Communications in Mathematics and Applications
Vol. 7, No. 2, pp. 73–80, 2016
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
http://www.rgnpublications.com

A Note on Two Classical Theorems of the Fourier Transform for Bounded Variation Functions

Research Article

Francisco J. Mendoza-Torres

Facultad de Cs. Físico Matemáticas, Benemérita Universidad Autónoma de Puebla, Puebla, Pue., 72570, México
francisco.mendoza@correo.buap.mx, fcojavier.mendozatorres@gmail.com

Abstract. Employing the Henstock-Kurzweil integral, we make simple proofs of the Riemann-Lebesgue lemma and the Dirichlet-Jordan theorem for functions of bounded variation which vanish at infinity.

Keywords. Riemann-Lebesgue lemma; Dirichlet-Jordan theorem; Bounded variation function; Henstock-Kurzweil integral

MSC. 42B10; 28C99; 26B30

Received: February 9, 2016 **Accepted:** July 8, 2016

1. Introduction

Given a function $f:\mathbb{R}\longrightarrow\mathbb{R}$ (or $\mathbb{C}$), its Fourier transform can be defined in $\omega\in\mathbb{R}$ if the function $f(\cdot)e^{-i(\cdot)\omega}$ is integrable. In this case, its Fourier transform is defined as

$$\widehat{f}(\omega)=\int_{-\infty}^{\infty} f(x)e^{-ix\omega}\,dx. \tag{1.1}$$

The existence of the Fourier transform of a function is dependent upon the type integral employed. We know that with respect to Lebesgue integral it satisfies: $f\in L^1(\mathbb{R})$ if and only if $\widehat{f}(\omega)$ is defined for all $\omega\in\mathbb{R}$. Even with the Lebesgue integral, the previous proposition is not always true if the function belongs to other spaces, for example, if it is of bounded variation on $\mathbb{R}$. In this case we must use nonabsolute improper integrals, like the Henstock-Kurzweil integral.

In this paper we present new proofs of the Riemann-Lebesgue lemma and the Dirichlet-Jordan theorem for bounded variation functions which vanish at infinity, $BV_0(\mathbb{R})$. In summary, we present a new proof of the following theorem.

Theorem. *If $f \in BV_0(\mathbb{R})$, then its Fourier transform $\widehat{f}$ is defined on $\mathbb{R}\setminus\{0\}$, belongs to $C_0(\mathbb{R}\setminus\{0\})$ and for each $x\in\mathbb{R}$:*

$$\lim_{M\to\infty,\delta\to 0}\frac{1}{2\pi}\int_{\delta<|\omega|<M} e^{ix\omega}\widehat{f}(\omega)d\omega = \frac{f(x+)+f(x-)}{2}.$$

$C_0(\mathbb{R}\setminus\{0\})$ is the space of continuous functions on $\mathbb{R}\setminus\{0\}$ which vanish at infinity, and $f(x\pm)$ are the lateral limits of f at x.

The space of bounded variation functions which vanish at infinity is not contained in any $L^p(\mathbb{R})$ when $p\in[1,\infty)$. For example, $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}\frac{1}{|x|^{\frac{1}{p}}} & \text{if } x\notin[-1,1],\\ 0 & \text{if } x\in[-1,1],\end{cases}$$

belongs to $BV_0(\mathbb{R})\backslash L^p(\mathbb{R})$.

Other examples are the functions $f_\alpha\chi_{\left[\pi^{\frac{1}{\alpha}},\infty\right)}$, with $0<\alpha<1$, where $f_\alpha(t)=\left[\frac{\operatorname{sen}(t^\alpha)}{t}\right]^{\frac{1}{p}}$ and $\chi_{\left[\pi^{\frac{1}{\alpha}},\infty\right)}$ is the characteristic function of $[\pi^{\frac{1}{\alpha}},\infty)$. In [5] it is proved that the functions within parenthesis are not Lebesgue integrable. They are improper Lebesgue integrable and belong to $BV_0([\pi^{\frac{1}{\alpha}},\infty))$. Therefore, the functions $f_\alpha\chi_{\left[\pi^{\frac{1}{\alpha}},\infty\right]}$ are Henstock-Kurzweil integrables and belong to $BV_0(\mathbb{R})$. On this space, the Riemann-Lebesgue lemma is proved in [4] and [12], and the Dirichlet-Jordan theorem is proved in [4], [8], [11] and [12]. We prove these theorems of a different way to those references, and employ the Henstock-Kurzweil integral. This helps us carry out the proofs in an easier way.

2. Preliminaries

We provide some basic definitions and results to develop our exposition.

2.1 Functions of Bounded Variation

A closed interval $I\subseteq\mathbb{R}$ may be bounded or unbounded. The function f is of bounded variation on a closed interval I if the set defined by the total variations of f over the compact intervals $J\subset I$ is uniformly bounded. The total variation of f on I is defined as

$$\operatorname{Var}(f;I)=\sup\{\operatorname{Var}(f;J): J \text{ is a compact interval contained in } I\}.$$

We denoted by $BV(I)$ the vector space of bounded variation functions on I. Some characteristics of $BV(I)$ are the following:

- Jordan decomposition: $f\in BV(I)$ if and only if there exist f_1 and f_2 which are increasing bounded functions such that $f=f_1-f_2$.
- If I is an unbounded interval, then $\lim\limits_{t\to\pm\infty} f(t)$ exists.

We will refer to $BV_0(I)$ as the subspace of functions $f\in BV(I)$ such that vanishing at $\pm\infty$.

2.2 The Henstock-Kurzweil Integral

The definition of Henstock-Kurzweil integral is available in [2] and [9]. We denote by $HK(I)$ the vector space of Henstock-Kurzweil integrable functions on I. In this section we present some basic concepts related to this integral which also are taken from the above references. To simplify, sometimes we say that a function is HK integrable. Over intervals, any Lebesgue integrable function is HK integrable and their values are equal. This inclusion is proper. The $HK(I)$ space is a semi-normed space with the Alexiewicz semi-norm

$$\|f\|_A = \sup_{[c,d]\subset I}\left|\int_c^d f(t)dt\right|. \tag{2.1}$$

Hake's theorem plays an important role in the HK-integral theory. We expose the case for $[a,\infty]$. The other cases are analogous.

Theorem 1 (Hake's Theorem)**.** *$f \in HK([a,\infty])$ if and only if, for all c, ε such that $c>a$, $c-a>\varepsilon>0$, it holds that $f \in HK([a+\varepsilon,c])$ and*

$$\lim_{\varepsilon\to 0;c\to\infty}\int_a^c f(t)dt$$

exists. The limit will be $\int_a^\infty f(t)dt$.

As a consequence of this theorem, the integrals in the improper Lebesgue sense will be integrable in the HK sense. Also the Cauchy principal value of a Henstock-Kurzweil integrable function and its HK-integral are equals. The reciprocal result is not true.

We say that a function, defined on the closed interval I, is locally Henstock-Kurzweil integrable if it is integrable on every compact interval contained in I. The class of these functions is denoted by $HK_{\text{loc}}(I)$.

The multipliers in $HK(I)$ are the functions of bounded variation. That is , if $f \in HK(I)$ and $\varphi \in BV(I)$, then $f\varphi \in HK(I)$. The product of two HK-integrable functions is not necessarily HK-integrable. With respect to this fact and for the case $[a,\infty]$ we give the next theorem. The results on $[-\infty,b]$ and $[-\infty,\infty]$ are analogous.

Theorem 2 (Multiplier Theorem)**.** *Let $[a,b]$ be a compact interval. If $f \in HK([a,b])$, $\varphi \in BV([a,b])$ and $F(x)=\int_a^x f(t)$, for $x \in [a,b]$, then $f\varphi \in HK([a,b])$ and*

$$\int_a^b f\varphi = F(b)\varphi(b) - \int_a^b F d\varphi. \tag{2.2}$$

If $f \in HK([a,\infty])$ and $\varphi \in BV([a,\infty])$, then $f\varphi \in HK([a,\infty])$ and

$$\int_a^\infty f\varphi = \lim_{b\to\infty}\left[F(b)\varphi(b) - \int_a^b F d\varphi\right]. \tag{2.3}$$

The integral on the right is a Stieljes integral.

Theorem 3 (Chartier-Dirichlet Test)**.** *Let f, $\varphi : [a,\infty] \to \mathbb{R}$ and suppose that: (a) $f \in HK([a,c])$ for each $c \geq a$, and $F(x) = \int_a^x f$ is bounded on $[a,\infty)$; and (b) φ is monotone with $\lim\limits_{x\to\infty}\varphi(x) = 0$. Then $f\varphi \in HK([a,\infty])$.*

The next result is in [6].

Lemma 4. *Let $I \subset \mathbb{R}$ be a closed interval. Let $f : \mathbb{R} \times I \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ functions such that $f(\cdot,t) \in HK_{\mathrm{loc}}(\mathbb{R})$ and $\varphi \in BV_0(\mathbb{R})$. Suppose that exists $A > 0$ such that for all $t \in I$ and for all interval $[a,b] \subset \mathbb{R}$:*

$$\left|\int_a^b f(x,t)\,dx\right| \le A.$$

Then $\int_a^b f(x,t)\varphi(x)\,dx$ converges uniformly to $\int_{-\infty}^{\infty} f(x,t)\varphi(x)\,dx$, with respect to $t \in I$, when $a \to -\infty$ and $b \to \infty$.

We recall the following two lemmas, see [1, Theorem 9.16; Theorem 11.8] and [3, Theorem 1].

Lemma 5 (Moore-Osgood Theorem)**.** *Let $s(m,n)$ be a double sequence in a complete metric space X. Assume that*:

(1) $\lim\limits_{n\to\infty} s(m,n)$ *exists uniformly in m;*

(2) $\lim\limits_{m\to\infty} s(m,n)$ *exists for each n.*

Then $\lim\limits_{m\to\infty}\lim\limits_{n\to\infty} s(m,n) = \lim\limits_{n\to\infty}\lim\limits_{m\to\infty} s(m,n) = \lim\limits_{m,n\to\infty} s(m,n).$

Lemma 6. *Let $\delta > 0$ be. If g is of bounded variation on $[0,\delta]$, then*

$$\lim_{M\to\infty} \frac{2}{\pi}\int_0^{\delta} g(t)\frac{\sin Mt}{t}\,dt = g(0+).$$

3. Our Classical Theorems in BV_0

The existence of the Fourier transform for functions in $BV_0(\mathbb{R})$ is proved in [4] and [10]. The respective Riemann-Lebesgue Lemma in $\mathbb{R}\setminus\{0\}$ is proved in [4] and [12]. We recall the proof on its existence. By Jordan decomposition, there exist f_1 and f_2 increasing bounded functions that tend to zero, when $x \to \infty$, such that $f = f_1 - f_2$. For each $\omega \neq 0$ fixed and for any compact interval $[a,b]$, the function $\varphi(t) = e^{\pm it\omega}$ satisfies

$$\left|\int_a^b e^{-it\omega}\,dt\right| = \left|\frac{-e^{-ib\omega} + e^{-ia\omega}}{\omega}\right| \le \frac{2}{\omega}\,. \tag{3.1}$$

Then, by Chartier-Dirichlet theorem, $f(\cdot)e^{\pm i(\cdot)\omega} = f_1 e^{\pm i(\cdot)\omega} - f_2 e^{\pm i(\cdot)\omega} \in HK([0,\infty])$. The case $[-\infty,0]$ is analogous. Therefore $\widehat{f}(\omega)$ is defined for all $\omega \in \mathbb{R}\setminus\{0\}$. In general the Fourier transform for functions in $BV_0(\mathbb{R})$ is not defined at $\omega = 0$. For example, if $f(t) = \frac{1}{t}$ for $t \in (-\infty,-1)\cup(1,\infty)$ and $f(t) = 0$ for $t \in [-1,1]$, then f belongs to $BV_0(\mathbb{R})$ and $\widehat{f}(0)$ does not exist.

The Riemann-Lebesgue Lemma

From Lemma 4, Hake's Theorem and inequality (3.1) the following corollaries are obtained.

Corollary 7. *If $f \in BV_0(\mathbb{R})$ and I is an interval contained in $\mathbb{R}\setminus\{0\}$, then the convergence from $\int_{-a}^{a} e^{-i\omega t} f(t)\,dt$ to $\widehat{f}(\omega)$, as $a \to \infty$, is uniform with respect to $\omega \in I$.*

Corollary 8. *Let $f \in BV_0(\mathbb{R})$ and $\beta > 0$. For any compact interval I contained in $\mathbb{R}\setminus\{0\}$, it satisfies*

$$\lim_{a\to\infty}\int_I\int_{-a}^{a} f(u)e^{-iu\omega}e^{i\beta\omega}\,du\,d\omega = \int_I \widehat{f}(\omega)e^{i\beta\omega}\,d\omega.$$

Lemma 9. *Let* $f \in BV_0(\mathbb{R})$ *and* $\omega \neq 0$, *then*

$$\widehat{f}(\omega) = -\frac{i}{\omega}\int_{-\infty}^{\infty} e^{-iu\omega}\,df(u).$$

Proof. Since $h(u) = e^{-iu\omega}$ belongs to $HK_{\text{loc}}(\mathbb{R})$, then using the Multiplier Theorem on $[a,b]$:

$$\begin{aligned}\int_a^b f(u)e^{-iu\omega}\,du &= \frac{i}{\omega}\left\{(e^{-ib\omega} - e^{-ia\omega})f(b) - \int_a^b \left[e^{-iu\omega} - e^{-ia\omega}\right]df(u)\right\}\\ &= \frac{i(e^{-ib\omega} - e^{-ia\omega})f(b)}{\omega} + \frac{ie^{-ia\omega}}{\omega}[f(b) - f(a)] - \frac{i}{\omega}\int_a^b e^{-iu\omega}df(u).\end{aligned}$$

Since $\widehat{f}(\omega)$ is defined as a Henstock-Kurzweil integral, then the proof is obtained when $a \to -\infty$ and $b \to \infty$. □

From this result we obtain the following corollary.

Corollary 10. *If* $f \in BV_0(\mathbb{R})$, *then for all* $\omega \in \mathbb{R}\setminus\{0\}$:

$$\left|\widehat{f}(\omega)\right| = \left|\int_{-\infty}^{\infty} e^{-i\omega t}f(t)dt\right| \leq \frac{V(f;\mathbb{R})}{|\omega|}.$$

Proof. From equality

$$\left|\widehat{f}(\omega)\right| = \frac{1}{|\omega|}\left|\int_{-\infty}^{\infty} e^{-iu\omega}df(u)\right|,$$

and from [7, p. 232] the proof is obtained. □

Now we prove the Riemann-Lebesgue Lemma in a different way as in [4] and [12].

Theorem 11. *If* $f \in BV_0(\mathbb{R})$, *then* $\widehat{f}$ *is continuous on* $\mathbb{R}\setminus\{0\}$ *and* $\lim_{|\omega|\to\infty}\widehat{f}(\omega) = 0$.

Proof. By Hake's Theorem:

$$\widehat{f}(\omega) = \lim_{a\to\infty}\int_{-a}^{a} e^{-i\omega t}f(t)dt. \tag{3.2}$$

By Corollary 7, the convergence on the right side of (3.2) is uniform with respect to $\omega \in \mathbb{R}\setminus\{0\}$. Furthermore, it is clear that for each $a > 0$:

$$\lim_{\omega'\to\omega}\int_{-a}^{a} e^{-i\omega' t}f(t)dt = \int_{-a}^{a} e^{-i\omega t}f(t)dt. \tag{3.3}$$

Considering the equalities (3.2), (3.3) and using Lemma 5 we can interchange the limits. Therefore

$$\begin{aligned}\lim_{\omega'\to\omega}\widehat{f}(\omega') &= \lim_{\omega'\to\omega}\lim_{a\to\infty}\int_{-a}^{a} e^{-i\omega' t}f(t)dt\\ &= \lim_{a\to\infty}\lim_{\omega'\to\omega}\int_{-a}^{a} e^{-i\omega' t}f(t)dt\\ &= \lim_{a\to\infty}\int_{-a}^{a} e^{-i\omega t}f(t)dt\\ &= \widehat{f}(\omega).\end{aligned}$$

So, $\widehat{f}$ is continuous on $\mathbb{R}\setminus\{0\}$. The convergence to zero of $\widehat{f}(\omega)$, as $\omega \to \pm\infty$, it follows from Corollary 10. $\square$

3.1 The Dirichlet-Jordan theorem

Suppose that $f \in BV_0(\mathbb{R})$, $M > \delta > 0$ and $\beta \in \mathbb{R}$. Since

$$\begin{aligned}\int_{0<|\delta|<M} e^{ix(-u+\beta)}\,dx &= \left(\int_{-M}^{-\delta} + \int_{\delta}^{M}\right) e^{ix(-u+\beta)}\,dx\\ &= 2\frac{\sin M(-u+\beta) - \sin\delta(-u+\beta)}{(-u+\beta)},\end{aligned}$$

and taking into account the Corollaries 7 and 8, it gets

$$\begin{aligned}\int_{0<|\delta|<M}\int_{-\infty}^{\infty} f(u)e^{-iux}e^{i\beta x}\,du\,dx &= \lim_{a\to\infty}\int_{0<|\delta|<M}\int_{-a}^{a} f(u)e^{-iux}e^{i\beta x}\,du\,dx\\ &= \lim_{a\to\infty}\int_{-a}^{a} f(u)\int_{0<|\delta|<M}\left[e^{ix(-u+\beta)}\,dx\right]du\\ &= \lim_{a\to\infty} 2\int_{-a}^{a} f(u)\frac{\sin M(-u+\beta)}{(-u+\beta)}\,du\\ &\quad + \lim_{a\to\infty} 2\int_{-a}^{a} f(u)\frac{\sin\delta(-u+\beta)}{(-u+\beta)}\,du. \end{aligned} \tag{3.4}$$

The previous expression is interesting because it shows us a relation between a principal value and integrals that exist in the HK sense.

Lemma 12. *If $f \in BV_0(\mathbb{R})$, then*

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} f(t)\frac{\sin\varepsilon(t-x)}{t-x}\,dt = 0.$$

Proof. We know that $\sin(\cdot)/(\cdot) \in HK(\mathbb{R})$. Therefore for any $\varepsilon \in [-1,1]$, $x \in \mathbb{R}$, and $b < c$:

$$\left|\int_b^c \frac{\sin\varepsilon(t-x)}{(t-x)}\,dx\right| = \left|\int_{\varepsilon(b-x)}^{\varepsilon(c-x)} \frac{\sin t}{t}\,dx\right| \le \left\|\frac{\sin(\cdot)}{(\cdot)}\right\|_A.$$

By Lemma 4, the convergence

$$\lim_{a\to\infty}\int_{-a}^{a} f(t)\frac{\sin\varepsilon(t-x)}{t-x}dt = \int_{-\infty}^{\infty} f(t)\frac{\sin\varepsilon(t-x)}{t-x}dt$$

is uniform with respect to $\varepsilon \in [-1,1]$. The function $\sin\varepsilon(t-x)/(t-x)$ is uniformly bounded with respect to $\varepsilon \in [-1,1]$, over any interval $[-a,a]$, and converge to 0 when $\varepsilon \to 0$. By Lebesgue Dominated Convergence theorem:

$$\lim_{\varepsilon\to\infty}\int_{-a}^{a} f(t)\frac{\sin\varepsilon(t-x)}{t-x}\,dt = 0. \tag{3.5}$$

Then, by Moore-Osgood theorem,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} f(t)\frac{\sin\varepsilon(t-x)}{t-x}\,dt &= \lim_{\varepsilon\to 0}\lim_{a\to\infty}\int_{-a}^{a} f(t)\frac{\sin\varepsilon(t-x)}{t-x}\,dt\\ &= \lim_{a\to\infty}\lim_{\varepsilon\to 0}\int_{-a}^{a} f(t)\frac{\sin\varepsilon(t-x)}{t-x}\,dt\\ &= 0.\end{aligned}$$

$\square$

Remark 13. Since $\sin(\cdot)/(\cdot) \in HK(\mathbb{R})$ and taking into account the relation (3.4), then from above lemma we have

$$\begin{aligned}\lim_{\delta\to 0}\int_{0<|\delta|<\alpha}\int_{-\infty}^{\infty} f(u)e^{-iux}e^{i\beta x}\,du\,dx &= 2\int_{-\infty}^{\infty} f(u)\frac{\sin M(-u+\beta)}{(-u+\beta)}\,du\\ &= 2\int_{-\infty}^{\infty} f(-u+\beta)\frac{\sin Mu}{u}\,du. \end{aligned}\tag{3.6}$$

Now, we prove the Dirichlet-Jordan theorem.

Theorem 14. *If $f \in BV_0(\mathbb{R})$, then, for each $\beta \in \mathbb{R}$:*

$$\lim_{M\to\infty,\delta\to 0}\frac{1}{2\pi}\int_{\delta<|\omega|<M} e^{i\beta x}\widehat{f}(x)dx = \frac{f(\beta+)+f(\beta-)}{2}. \tag{3.7}$$

Proof. Let $\gamma > 0$ be. Because of $f(-\cdot+\beta)\chi_{(-\infty,-\gamma]\cup[\gamma,\infty)}/(\cdot)$ belongs to $BV_0(\mathbb{R})$, then by the Riemann-Lebesgue lemma:

$$\lim_{M\to\infty}\int_{-\infty}^{\infty}\frac{f(-u+\beta)\chi_{(-\infty,-\gamma]\cup[\gamma,\infty)}(u)}{u}\sin Mu\,du = 0. \tag{3.8}$$

On the other hand

$$\begin{aligned}\int_{-\infty}^{\infty} f(-u+\beta)\chi_{(-\gamma,\gamma)}\frac{\sin Mu}{u}\,du &= \int_{-\gamma}^{0} f(-u+\beta)\frac{\sin Mu}{u}\,du + \int_{0}^{\gamma} f(-u+\beta)\frac{\sin Mu}{u}\,du \\ &= \int_{0}^{\gamma}\left[f(u+\beta)+f(-u+\beta)\right]\frac{\sin Mu}{u}\,du.\end{aligned}\tag{3.9}$$

From (3.8), (3.9) and by Lemma 6, we get

$$\lim_{M\to\infty} 2\int_{-\infty}^{\infty} f(-u+\beta)\frac{\sin Mu}{u}\,du = \pi\left[f(\beta+)+f(\beta-)\right].$$

Applying (3.6) we then conclude the proof. □

4. Conclusion

The integral of Henstock-Kurzweil has been little used in the Fourier Analysis. It is possible that E. Talvila [10] was the first who made one study over this subject employing the HK integral. At this paper we show the application from this integral in the proof of two fundamental theorems of this important area. An intention is to spread the use of this integral.

Acknowledgement

The author express his sincere gratitude to Nancy Keranen for her excellent support. He thanks the referee for his valuable comments and suggestions. This paper was supported partially by VIEP-BUAP, Puebla, México.

Competing Interests

The author declare that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] M.T. Apostol, *Mathematical Analysis*, Addison-Wesley Publishing Company, Reading, Massachusetts (1991).

[2] R.G. Bartle, *A Modern Theory of Integration*, Graduate Studies in Mathematics, Vol. 32, American Mathematical Society, Providence (2001).

[3] Z. Kadelburg and M. Marjanović, Interchanging two limits, *Enseign. Math.* 8 (2005), 15–29. http://elib.mi.sanu.ac.rs/files/journals/tm/14/tm812.pdf

[4] F.J. Mendoza Torres, On pointwise inversion of the Fourier transform of BV_0 functions, *Ann. Funct. Anal.* 2 (2010), 112–120. http://www.emis.de/journals/AFA/AFA-tex_v1_n2_a12.pdf

[5] F.J. Mendoza Torres, J.A. Escamilla Reyna and S. Sánchez Perales, Some results about the Henstock-Kurzweil Fourier transform, *Math. Bohem.* 134 (4) (2009), 379-386. http://mb.math.cas.cz/full/134/4/mb134_4_5.pdf

[6] F.J. Mendoza Torres and M.G. Morales Macías, On the Convolution Theorem for the Fourier transform of BV_0 Functions, *J. Class. Anal.* 7 (1) (2015), 63–71. doi:10.7153/jca-07-06

[7] I.P. Natanson, *Theory of Functions of A Real Variable*, Vol. **1**, Rev. ed., Frederick Ungar Publishing, New York (1961).

[8] M. Riez and A.E. Livingston, A short proof of a classical theorem in the theory of Fourier integrals, *Amer. Math. Montly* 62 (1955), 434-437. https://www.researchgate.net/publication/266555117

[9] Ch. Swartz, *Introduction to Gauge Integrals*, World Scientific Publishing Co., Singapore (2001).

[10] E. Talvila, *Henstock-Kurzweil Fourier transforms. Illinois J. Math.* 46 (4) (2002), 1207–1226. http://www.math.ualberta.ca/~etalvila/publishedversions/TalvilaIJMfourier.pdf

[11] R.M. Trigub and E.S. Bellinsky, *Fourier Analysis and Approximation of Functions*, Kluwer Academic Publishers, Dordrecht (2004).

[12] A. Zygmund, *Trigonometric Series*, Vols. **I** and **II**, 3rd edition, Cambridge University Press, Cambridge (2002).