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Research Article

On Complex η-Einstein Normal Complex Contact Metric Manifolds

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Abstract. The aim of this paper is focusing on η-Einstein geometry of normal complex contact metric manifolds. We give the definition of complex η-Einstein normal complex contact metric manifolds. In addition, we study on concircular curvature tensor $\mathbb{Z}$ on a normal complex contact metric manifold which satisfy $\mathbb{Z}(U,X)\cdot\mathbb{Z}=0$ and $\mathbb{Z}(V,X)\cdot\mathbb{Z}=0$. Also, we prove that a projectively semi-symmetric normal complex contact metric manifold is complex η-Einstein.

Keywords. Normal complex contact metric manifold; Concircular curvature tensor; Projectively semi-symmetric; Complex η-Einstein

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1. Introduction

The Riemannian geometry of contact manifolds are studied widely at last 60 years and could be divided into two parts: reel and complex. While there are rather than works on the Riemannian geometry of reel contact manifolds the complex contact manifolds are still infancy. One can think that is it possible to transfer all results from real contact manifolds to complex unchanged. But it is not possible, so the Riemannian geometry of complex contact manifolds should be studied independently. Also complex contact manifolds have some different results from real case and they have good applications in optimal control of entanglements [12].

Complex contact manifolds were introduced by Kobayashi [13]. Ishihara and Konishi [9, 10, 11] studied contact 3-structures on a Riemann manifold and they proved that if one of these contact 3-structures is normal then there is a complex contact structure on the fibering manifold. They also presented normality condition which one is called IK-normality in the literature. On the other hand same authors proved that if a complex contact metric manifold is IK-normal then the complex structure is Kähler [10]. Unfortunately the complex Heisenberg group, whose real analogue (the real Heisenberg group) is also a well-known normal real contact metric manifold, is not IK-normal. For this reason in 2000, Korkmaz [14] gave a weaker version of normality. This notion of normality is called *normal complex contact manifolds* which we use here. Also she examined curvature properties and defined $\mathcal{GH}$-sectional curvature. The present authors [16] obtained some results on Riemann curvature and Ricci curvature of normal complex contact metric manifolds. Foreman [7] studied on complex contact manifolds and hyperkähler manifolds. In addition, he obtained some results on IK-normal complex contact metric manifolds [7]. One of the present authors and Blair [2] worked on energy distributions for 3-Sasakian and normal complex contact metric manifolds, they also studied the Boothby-Wang fibration of the Iwasawa manifold as a critical point of the energy [15]. Imada [8] also worked on IK-normal complex contact metric manifolds and gave new examples.

The notion of curvature has the center role in the Riemann geometry of manifolds. There are some special curvature tensors which are named projective, conformal, concircular and conharmonic and the flatness of manifolds can be determined by these tensors. Blair and Molina [4] studied on conformal and Bochner tensor on normal complex contact metric manifolds. They showed that normal complex contact metric manifolds can not be conformally flat and under Bochner flatness condition a normal complex contact metric manifold is isometric to $\mathbb{CP}^{2n+1}$. Yildirim [18] proved that with $\kappa < 1$ a complex (κ,μ)-space can not be conformally flat or conharmonically flat. Quasi-conformal, concircular and conharmonic flatness of normal complex contact metric manifolds are studied by presents authors [17] and they proved there are no normal complex contact metric manifolds under the flatness condition of these tensors.

In this article, we introduced complex η-Einstein normal complex contact metric manifolds and we obtain some results on these manifolds. We study on concircular curvature tensor $\mathcal{Z}$ in a normal complex contact metric manifold and we obtain a form for Riemann curvature tensor if the manifold satisfy $\mathcal{Z}(U,X)\mathcal{Z}=0$, $\mathcal{Z}(V,X)\mathcal{Z}=0$. Also, we prove that a projectively semi-symmetric normal complex contact metric manifold is complex η-Einstein.

2. Preliminaries

Definition 2.1 ([13])**.** Let M be a $(2m+1)$-complex dimensional complex manifold and $\mathcal{A}=\{\mathcal{O},\mathcal{O}',\dots\}$ be an open covering by coordinate neighbourhoods with following conditions:

(i) There is a holomoprhic 1-form η on each $\mathcal{O}$ such that $\eta\wedge(d\eta)^m\neq 0$,

(ii) There is a holomoprhic function $\lambda:\mathcal{O}\cap\mathcal{O}'\to\mathbb{C}\setminus\{0\}$ such that $\eta'=\lambda\eta$.

Then (M,η) is called complex contact manifold.

The equation $\eta = 0$ determines a non-integrable subbundle $\mathcal{H}$, it is called the complex contact subbundle or the horizontal subbundle. Ishihara and Konishi [10] proved following theorem for existence of complex almost contact structure on M.

Theorem 2.2 ([10])**.** *An odd complex dimensional complex contact manifold admits always a complex almost contact structure of class C^∞.*

Thus complex almost contact metric structure is given by:

Definition 2.3 ([10])**.** Let M be a complex manifold, g be Hermitian metric and J be complex structure on M. For an open covering $\mathcal{A} = \{\mathcal{O}, \mathcal{O}', \ldots\}$ of M consisting of coordinate neighbourhoods, M is called a *complex almost contact metric manifold* if following two conditions are satisfied:

(i) In each $\mathcal{O} \in \mathcal{A}$, there are given 1-forms u and $v = u \circ J$, with dual vector fields U and $V = -JU$ and $(1,1)$ tensor fields G and $H = GJ$ such that

$$H^2 = G^2 = -I + u \otimes U + v \otimes V,$$
$$GJ = -JG, \quad GU = 0, \quad g(X,GY) = -g(GX,Y),$$
$$GV = 0, \quad g(U,U) = 1.$$

(ii) There are functions a and b on $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$ such that $a^2 + b^2 = 1$, then

$$u' = au - bv, \qquad v' = bu + av,$$
$$G' = aG - bH, \quad H' = bG + aH.$$

For all X in $\mathcal{H}$, let define a unit vector field U locally by $du(U,X) = 0$ and $u(U) = 1$, $v(U) = 0$. At that time a global subbundle $\mathcal{V}$ can be defined which locally spanned by U and $V = -JU$ and it is called vertical subbundle. Also $TM \cong \mathcal{H} \oplus \mathcal{V}$.

Let $\eta = u - iv$ be a local contact form, G and H be local fields are related to du, dv and X, Y be vector fields on M then, we have

$$du(X,Y) = g(X,GY) + (\sigma \wedge v)(X,Y),$$
$$dv(X,Y) = g(X,HY) - (\sigma \wedge u)(X,Y),$$

where $\sigma(X) = g(\nabla_X U, V)$ [10].

Ishihara and Konishi studied on normality of complex almost contact metric manifolds. They proved that a complex almost contact metric manifold admits a complex contact structure if it is normal [11]. For normality they defined two local tensors as following:

$$\begin{aligned}\mathcal{S}(X,Y) = {} & [G,G](X,Y) + 2g(X,GY)U - 2g(X,HY)V + 2(v(Y)HX - v(X)HY) + \sigma(GY)HX \\ & - \sigma(GX)HY + \sigma(X)GHY - \sigma(Y)GHX,\end{aligned}$$
$$\begin{aligned}\mathcal{T}(X,Y) = {} & [H,H](X,Y) - 2g(X,GY)U + 2g(X,HY)V + 2(u(Y)GX - u(X)GY) + \sigma(HX)GY \\ & - \sigma(HY)GX + \sigma(X)GHX - \sigma(Y)GHX,\end{aligned}$$

where $[G,G]$ and $[H,H]$ denote the Nijenhuis tensors of G and H, respectively. These tensors are called the torsion tensors of the given complex almost contact metric structure.

Definition 2.4 ([10])**.** A complex contact metric manifold M is called normal if torsion tensors $\mathcal{S}$ and $\mathcal{T}$ vanish.

When the manifold is normal in this sense it is called IK-normal. IK-ormal was meant to be a complex analogue of normal real contact manifolds. In addition, Ishihara and Konishi [10, 11] proved following proposition.

Proposition 2.5 ([10])**.** *IK-normal complex contact metric manifold has Kählerian structure, i.e.* $\nabla J = 0$.

It seems from above proposition IK-normality get forced structure to be Kähler. The most well-known complex contact manifold $\mathbb{CP}^{2m+1}$ is indeed IK-normal. However, this definition of normality also did not include the complex Heisenberg group [7]. Korkmaz expanded the definition of normality and in this case the complex Heisenberg group is normal.

Definition 2.6 ([14])**.** A complex contact metric manifold M is called normal if it satisfied the following conditions:

$$S\,|_{\mathcal{H}\wedge\mathcal{H}} = S\,|_{\mathcal{H}\wedge\mathcal{V}} = T\,|_{\mathcal{H}\wedge\mathcal{H}} = T\,|_{\mathcal{H}\wedge\mathcal{V}} = 0.$$

We use this notion of normality in this paper. Also in [16] we proved a condition for normality.

Theorem 2.7 ([16])**.** *A complex contact metric manifold is normal if and only if the covariant derivative of G and H have the following forms:*

$$(\nabla_X G)Y = \sigma(X)HY - 2v(X)JY - u(Y)X - v(Y)JX + v(X)(2JY_0 - (\nabla_U J)GY_0) + g(X,Y)U \\ + g(JX,Y)V - d\sigma(U,V)v(X)(u(Y)V - v(Y)U) \tag{2.1}$$

and

$$(\nabla_X H)Y = -\sigma(X)GY + 2u(X)JY + u(Y)JX - v(Y)X + u(X)(-2JY_0 - (\nabla_U J)GY_0) - g(JX,Y)U \\ + g(X,Y)V + d\sigma(U,V)u(X)(u(Y)V - v(Y)U) \tag{2.2}$$

where $Y = Y_0 + u(Y)U + v(Y)V$, $Y_0 \in \mathcal{H}$.

From this theorem on a normal complex contact metric manifold we have

$$(\nabla_X J)Y = -2u(X)HY + 2v(X)GY + u(X)(2HY_0 + (\nabla_U J)Y_0) + v(X)(-2GY_0 + (\nabla_U J)JY_0).$$

For a horizontal vector field X on M and $a^2 + b^2 = 1$ the plane section generated by X and $Y = aGX + bHX$ is called a $\mathcal{GH}$*-section.* Then $\mathcal{GH}$*-sectional curvature* is defined by $\mathcal{GH}_{a,b}(X) = K(X, aGX + bHX)$, where $K(X,Y)$ is the sectional curvature of the plane section spanned by X and Y. We assume that $\mathcal{GH}$*-sectional curvature* $\mathcal{GH}_{a,b}(X)$ does not depend the choice of a and b, and we will use $\mathcal{GH}(X)$ notation. Under this assumption Korkmaz [14] showed a relation between the holomorphic sectional curvature and $\mathcal{GH}$-sectional curvature:

$$K(X,JX) = \mathcal{GH}(X) + 3. \tag{2.3}$$

We will use some curvature properties which we list here [14]. First of all, for U and $V = -JU$ vertical vector fields we have

$$R(U,V,V,U) = R(V,U,U,V) = -2d\sigma(U,V). \tag{2.4}$$

Secondly, for X and Y horizontal vector fields, we have the followings:

$$R(X,U)U = X, \qquad R(X,V)V = X, \tag{2.5}$$

$$R(X,Y)U = 2(g(X,JY) + d\sigma(X,Y))V, \tag{2.6}$$

$$R(X,Y)V = -2(g(X,JY) + d\sigma(X,Y))U, \tag{2.7}$$

$$R(X,U)V = \sigma(U)GX + (\nabla_U H)X - JX, \tag{2.8}$$

$$R(X,V)U = -\sigma(V)HX + (\nabla_V G)X + JX, \tag{2.9}$$

$$R(X,U)Y = -g(X,Y)U - g(JX,Y)V + d\sigma(Y,X)V, \tag{2.10}$$

$$R(X,V)Y = -g(X,Y)V + g(JX,Y)U - d\sigma(Y,X)U, \tag{2.11}$$

$$R(U,V)X = JX. \tag{2.12}$$

Details about complex contact geometry could be founded in [3, §12, p. 233]. On the other hand for X and Y horizontal vector fields Ricci curvature tensor of a $(2m+1)$-complex dimensional normal complex contact metric manifold has the followings [16]:

$$\rho(GX,GY) = \rho(HX,HY) = \rho(X,Y), \tag{2.13}$$

$$\rho(U,U) = \rho(V,V) = 4m - 2d\sigma(U,V) \text{ and } \rho(U,V) = 0, \tag{2.14}$$

$$\rho(X,U) = \rho(X,V) = 0. \tag{2.15}$$

Since $TM = \mathcal{H} \oplus \mathcal{V}$, an arbitrary vector field X can be written by $X = X_0 + u(X)U + v(X)V$, where $X_0 \in \mathcal{H}$. By using this we have

$$\rho(X,Y) = \rho(X_0,Y_0) + (4m - 2d\sigma(U,V))(u(X)u(Y) + v(X)v(Y)). \tag{2.16}$$

3. Complex η-Einstein Normal Complex Contact Metric Manifolds

n-dimensional a real almost contact metric manifold (M,η,g) is called an η-Einstein if the Ricci tensor ρ satisfies

$$\rho = \alpha g + \beta\eta \otimes \eta, \quad \text{for } \alpha,\beta \in C^\infty(M).$$

In this section, we give the definition a complex η-Einstein manifold which is the complex analogue of η-Einstein in real case and we get some results.

Definition 3.1. Let (M,G,H,U,V,u,v,g) be a normal complex contact metric manifold and $\eta = u - iv$. If for α and β smooth functions on M the Ricci tensor satisfies

$$\rho = \alpha g + \beta(u \otimes U + v \otimes V), \tag{3.1}$$

then M is called a complex η-Einstein.

Remark 3.2. From (2.14) it is easily seen that on a complex η-Einstein normal complex contact metric manifold we have

$$\alpha + \beta = 4m - 2d\sigma(U,V).$$

If $\beta = 0$ then the manifold is Einstein and we get following;

Corollary 3.3. *If a* $(2m+1)$*-complex dimensional normal complex contact metric manifold is Einstein then* $\rho(X,Y) = (4m - 2d\sigma(U,V))g(X,Y)$.

In view of Definition 2.3 and from (3.1), we have

$$QX = \alpha X + \beta(u(X)U + v(X)V), \tag{3.2}$$

where Q is the Ricci operator is defined by $\rho(X,Y) = g(QX,Y)$. On the other hand contracting (3.2) with respect to X and using Definition 2.3, we have scalar curvature

$$\tau = 4m\alpha + 8m - 4d\sigma(U,V). \tag{3.3}$$

Thus we obtain following relations

$$\alpha = \frac{\tau - 8m + 4d\sigma(U,V)}{4m},$$

$$\beta = \frac{16m^2 - (8m+4)d\sigma(U,V) + 8m - \tau}{4m}.$$

Let X_0, Y_0 be two horizontal vector fields on M. From (3.1), we get

$$\rho(X_0,Y_0) = (4m - 2d\sigma(U,V))g(X_0,Y_0).$$

Thus we have:

Corollary 3.4. *If horizontal bundle of a normal complex contact metric manifold has Einstein metric then the manifold is complex* η*-Einstein.*

Now, we give an example for complex η-Einstein manifolds. For this recall some results from [14].

Theorem 3.5 ([14])**.** *Let* M *be a normal complex contact metric manifold with complex dimension greater than or equal to* 5. *If the* $\mathcal{GH}$*-sectional curvature is independent of the choice of the* $\mathcal{GH}$*-section at each point then it is constant on the manifold and the curvature tensor is given by*

$$R(X,Y)Z = \frac{s+3}{4}[g(Y,Z)X - g(X,Z)Y + g(Z,JY)JX + g(X,JZ)JY + 2g(X,JY)JZ] \tag{3.4}$$

$$
\begin{aligned}
&+\frac{s-1}{4}[(u(X)u(Z)+v(X)v(Z))Y-(u(Y)u(Z)+v(Y)v(Z))X\\
&+4u\wedge v(X,Y)JZ+2u\wedge v(X,Z)JY+2u\wedge v(X,Y)JZ\\
&+2g(X,GY)GZ+g(X,GZ)GY+g(X,GY)GZ\\
&+2g(X,HY)HZ+g(X,HZ)HY+g(X,HY)HZ\\
&+[u(Y)g(X,Z)-u(X)g(Y,Z)+v(X)g(Z,JY)\\
&+v(Y)g(X,JZ)+2v(X)g(Z,JY)]U\\
&+[v(Y)g(X,Z)-v(X)g(Y,Z)-u(X)g(Z,JY)\\
&-u(Y)G(X,JZ)-2u(X)g(Z,JY)]V]\\
&-\frac{4}{3}(s+1+d\sigma(U,V))[(v(X)u\wedge v(Z,Y)+v(Y)u\wedge v(X,Z)\\
&+2v(Z)u\wedge v(X,Y))U-(u(X)u\wedge v(Z,Y)+u(Y)u\wedge v(X,Z)\\
&+2u(Z)u\wedge v(X,Y))V]
\end{aligned}
$$

where s is the constant $\mathcal{GH}$-sectional curvature.

Definition 3.6 ([14])**.** A normal complex contact metric manifold M of constant $\mathcal{GH}$-sectional curvature is called complex contact space form.

We can locally choose orthonormal vectors $E_1,\dots,E_n$ in $\mathcal{H}$ such that

$$\{E_i, GE_i, HE_i, JE_i, U, V : 1\le i\le n\}$$

is an orthonormal basis of the vector field space of M. Then the Ricci tensor has the following form

$$
\begin{aligned}
\rho(X,Y)=\sum_{i=1}^{n}&[g(R(E_iX)Y,E_i)+g(R(GE_i,X)Y,GE_i)\\
&+g(R(HE_i,X)Y,HE_i)+g(R(JE_i,X)Y,JE_i)]\\
&+g(R(U,X)Y,U)+g(R(V,X)Y,V). \qquad (3.5)
\end{aligned}
$$

If manifold has constant $\mathcal{GH}$-sectional curvature then Ricci tensor ρ and scalar curvature τ are given by

$$
\begin{aligned}
\rho(X,Y)=&((m+2)s+3m+2)g(X,Y)\\
&+(-(m-2)s+m-2-2d\sigma(U,V))(u(X)u(Y)+v(X)v(Y)) \qquad (3.6)
\end{aligned}
$$

and

$$\tau=4m(m-2)s+4m(3m-4)-4d\sigma(U,V)$$

From the definition of complex η-Einstein manifold and by consider (3.6) we have following:

Corollary 3.7. *A complex contact space form is complex η-Einstein.*

Blair, Baikousis and Gouli-Andreou [1] obtained contact structure on Iwasawa manifold. In addition, present authors [16] computed some curvature identities on the Iwasawa manifold.

Example 3.8. Let a subgroup $GL(3,\mathbb{C})$ is defined by

$$H_{\mathbb{C}} = \left\{ \begin{pmatrix} 1 & c_{12} & c_{13} \\ 0 & 1 & c_{23} \\ 0 & 0 & 1 \end{pmatrix} : c_{12}, c_{13}, c_{23} \in \mathbb{C} \right\} \simeq \mathbb{C}^3.$$

$H_{\mathbb{C}}$ is called complex Heisenberg group. Blair, Baikousis and Gouli-Andreou defined the following complex contact metric structure on $H_{\mathbb{C}}$ in [1]. Let w_1, w_2, w_3 be the coordinates on $H_{\mathbb{C}} \simeq \mathbb{C}^3$ defined by $w_1(C) = c_{23}$, $w_2(C) = c_{12}$, $w_3(C) = c_{13}$ for C in $H_{\mathbb{C}}$. Here $H_{\mathbb{C}} \simeq \mathbb{C}^3$ and $\alpha = \frac{1}{2}(dw_3 - w_2 dw_1)$ is global, so the structure tensors may be taken globally. For the standard almost complex structure J on $\mathbb{C}^3$ a complex almost contact structure on $H_{\mathbb{C}}$ may be give as follows. Since α is holomorphic, by setting $\alpha = u - iv$, $v = u \circ J$ and $4\frac{\partial}{\partial w_3} = U + iV$ then $u(X) = g(U,X)$ and $v(X) = g(V,X)$. Therefore in real coordinates, G and H are given by

$$G = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & y_2 & 0 & 0 \\ 0 & 0 & y_2 & -x_2 & 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -y_2 & x_2 & 0 & 0 \\ 0 & 0 & x_2 & y_2 & 0 & 0 \end{bmatrix}.$$

Then relative to the coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ the Hermitian metric

$$g = \frac{1}{4} \begin{bmatrix} 1 + x_2^2 + y_2^2 & 0 & 0 & 0 & -x_2 & -y_2 \\ 0 & 1 + x_2^2 + y_2^2 & 0 & 0 & y_2 & -x_2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -x_2 & y_2 & 0 & 0 & 1 & 0 \\ -y_2 & -x_2 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now let

$$\Gamma = \left\{ \begin{pmatrix} 1 & \gamma_2 & \gamma_3 \\ 0 & 1 & \gamma_1 \\ 0 & 0 & 1 \end{pmatrix} \middle| \gamma_k = p_k + i q_k,\ p_k, q_k \in \mathbb{Z} \right\}.$$

Γ is subgroup of $H_{\mathbb{C}} \simeq \mathbb{C}^3$, the 1-form $dw_3 - w_2 dw_1$ is invariant under the action on Γ and with $\xi = U \wedge V$, hence the quotient $H_{\mathbb{C}}/\Gamma$ is a compact complex contact manifold with a global complex contact form. $H_{\mathbb{C}}/\Gamma$ is known the *Iwasawa manifold*. From [1] since

$$(\nabla_X G)Y = -2v(X)JY - u(Y)X - v(Y)JX + g(X,Y)U + g(JX,Y)V$$

and

$$(\nabla_X H)Y = 2u(X)JY + u(Y)JX - v(Y)X - g(JX,Y)U + g(X,Y)V$$

and take into account (2.1) and (2.2) the Iwasawa manifold is normal and it has constant holomorphic sectional curvature, -3 [14]. In this case from (3.6), we have

$$\rho(X,Y) = -4g(X,Y) + 8(u(X)u(Y) + v(X)v(Y)). \tag{3.7}$$

Therefore the Iwasawa manifold is complex η-Einstein normal complex contact metric manifold.

4. Normal Complex Contact Metric Manifolds Satisfy $\mathcal{Z}(U,X)\cdot\mathcal{Z}=0$, $\mathcal{Z}(V,X)\cdot\mathcal{Z}=0$

Concircular curvature tensor $\mathcal{Z}$ of a $(2m+1)$-complex dimensional normal complex contact metric manifold M is defined by

$$\mathcal{Z}(X,Y)Z = R(X,Y)Z - \frac{\tau}{(4m+2)(4m+1)}[g(Y,Z)X - g(X,Z)Y] \tag{4.1}$$

for X, Y, Z are vector fields on M, where Q is the Ricci tensor and τ is the scalar curvature.

There are several works in literature about this tensor. Blair, Jeong-Sik Kim and Tripathi [6] studied on some certain conditions about concircular curvature tensor on a contact metric manifold. In this section we examine normal complex contact metric manifolds under $\mathcal{Z}(U,X)\cdot\mathcal{Z}=0$, $\mathcal{Z}(V,X)\cdot\mathcal{Z}=0$ conditions and we obtain some results.

For next computations we give some result about concircular curvature tensor which can be easily obtained from the definition of $\mathcal{Z}$ and curvature properties.

Lemma 4.1. *Let M be a $(2m+1)$-complex dimensional normal complex contact metric manifold. Then for concircular curvature tensor $\mathcal{Z}$ we have*

$$\mathcal{Z}(V,U,U,V) = \mathcal{Z}(U,V,V,U) = -\left(2d\sigma(U,V) + \frac{\tau}{(4m+2)(4m+1)}\right), \tag{4.2}$$

$$\mathcal{Z}(X,U)U = \left(1 - \frac{\tau}{(4m+2)(4m+1)}\right)X, \tag{4.3}$$

$$\mathcal{Z}(X,Y)U = R(X,Y)U, \tag{4.4}$$

$$\mathcal{Z}(X,Y)V = R(X,Y)V, \tag{4.5}$$

$$\mathcal{Z}(X,U)V = R(X,U)V, \tag{4.6}$$

$$\mathcal{Z}(X,V)U = R(X,V)U, \tag{4.7}$$

$$\mathcal{Z}(U,V)X = JX, \tag{4.8}$$

$$\mathcal{Z}(U,X)Y = \left(1 - \frac{\tau}{(4m+2)(4m+1)}\right)g(X,Y)U + (g(JX,Y) - d\sigma(Y,X))V, \tag{4.9}$$

$$\mathcal{Z}(V,X)Y = \left(1 - \frac{\tau}{(4m+2)(4m+1)}\right)g(X,Y)V - (g(JX,Y) - d\sigma(Y,X))U, \tag{4.10}$$

where X and Y are horizontal vector fields on M.

Theorem 4.2. *If a* $(2m+1)$*-complex dimensional normal complex contact metric manifold* M *satisfies* $\mathcal{Z}(U,X)\cdot\mathcal{Z}=0$ *and* $\mathcal{Z}(V,X)\cdot\mathcal{Z}=0$ *then the Riemann curvature has following form:*

$$\begin{aligned}R(X,Y)Z &= g(Y,Z)X-g(X,Z)Y\\ &\quad -\frac{(4m+2)(4m+1)}{(4m+2)(4m+1)-\tau}\{[g(JY,Z)\ -d\sigma(Z,Y)](\sigma(U)GX+(\nabla_U H)X-JX)\\ &\quad -[g(JX,Z)-d\sigma(Z,X)](\sigma(U)GY+(\nabla_U H)Y-JY)\\ &\quad -[g(JX,Y)-d\sigma(Y,X)]JZ\}, \end{aligned} \tag{4.11}$$

where X,Y *and* Z *are horizontal vector fields on* M*.*

Proof. Let X,Y,W and Z be arbitrary vector fields on M with $\mathcal{Z}(U,X).\mathcal{Z}=0$ condition and then, we have

$$\begin{aligned}0&=\mathcal{Z}(U,X)\cdot\mathcal{Z}(W,Y)Z-\mathcal{Z}(\mathcal{Z}(U,X)W,Y)Z\\ &\quad-\mathcal{Z}(W,\mathcal{Z}(U,X)Y)Z-\mathcal{Z}(W,Y)\mathcal{Z}(U,X)Z.\end{aligned} \tag{4.12}$$

By setting $W=U$ and $X=X_0$, $Y=Y_0$, $Z=Z_0$, $X_0,Y_0,Z_0\in\mathcal{H}$ in (4.12), we get

$$\begin{aligned}0&=u(\mathcal{Z}(U,X_0)\cdot\mathcal{Z}(U,Y_0)Z_0)-u(\mathcal{Z}(\mathcal{Z}(U,X_0)U,Y_0)Z_0)\\ &\quad u(\mathcal{Z}(U,\mathcal{Z}(U,X_0)Y_0)Z_0)-u(\mathcal{Z}(U,Y_0)\mathcal{Z}(U,X_0)Z_0).\end{aligned} \tag{4.13}$$

From (4.3), (4.6) and (4.10) we have

$$\begin{aligned}\left(1-\frac{\tau}{(4m+2)(4m+1)}\right)\mathcal{Z}(X_0,Y_0)Z_0 &= \left(1-\frac{\tau}{(4m+2)(4m+1)}\right)^2 g(Y_0,Z_0)X_0\\ &\quad+[g(JY_0,Z_0)-d\sigma(Z_0,Y_0)]R(X_0,U)V\\ &\quad-[g(JX_0,Y_0)-d\sigma(Y_0,X_0)]JZ_0\\ &\quad+\left(1-\frac{\tau}{(4m+2)(4m+1)}\right)^2 g(X_0,Z_0)W_0\\ &\quad-[g(JX_0,Z_0)-d\sigma(Z_0,X_0)]R(Y_0,U)V.\end{aligned}$$

Then from the definition of $\mathcal{Z}$ and (2.8), we obtain (4.11). Similarly, by setting $W=V$ in (4.13) and following same steps we can obtain (4.11). □

As a consequence of above theorem, we have following;

Corollary 4.3. *Under same assumption in Theorem* 4.2 *for horizontal* X*,* Y *vector fields the sectional curvature of* M *is following:*

$$k(X,Y)=1+\frac{(4m+2)(4m+1)}{(4m+2)(4m+1)-\tau}[g(X,JY)-d\sigma(X,Y)]d\sigma(X,Y).$$

5. Projectively Semi-Symmetric Normal Complex Contact Metric Manifolds

Symmetry in complex contact manifolds was studied by Blair and Mihai [5]. In this section we give a result for projectively semi-symmetric normal complex contact manifolds.

Let M be a $(2m+1)$-complex dimensional normal complex contact metric manifold. Projective curvature tensor of M is defined by

$$\mathcal{P}(X,Y)Z = R(X,Y)Z - \frac{1}{4m+1}\left[\rho(Y,Z)X - \rho(X,Z)Y\right],$$

where X,Y,Z are arbitrary vector fields on M. Normal complex contact metric manifolds satisfying $R(X,Y)\cdot\mathcal{P} = 0$ is called projectively semi-symmetric.

Lemma 5.1. *Projective curvature of a normal complex contact metric manifold has followings:*

$$\mathcal{P}(U,V,V,U) = \mathcal{P}(V,U,U,V) = -\frac{4m(1+2d\sigma(U,V))}{4m+1}, \tag{5.1}$$

$$\mathcal{P}(X,U)U = \mathcal{P}(X,V)V = \frac{1+2d\sigma(U,V)}{4m+1}X, \tag{5.2}$$

$$\mathcal{P}(X,Y)U = R(X,Y)U, \tag{5.3}$$

$$\mathcal{P}(X,Y)V = R(X,Y)V, \tag{5.4}$$

$$\mathcal{P}(U,X)Y = R(U,X)Y + \frac{1}{4m+1}\rho(X,Y)U, \tag{5.5}$$

$$\mathcal{P}(V,X)Y = R(V,X)Y + \frac{1}{4m+1}\rho(X,Y)V, \tag{5.6}$$

where X and Y are horizontal vector fields on M.

Theorem 5.2. *Projectively semi-symmetric normal complex contact metric manifold is a complex η-Einstein manifold.*

Proof. Let M be a projectively semi-symmetric normal complex contact metric manifold, i.e., $R\cdot\mathcal{P} = 0$. Then for arbitrary vector fields X,Y,Z,W and T on M, we have

$$\begin{aligned} 0 &= R(X,Y)\cdot\mathcal{P}(Z,W)T - \mathcal{P}(R(X,Y)Z,W)T \\ &\quad - \mathcal{P}(Z,R(X,Y)W)T - \mathcal{P}(Z,W)R(X,Y)T. \end{aligned} \tag{5.7}$$

By setting $X = Z = T = U$ and $Y = Y_0, W = W_0$, $Y_0, W_0 \in \mathcal{H}$ in (5.7), we get

$$\begin{aligned} 0 &= u(R(U,Y_0)\mathcal{P}(U,W_0)U) - u(\mathcal{P}(R(U,Y_0)U,W_0)U) \\ &\quad - u(\mathcal{P}(U,R(U,Y_0)W_0)U) - u(\mathcal{P}(U,W_0)R(U,Y_0)U). \end{aligned}$$

From (2.5), (2.10), (5.2), (5.3) and (5.5), we have

$$0 = \frac{1+2d\sigma(U,V)}{4m+1}g(Y_0,W_0) - \left(-g(Y_0,W_0) + \frac{1}{4m+1}\rho(Y_0,W_0)\right)$$

and

$$\rho(Y_0,W_0) = (4m+2+2d\sigma(U,V))g(Y_0,W_0). \tag{5.8}$$

Similarly choosing $X = Z = T = V$ and $Y = Y_0$, $W = W_0$, $Y_0, W_0 \in \mathcal{H}$ in (5.7) and by same computation we have (5.8). Thus, we get

$$\rho(Y,W) = (4m+2+2d\sigma(U,V))g(Y,W) - (2+4d\sigma(U,V))(u(Y)u(W) + v(Y)v(W)).$$

So the manifold is complex η-Einstein. □

6. Conclusion

Contact geometry have lots of applications in all branches of theoretical physics; from mechanics, thermodynamics and electrodynamics to optics, gauge fields and gravity, physics of liquid crystals to quantum mechanics and quantum computers, etc. Also complex contact manifolds has several applications in theoretical physics. There are lots of open problems in the Riemannian geometry of complex contact manifolds. In this paper we give the definition of complex η-Einstein manifold. We examined certain condition for concircular curvature in a normal complex contact metric manifolds. For future works and searching new problems in complex contact geometry our definition and results will be used. Another tensors could be study under certain conditions for normal complex contact metric manifolds.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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