) > 0$ *for all* $j = 1,\ldots,s$, *then*

$$\mathcal{G}_t\left(\boxtimes_{j=1}^{s} A_{1j},\ldots,\boxtimes_{j=1}^{s} A_{rj}\right) \leqslant \boxtimes_{j=1}^{s} \mathcal{H}_t(A_{1j},\ldots,A_{rj}). \tag{28}$$

Proof. It follows from Lemma 5 that

$$A_t(A_{1j},\ldots,A_{rj}) \leqslant \mathcal{G}_t\left(A_{1j},\ldots,A_{rj}\right) \leqslant \mathcal{H}_t(A_{1j},\ldots,A_{rj})$$

for each $j = 1,\ldots,s$. Since $A_t(A_{1j},\ldots,A_{rj}) \geqslant 0$ for all $j = 1,\ldots,s$, we have by Lemmas 1 and 5 that

$$\mathcal{G}_t\left(\boxtimes_{j=1}^{s} A_{1j},\ldots,\boxtimes_{j=1}^{s} A_{rj}\right) = \boxtimes_{j=1}^{s} \mathcal{G}_t\left(A_{1j},\ldots,A_{rj}\right) \geqslant \boxtimes_{j=1}^{s} \mathcal{A}_t(A_{1j},\ldots,A_{rj}).$$

Since $\mathcal{H}_t(A_{1j},\ldots,A_{rj}) > 0$ for all $j = 1,\ldots,s$, we obtain by Lemma 1 that

$$\boxtimes_{j=1}^{s} \mathcal{H}_t(A_{1j},\ldots,A_{rj}) > 0.$$

The proof is complete by applying Lemma 5 and Corollary 4. □

Theorem 6. *Let* $A_{ij} \in \mathcal{B}(\mathbb{H})$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ *be compatible positive invertible operators and* t_i $(1 \leqslant i \leqslant r)$ *be real numbers such that* $t_1 > 0, t_i < 0$ $(2 \leqslant i \leqslant r)$ *and* $\sum_{i=1}^{r} t_i = 1$. *Then*

$$\mathcal{A}_t\left(\boxtimes_{j=1}^{s} A_{1j},\ldots,\boxtimes_{j=1}^{s} A_{rj}\right) \leqslant \boxtimes_{j=1}^{s} \mathcal{G}_t(A_{1j},\ldots,A_{rs}). \tag{29}$$

If $\mathcal{H}_t\left(\boxtimes_{j=1}^{s} A_{1j},\ldots,\boxtimes_{j=1}^{s} A_{rj}\right) > 0$, *then*

$$\boxtimes_{j=1}^{s} \mathcal{G}_t(A_{1j},\ldots,A_{rs}) \leqslant \mathcal{H}_t\left(\boxtimes_{j=1}^{s} A_{1j},\ldots,\boxtimes_{j=1}^{s} A_{rj}\right). \tag{30}$$

Proof. By applying Lemma 5 and Corollary 4, we get the results. □

Corollary 9. *Let* $A_{ij} \in \mathcal{B}(\mathbb{H})$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ *be compatible positive invertible operators and* t_i $(1 \leqslant i \leqslant r)$ *be real numbers such that* $t_1 > 0, t_i < 0$ $(2 \leqslant i \leqslant r)$ *and* $\sum_{i=1}^{r} t_i = 1$. *Then*

$$\mathcal{A}_t\left(\boxdot_{j=1}^{s} A_{1j},\ldots,\boxdot_{j=1}^{s} A_{rj}\right) \leqslant \boxdot_{j=1}^{s} \mathcal{G}_t(A_{1j},\ldots,A_{rj}) \leqslant \boxdot_{j=1}^{s} \mathcal{H}_t(A_{1j},\ldots,A_{rj}). \tag{31}$$

Proof. This result is a direct consequence of Theorem 6 and Lemmas 2 and 4. □

6. Sagae-Tanabe Spectral Geometric Mean

We introduce the following definition:

Definition 6. Let $r \in \mathbb{N} - \{1\}$. For each $1 \leqslant i \leqslant r$, let $A_i \in \mathcal{B}(\mathbb{H})$ be a positive invertible operator. Let $t = (t_1,\ldots,t_r)$ where $t_i \in [0,1]$ for each $1 \leqslant i \leqslant r$ and $\sum_{i=1}^{r} t_i = 1$. Let $\alpha_i = 1 - (t_{i+1}/\sum_{j=1}^{i+1} t_j)$

for each $1 \leqslant i \leqslant r-1$. The *Sagae-Tanabe spectral geometric mean* of $A_1,\dots,A_r$ is defined by

$$\mathcal{G}_t^{sp}(A_1,\dots,A_r) = \mathcal{G}_\alpha^{sp}(A_1,\dots,A_r)$$

where $\alpha = (\alpha_1,\dots,\alpha_{r-1})$.

Proposition 4. *Let A_i and B_i $(1 \leqslant i \leqslant r, r \geqslant 2)$ be compatible positive operators in $\mathcal{B}(\mathbb{H})$ and let $t = (t_1,\dots,t_r)$ where $t_i \in [0,1]$ for $i = 1,\dots,r-1$ and $\sum_{i=1}^r t_i = 1$. Then*

$$\mathcal{G}_t^{sp}(A_1 \boxtimes B_1,\dots,A_r \boxtimes B_r) = \mathcal{G}_t^{sp}(A_1,\dots,A_r) \boxtimes \mathcal{G}_t^{sp}(B_1,\dots,B_r) \tag{32}$$

$$\mathcal{G}_t^{sp}\left(\boxtimes_{i=1}^{r} A_i, \boxtimes_{i=1}^{r} B_i\right) = \boxtimes_{i=1}^{r} \mathcal{G}_t^{sp}(A_i,B_i). \tag{33}$$

Proof. Let $\alpha_i = 1-(t_{i+1}/\sum_{j=1}^{i+1} t_j)$ for each $1 \leqslant i \leqslant r-1$ and denote $\alpha = (\alpha_1,\dots,\alpha_{r-1})$. By Definition 6, we have

$$\mathcal{G}_t^{sp}(A_1 \boxtimes B_1,\dots,A_r \boxtimes B_r) = \mathcal{G}_\alpha^{sp}(A_1 \boxtimes B_1,\dots,A_r \boxtimes B_r)$$

$$\mathcal{G}_t^{sp}\left(\boxtimes_{i=1}^{r} A_i, \boxtimes_{i=1}^{r} B_i\right) = \mathcal{G}_\alpha^{sp}\left(\boxtimes_{i=1}^{r} A_i, \boxtimes_{i=1}^{r} B_i\right).$$

By Corollary 4, we get (32). Applying Corollary 3, we obtain (33). □

Corollary 10. *Let $A_{ij} \in \mathcal{B}(\mathbb{H})$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s,\ r \geqslant 2)$ be compatible positive invertible operators and let $t = (t_1,\dots,t_r)$ where $t_i \in [0,1]$ for $i = 1,\dots,r-1$ and $\sum_{i=1}^r t_i = 1$. Then*

$$\mathcal{G}_t^{sp}\left(\boxtimes_{j=1}^{s} A_{1j},\dots,\boxtimes_{j=1}^{s} A_{rj}\right) = \boxtimes_{j=1}^{s} \mathcal{G}_t^{sp}(A_{1j},\dots,A_{rj}). \tag{34}$$

Proof. From (32), we have

$$\mathcal{G}_t^{sp}(A_{11} \boxtimes A_{12},\dots,A_{r1} \boxtimes A_{r2}) = \mathcal{G}_t^{sp}(A_{11},\dots,A_{r1}) \boxtimes \mathcal{G}_t^{sp}(A_(12),\dots,A_{r2}).$$

We obtain (34) by induction on s. □

7. Conclusion

Several relations between metric/spectral/Sagae-Tanabe geometric means for several positive operators and Tracy-Singh products are established in terms of identities and inequalities. In particular, we obtain noncommutative arithmetic-geometric-harmonic means inequalities and their reverses. Moreover, we define the weighted Sagae-Tanabe spectral geometric mean for several positive operators and deduce its properties related to Tracy-Singh products.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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