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Research Article

Characterization of (m,n)-High-Ideals of Posemigroups when they are 0^*-Minimal

M.M. Shamivand, H. Doostie* and and H. Rasouli

Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
***Corresponding author:** doostieh@yahoo.com

Abstract. A semigroup S is called a posemigroup if S is equipped with a partial ordering relation "$\leq$" such that $a \leq b$ in S implies $xa \leq xb$ and $ax \leq bx$, for all $x \in S$. In this paper, we define an equivalence relation $\mathbf{B}^*$ on a posemigroup S and introduce the concept of (m,n)-high-ideals by generalizing the concept of (m,n)-ideals in a posemigroup, for two non-negative integers m and n. As a result of this definition we get a relationship between 0^*-minimal (m,n)-high-ideals and (m,n)-regular posemigroups. A necessary and sufficient condition for a posemigroup S to be an (m,n)-regular posemigroup is given as well.

Keywords. Posemigroup, (m,n)-regular, High-bi-ideal, 0^*-minimal, (m,n)-high-ideal

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1. Introduction

In 1952 Good [7] first defined the notion of bi-ideals of a semigroup. Also, the concept of the (m,n)-ideal in semigroups was given by Lajos [13] as a generalization of one-sided ideals of semigroups. Thereafter, the notion of the generalized bi-ideal was introduced in semigroups by Lajos in [14] as a generalization of bi-ideals of semigroups. Moreover, the concept of ideals was studied in rings and posemirings. For example, Pawar in 2015 introduced a class of ideals that lies between essential ideals and semi-essential ideals (see [15]). Also, Gan introduced some properties of ideals in posemirings [6]. Let m, n be non-negative integers. A subsemigroup A of a semigroup S is called an (m,n)-ideal of S if $A^m S A^n \subseteq A$. Here, $A^0 S = S A^0 = S$. Furthermore, the theory of (m,n)-ideals in other structures have also been studied by many authors(see [1], [2]). A semigroup S is said to be (m,n)-regular [12] if for any a in S, there exsist x in S such that $a = a^m x a^n$. In 1979 Bogdanović [3] studied some properties of (m,n)-ideals and (m,n)-regularity of S. Indeed, the author characterized when every (m,n)-ideal of an (m,n)-ideal A of S is an (m,n)-ideal of S. Moreover, (m,n)-regularity of S was discussed. In the present paper, using the concept 0f (m,n)-ideals in posemigroups defined by Changphas in [5]. We extend the results in [12], for posemigroups. Kaap [8] introduced an equivalence relation $\mathbf{B}$ on semigroup S by, for $a,b \in S$, $a\mathbf{B}b$ if $a = b$ or $a \in bSb$ and $b \in aSa$; using this relation the author characterized 0-minimal bi-ideal of S. Tilidetzke [16] generalized Kapp's results; the author introduced an equivalence relation $\mathbf{B}_m^n$ on S where m,n are non-negative integers by, for $a,b \in S$, $a\mathbf{B}_m^n b$ if $a = b$ or $a \in b^m S b^n$ and $b \in a^m S a^n$; using the relation $\mathbf{B}_m^n$, 0-minimal (m,n)-ideals of S are characterized.

Throughout the paper S stands for a posemigroup. Following [10] we recall the definitions of $(A]$ and AB as:

$$(A] := \{s \in S \mid s \le a, \text{ for some } a \in A\},$$

$$AB := \{ab \mid a \in A, b \in B\},$$

for subposets A and B of a posemigroup S. Changphas [5] generalized Tilidetzeke's results; the author introduced and equivalence relation $\mathbf{B}_m^n$ on posemigroup S where m,n are non-negative integers by, for $a,b \in S$, $a\mathbf{B}_m^n b$ if and only if $a = b$ or $a \le b^m v b^n$ and $b \le a^m u a^n$ for some $u,v \in S$, that is,

$$a \in (b^m S b^n] \quad \text{and} \quad b \in (a^m S a^n].$$

By a left ideal of a posemigroup S we mean a non-empty subset L of S satisfying $SL \subseteq L$ and $(L] \subseteq L$. A right ideal may be defined in a similar way. A two sided ideal of S is a left as well as a right ideal of S. For every non-empty subset A of S, let

$$(A]^* := \{s \in S | s \le a^n, \text{ for some } a \in A, n \in \mathbb{N}\}. \tag{$*$}$$

If there is an element 0 of a posemigroup S such that $x0 = 0x = 0$ and $0 \le x$ for all $x \in S$, we call 0 a zero element of S. A left ideal L of a posemigroup S with zero 0 is said to be 0-minimal if there is no a left ideal L' of S such that $\{0\} \subset L' \subset L$. For a right ideal of S, we can be defined similarly. Recall from [10, 11] that a non-empty subset B of S is called a bi-ideal of S if $BSB \subseteq B$ and $(B] \subseteq B$. Changphas defined (m,n)-ideals in posemigroups as follows:

Let m, n be non-negative integers. A subposemigroup A of posemigroup S is called an (m,n)-ideal of S if the following conditions hold:

(i) $A^m S A^n \subseteq A$,

(ii) $(A] \subseteq A$, that is, for $x \in A$ and $y \in S$, $y \le x$ implies $y \in A$.

In particular, A is called a bi-ideal of S if $m = n = 1$. It is clear that if A be a bi-ideal of S, then A is an (m,n)-ideal of S. Bussaban [4] show that if A be a non-empty subset of a posemigroup S, then the intersection of all (m,n)-ideals contaning A of S, denoted by $[A]_{m,n}$, is an (m,n)-ideal contaning A of S, and is of the form

$$[A]_{m,n} = (A \cup A^2 \cup \ldots \cup A^{m+n} \cup A^m S A^n].$$

For an element a of a posemigroup S, we write $[\{a\}]_{m,n}$(or simply $[a]_{m,n}$) by:

$$[a]_{m,n} = (a \cup a^2 \cup \ldots \cup a^{m+n} \cup a^m S a^n].$$

Here, we generalize the notion of $\mathbf{B}_m^n$ on a posemigroup S. The results obtained in [8, 16] and [5] become then special cases. Let S be a posemigroup. A non-empty subset B of S is said to be a *high-bi-ideal* of S if $BSB \subseteq B$ and $(B]^* \subseteq B$. Now relation $(\mathbf{B}^*)_m^n$ on S there m,n are non-negative integers, as follows:

For $a,b \in S$, let $a(\mathbf{B}^*)_m^n b$ if and only if $a = b$ or $a \in (b^m S b^n]^*$ and $b \in (a^m S a^n]^*$. Also a subposemigroup A of S is called a (m,n)-*high-ideal* of S if the following conditions hold:

(i) $A^m S A^n \subseteq A$,

(ii) $(A]^* = A$, that is, for $x \in A$ and $y \in S$, $y \le x^n$ implies $y \in A$.

We denote the high-bi-ideal generated by an element $a \in S$ by $B^*(a)$. One can easily show that $B^*(a) = (a \cup a^2 \cup aSa]^*$. If A is a non-empty subset of posemigroup S, then the intersection of all (m,n)-high-ideals contaning A of S, denoted by $[A]^*_{m,n}$, is a (m,n)-high-ideal contaning A of S and is of the form

$$[A]^*_{m,n} = (A \cup A^2 \cup \ldots \cup A^{m+n} \cup A^m S A^n]^*.$$

(see Lemma 2.4). For an element a of a posemigroup S, we write $[\{a\}]^*_{m,n}$ (or simply $[a]^*_{m,n}$) by:

$$[a]^*_{m,n} = (a \cup a^2 \cup \ldots \cup a^{m+n} \cup a^m S a^n]^*. \qquad (**)$$

A posemigroup S with zero 0 is called nilpotent if $S^l = 0$, for some positive integer l. Also, the center of posemigroup S is defined by

$$Z(S) = \{x \in S \mid \forall\ y \in S : xy = yx\}.$$

Let m,n be non-negative integers. An element a of posemigroup S is said to be (m,n)-regular [5] if $a \le a^m x a^n$ for some $x \in S$. If every element of S be (m,n)-regular, then S is called an (m,n)-regular posemigroup. A (m,n)-high-ideal A of a posemigroup S with zero 0 is said to be 0^*-*minimal* if there is no a (m,n)-high-ideal A' of S such that $\{0\} \subset A' \subset A$. We consider 0^*-minimal (m,n)-high-ideals for regular posemigroups. Our main results concerning the 0^*-minimal (m,n)-high-ideals which are the generalization of the 0-minimal (m,n)-ideals of semigroups, are:

Proposition 1.1. *Let S be a posemigroup with zero 0. Then:*

(i) *If A be a (m,n)-high-ideal of S, then A is 0^*-minimal if and only if A is one non-zero $(\boldsymbol{B}^*)_m^n$-class union $\{0\}$.*

(ii) *If $m,n \ge 1$ and A be a 0^*-minimal (m,n)-high-ideal of S such that, $(A^2]^* \ne \{0\}$, then A is a 0^*-minimal high-bi-ideal of S.*

Proposition 1.2. *Let S be a posemigroup with zero 0 and R,L are 0^*-minimal right-ideal and 0^*-minimal left-ideal of S respectively. Then:*

(i) *If S contains no non-zero nilpotent (m,n)-high-ideals, then $(RL]^* = \{0\}$ or $(RL]^*$ is a 0^*-minimal (m,n)-high-ideal of S.*

(ii) *If $(R^mL^n]^* \subseteq Z(S)$, then $(R^mL^n]^* = \{0\}$ or $(R^mL^n]^*$ is a 0^*-minimal (m,n)-high-ideal of S.*

Proposition 1.3. *Let S be an (m,n)-regular posemigroup with zero 0 and A and B be 0^*-minimal $(m,0)$-high-ideal and 0^*-minimal $(0,n)$-high-ideal of S respectively. Then:*

(i) *If $(AB]^* \subseteq A \cap B$, then $(AB]^* = \{0\}$ or $(AB]^*$ is a 0^*-minimal (m,n) high-ideal of S.*

(ii) *$A \cap B = \{0\}$ or $A \cap B$ is a 0^*-minimal (m,n)-high-ideal of S.*

Proposition 1.4. *Let S be a posemigroup. Then, S is (m,n)-regular if and only if for every a in S, we have:*

$$[a]^*_{m,n} = (a^mSa^n]^*.$$

2. The Proofs

To prove our assertions first we have to give certain preliminary results concerning the notions of Section1. First, we give a generalized results of [9].

Lemma 2.1. *For a posemigroup S and two non-empty subset A and B of S,*

(i) $A \subseteq (A]^*$.

(ii) *If $A \subseteq B$, then $(A]^* \subseteq (B]^*$.*

(iii) $(A]^*(B]^* \subseteq (AB]^*$ *for any two subposemigroups A and B of S.*

(iv) $((A]^*(B]^*]^* = (AB]^*$.

(v) $(A \cup B]^* = (A]^* \cup (B]^*$.

Proof. (i), (iv) and (v) are evident.

(ii) Let $t \in (A]^*$. Then there exists $a \in A$ and $n \in \mathbb{N}$ such that $t \leq a^n$. Since $A \subseteq B$, there exists $a \in B$ and $n \in \mathbb{N}$ such that $t \leq a^n$. Thus $t \in (B]^*$.

(iii) Take any $x \in (A]^*(B]^*$. This implies that $x = ab$ for some $a \in (A]^*$ and $b \in (B]^*$. Then $a \leq h^m$ and $b \leq k^n$ for some $h \in A$, $k \in B$ and $m,n \in \mathbb{N}$. It follows that $ab \leq h^mk^n$. Since $h^m \in A$ and $k^n \in B$, we obtain $h^mk^n \in AB$. Therefore, $ab \leq h^mk^n \in AB$ showing that $x \in (AB]^*$. □

Lemma 2.2. *For a posemigroup S, the following conditions hold:*

(i) *The relation $(\boldsymbol{B}^*)^n_m$ is an equivalence relation on S.*

(ii) *If A be a (m,n)-high-ideal of S, then*

$$A = \bigcup_{a \in A} (\boldsymbol{B}^*)^n_m(a)$$

where $(\boldsymbol{B}^*)^n_m(a)$ denote the $(\boldsymbol{B}^*)^n_m$-class contaning a in S.

(iii) *If A is a (m,n)-high-ideal of posemigroup S with zero 0 such that A be a single non- zero $(\boldsymbol{B}^*)^n_m$-class union $\{0\}$, then A is a 0^*-minimal (m,n)-high-ideal of S.*

Proof. (i) We show that $(\mathbf{B}^*)^n_m$ is transitive. The reflexive and symetric properties are evident. Let $a,b,c \in S$ such that $a(\mathbf{B}^*)^n_m b$ and $b(\mathbf{B}^*)^n_m c$. There are four cases to consider:

(a) $a = b$ and $b = c$.

(b) $a = b$, $b \in (c^m S c^n]^*$ and $c \in (b^m S b^n]^*$.

(c) $b = c$, $a \in (b^m S b^n]^*$and $b \in (a^m S a^n]^*$.

(d) $a \in (b^m S b^n]^*$, $b \in (a^m S a^n]^*$, $b \in (c^m S c^n]^*$ and $c \in (b^m S b^n]^*$.

(a), (b) and (c) by definition of $(\mathbf{B}^*)^n_m$ implies $a(\mathbf{B}^*)^n_m c$.

If (d) holds, then

$$a \in (b^m S b^n]^* \subseteq (((c^m S c^n]^*)^m S((c^m S c^n]^*)^n]^* \subseteq (c^m S c^n]^*,$$
$$c \in (b^m S b^n]^* \subseteq (((a^m S a^n]^*)^m S((a^m S a^n]^*)^n]^* \subseteq (a^m S a^n]^*.$$

Therefore, $a(\mathbf{B}^*)^n_m c$. Hence, $(\mathbf{B}^*)^n_m$ is an equivalence relation on S.

(ii) Let A is a (m,n)-high-ideal of posemigroup S. By (i), $A \subseteq \bigcup_{a\in A} (\mathbf{B}^*)^n_m(a)$. Conversely, let $x \in \bigcup_{a\in A} (\mathbf{B}^*)^n_m(a)$, hence $x \in (\mathbf{B}^*)^n_m(a)$ for some $a \in A$.

Thus, $x \in (a^m S a^n]^* \subseteq (A^m S A^n]^* \subseteq (A]^* = A$. So, $\bigcup_{a\in A} (\mathbf{B}^*)^n_m(a) \subseteq A$.

(iii) This follows directly from (ii). □

Lemma 2.3. *Let S be a posemigroup with zero 0. For any $a,b \in S$, $a(\boldsymbol{B}^*)^n_m b$ if and only if $[a]^*_{m,n} = [b]^*_{m,n}$.*

Proof. It is clear that if $a(\mathbf{B}^*)^n_m b$, then $[a]^*_{m,n} = [b]^*_{m,n}$. Conversely, let $[a]^*_{m,n} = [b]^*_{m,n}$. Then,

$$(a \cup a^2 \cup \ldots \cup a^{m+n} \cup a^m S a^n]^* = (b \cup b^2 \cup \ldots \cup b^{m+n} \cup b^m S b^n]^*.$$

Let, $a \neq b$. There are four cases to consider:

(a) $a \leq b^s$, for some $1 < s \leq m+n$ and $b \leq a^t$, for some $1 < t \leq m+n$. By assumption, we have $a \in (b^m S b^n]^*$ and $b \in (a^m S a^n]^*$.

(b) $a \leq b^s$, for some $1 < s \leq m+n$ and $b \in (a^m S a^n]^*$. By relation $(*)$, we have $a \in (b^m S b^n]^*$.

(c) $b \leq a^t$, for some $1 < t \leq m+n$ and $a \in (b^m S b^n]^*$. This case is similar to (b). So $b \in (a^m S a^n]^*$.

(d) $a \in (b^m S b^n]^*$ and $b \in (a^m S a^n]^*$. This cases immediately implies $a(\mathbf{B}^*)^n_m b$. □

Lemma 2.4. *Let A be a non-empty subset of a posemigroup S. Then, the intersection of all (m,n)-high-ideals contaning A of S, denoted by $[A]^*_{m,n}$, is a (m,n)-high-ideal contaning A of S and is of the form*

$$[A]^*_{m,n} = (A \cup A^2 \cup \ldots \cup A^{m+n} \cup A^m S A^n]^*.$$

Proof. Let $\{A_i | i \in I\}$ be the set of all (m,n)-high-ideals containing A of S. Then, $\bigcap_{i\in I} A_i$ is subsemigroup contaning A of S. For $j \in I$, we have

$$\Big(\bigcap_{i\in I} A_i\Big)^m S \Big(\bigcap_{i\in I} A_i\Big)^n \subseteq A_j^m S A_j^n \subseteq A_j.$$

Then,

$$\Big(\bigcap_{i\in I} A_i\Big)^m S \Big(\bigcap_{i\in I} A_i\Big)^n \subseteq \bigcap_{i\in I} A_i.$$

But,

$$\Big(\bigcap_{i\in I} A_i\Big]^* \subseteq \bigcap_{i\in I} (A_i]^* = \bigcap_{i\in I} A_i \subseteq \Big(\bigcap_{i\in I} A_i\Big]^*,$$

hence $\bigcap_{i\in I} A_i$ is a (m,n)-high-ideal of S. Clearly, $\Big(\bigcup_{i=1}^{m+n} A^i \cup A^m S A^n\Big]^*$ is a subsemigroup of S. We now consider:

$$\begin{aligned}
\Big(\Big(\bigcup_{i=1}^{m+n} A^i \cup A^m S A^n\Big]^*\Big)^m S &= \Big(\Big(\bigcup_{i=1}^{m+n} A^i \cup A^m S A^n\Big]^*\Big)^{m-1} \Big(\bigcup_{i=1}^{m+n} A^i \cup A^m S A^n\Big]^* S \\
&\subseteq \Big(\Big(\bigcup_{i=1}^{m+n} A^i \cup A^m S A^n\Big]^*\Big)^{m-1} (AS]^* \\
&= \Big(\Big(\bigcup_{i=1}^{m+n} A^i \cup A^m S A^n\Big]^*\Big)^{m-2} \Big(\bigcup_{i=1}^{m+n} A^i \cup A^m S A^n\Big]^* (AS]^* \\
&\subseteq \Big(\Big(\bigcup_{i=1}^{m+n} A^i \cup A^m S A^n\Big]^*\Big)^{m-2} (A^2 S]^* \\
&\subseteq (A^m S]^*.
\end{aligned}$$

Similarly,

$$S\Big(\Big(\bigcup_{i=1}^{m+n} A^i \cup A^m S A^n\Big]^*\Big)^n \subseteq (S A^n]^*.$$

So,

$$\begin{aligned}
\Big(\Big(\bigcup_{i=1}^{m+n} A^i \cup A^m S A^n\Big]^*\Big)^m S \Big(\Big(\bigcup_{i=1}^{m+n} A^i \cup A^m S A^n\Big]^*\Big)^n &\subseteq (A^m S A^n]^* \\
&\subseteq \Big(\bigcup_{i=1}^{m+n} A^i \cup A^m S A^n\Big]^*.
\end{aligned}$$

Hence, $\Big(\bigcup_{i=1}^{m+n} A^i \cup A^m S A^n\Big]^*$ is a (m,n)-high-ideal contaning A of S and

$$[A]^*_{m,n} \subseteq (A \cup A^2 \cup \ldots \cup A^{m+n} \cup A^m S A^n]^*.$$

Finally, since $(A^m S A^n]^* \subseteq (([A]^*_{m,n})^m S ([A]^*_{m,n})^n]^* \subseteq [A]^*_{m,n}$, so

$$(A \cup A^2 \cup \ldots \cup A^{m+n} \cup A^m S A^n]^* \subseteq [A]^*_{m,n}.$$

Therefore,

$$[A]^*_{m,n} = (A \cup A^2 \cup \ldots \cup A^{m+n} \cup A^m S A^n]^*.$$ □

We are now ready to prove the propositions.

Proof of Proposition 1.1. (i) Let A is 0^*-minimal (m,n)-high-ideal of S and $a,b \in A \setminus \{0\}$ such that $a \neq b$. This fact that $\{0\} \subset [a]^*_{m,n} \subseteq A$ and the minimality of A implies, $[a]^*_{m,n} = A$. Similarly, $[b]^*_{m,n} = A$. So, $[a]^*_{m,n} = [b]^*_{m,n}$. Now, by Lemma 2.3, $a(\mathbf{B}^*)^n_m b$. The converse from case (i) follows by Lemma 2.2 (iii).

(ii) Let $\{0\} \subset B \subseteq A$, for some high-bi-ideal B of S. Then $A = B$ since, A is a (m,n)-high-ideal of S. We have, A is a 0^*-minimal high-bi-ideal of S, for, assume that there is no a high-bi-ideal B of S such that $\{0\} \subset B \subseteq A$. Since $\{0\} \subset (A^2]^* \subseteq A$ and $(A^2]^*$ is a (m,n)-high-ideal of S, then by the minimality of A, $(A^2]^* = A$. On the other hand,

$$ASA = (A^2]^* S (A^2]^* \subseteq (A^2 S A^2]^* \subseteq A.$$

Therefore, by assumption, A is a 0^*-minimal high-bi-ideal of S. □

Proof of Proposition 1.2. (i) Let S contains no non-zero nilpotent (m,n)-high-ideal and R,L are 0^*-minimal right-ideal and 0^*-minimal left-ideal of S respectively, such that $(RL]^* \neq \{0\}$. We have

$$(RL]^*(RL]^* \subseteq (RLRL]^* \subseteq (RSL]^* \subseteq (RL]^*,$$

$$(RL]^* S (RL]^* \subseteq (RLSRL]^* \subseteq (RSL]^* \subseteq (RL]^*.$$

Then $(RL]^*$ is a high-bi-ideal of S, hence, $(RL]^*$ is a (m,n)-high-ideal of S. Now, let A is a (m,n)-high-ideal of S and $\{0\} \subset A \subseteq (RL]^*$. So, $A^m \neq \{0\}$ and $A^n \neq \{0\}$. Since $(RL]^* \subseteq R \cap L$, hence, $A \subseteq R \cap L$. But, $\{0\} \subset (A^m \cup A^m S]^* \subseteq R$, that by minimality of R implies $(A^m \cup A^m S]^* = R$. Similarly, $(A^n \cup SA^n]^* = L$. So,

$$\begin{aligned} A \subseteq (RL]^* &= ((A^m \cup A^m S]^* (A^n \cup SA^n]^*]^* \\ &= ((A^m \cup A^m S)(A^n \cup SA^n)]^* \\ &\subseteq (A^m S A^n]^* \subseteq A. \end{aligned}$$

Hence, $A = (RL]^*$.

(ii) Let R,L are 0^*-minimal right-ideal and 0^*-minimal left-ideal of S respectively, such that $(R^m L^n]^* \subseteq Z(S)$ and $(R^m L^n]^* \neq \{0\}$. Then, $(R^m]^* \neq \{0\}$ and $(L^n]^* \neq \{0\}$, since, $\{0\} \subset (R^m]^* \subseteq R$. By minimality of R we have $(R^m]^* = R$. Similarly, $(L^n]^* = L$. So, $(R^m L^n]^* = ((R^m]^*(L^n]^*]^* = (RL]^*$, that is a high-bi-ideal of S, hence, $(R^m L^n]^*$ is a (m,n)-high-ideal of S. Now, let A is a (m,n)-high-ideal of S and $\{0\} \subset A \subseteq (R^m L^n]^*$. Since, $(RL]^* \subseteq R \cap L$ so, $A \subseteq R$ and $A \subseteq L$. But, $(A \cup AS]^* \subseteq (R \cup RS]^* \subseteq R$ that implies $(A \cup AS]^* = R$. Similarly, $(A \cup SA]^* = L$. By $(R^m L^n]^* \subseteq Z(S)$, we have

$$\begin{aligned} A \subseteq (R^m L^n]^* &= (((A \cup AS]^*)^m ((A \cup SA]^*)^n]^* \\ &\subseteq ((A \cup AS)^m (A \cup SA)^n]^* \\ &\subseteq (A^{m+n} \cup A^m S A^n]^* \\ &\subseteq A. \end{aligned}$$

Hence, $A = (R^m L^n]^*$. So, $(R^m L^n]^*$ is a 0^*-minimal (m,n)-high-ideal of S. □

Proof of Proposition 1.3. (i) Let A and B be 0^*-minimal $(m,0)$-high-ideal and 0^*-minimal $(0,n)$-high-ideal of S respectively, such that, $(AB]^* \subseteq A \cap B$ and $(AB]^* \neq \{0\}$. Since, $(AB]^*(AB]^* \subseteq AB \subseteq (AB]^*$ and,

$$\begin{aligned}((AB]^*)^m S((AB]^*)^n &\subseteq A^m S((AB]^*)^n \\ &\subseteq AB^n \\ &\subseteq AB \subseteq (AB]^*,\end{aligned}$$

so, $(AB]^*$ is a (m,n)-high-ideal of S. Now, let C be a (m,n)-high-ideal of S such that, $\{0\} \subset C \subseteq (AB]^*$. Then $C \subseteq A$ and $C \subseteq B$. But, $C \subseteq (C^m SC^n]^*$ follows $(C^m S]^* \neq \{0\}$ and $(SC^n]^* \neq \{0\}$. Since, $(C^m S]^* \subseteq (A^m S]^* \subseteq A$ so, $(C^m S]^* = A$. Similarly $(SC^n]^* = B$. Now,

$$\begin{aligned}C &\subseteq (AB]^* \\ &\subseteq ((C^m S]^*(SC^n]^*]^* \\ &= (C^m SC^n]^* \subseteq C.\end{aligned}$$

Thus, $(AB]^* = C$. Hence, $(AB]^*$ is 0^*-minimal.

(ii) Let A and B be 0^*-minimal $(m,0)$-high-ideal and 0^*-minimal $(0,n)$-high-ideal of S respectively and $A \cap B \neq \{0\}$. We have,

$$(A \cap B)^m S(A \cap B)^n \subseteq (A^m S)B^n \subseteq AB^n \subseteq B,$$

$$(A \cap B)^m S(A \cap B)^n \subseteq A^m (SB^n) \subseteq A^m B \subseteq A.$$

Hence, $(A \cap B)^m S(A \cap B)^n \subseteq A \cap B$. So, $A \cap B$ is a (m,n)-high-ideal of S. Now, let C be a (m,n)-high-ideal of S such that, $\{0\} \subset C \subseteq A \cap B$. Then, $C \subseteq A$ and $C \subseteq B$. But, $C \subseteq C^m SC^n$ follows $C^m S \neq \{0\}$ and $SC^n \neq \{0\}$. Since, $C^m S \subseteq A^m S \subseteq A$ so, $C^m S = A$. Similarly $SC^n = B$. Now,

$$\begin{aligned}C &\subseteq A \cap B \\ &= C^m S \cap SC^n \\ &= C^m SC^n \subseteq C.\end{aligned}$$

Thus, $A \cap B = C$. Hence, $A \cap B$ is 0^*-minimal. □

Proof of Proposition 1.4. Let S is (m,n)-regular. If $a \in S$ and $x \in [a]^*_{m,n}$ then, by relation $(**)$, $x \leq y^t$ for some y in $(a \cup a^2 \cup \ldots \cup a^{m+n} \cup a^m Sa^n)$ and $t \in \mathbb{N}$. If $y \in a^m Sa^n$ then, $y^t \in (a^m Sa^n)^t = (a^m Sa^n)(a^m Sa^n)\ldots(a^m Sa^n) = a^m Sa^{n+m} S \ldots Sa^{n+m} Sa^n \subseteq a^m Sa^n$ so, $y \in (a^m Sa^n]^*$ and hence, $x \in (a^m Sa^n]^*$. If $y \in \bigcup_{i=1}^{m+n} a^i$ then, $y = a^k$ for some $k \in \{1,2,\ldots,m+n\}$. So,

$$\begin{aligned}x \in (a^k]^* &\subseteq (((a^m Sa^n]^*)^k]^* \\ &\subseteq ((a^m Sa^n]^*]^* \\ &= (a^m Sa^n]^*.\end{aligned}$$

Therefore, $[a]^*_{m,n} \subseteq (a^m Sa^n]^*$. On the other hand, $(a^m Sa^n]^* \subseteq [a]^*_{m,n}$. So, $[a]^*_{m,n} = (a^m Sa^n]^*$. Conversely, if $a \in S$ then, $a \in [a]^*_{m,n} = (a^m Sa^n]^*$. Hence, S is (m,n)-regular. □

3. Examples

In this section we give two posemigroups where, one is $(1,1)$-regular and the other justifying the conditions of the Proposition 1.4.

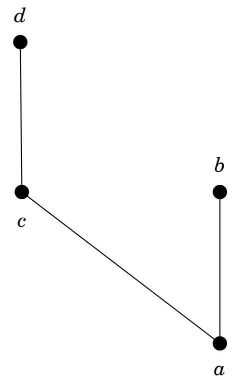
Example 3.1. Let $S = \{a,b,c,d,\}$ be a posemigroup with the multiplication and the order relation defined by:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	c	c	c	c
d	c	d	c	d

$$\leq := \{(a,a),(a,b),(a,c),(a,d),(b,b),(c,c),(c,d)(d,d)\}.$$

We give the covering relation and the figure of S by

$$\preceq := \{(a,b),(a,c),(c,d)\}$$



So, S is (m,n)-regular for any integer $m,n \geq 1$.

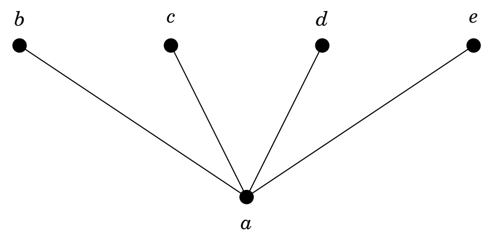
Example 3.2. Let $S = \{a,b,c,d,e\}$ be a posemigroup with the multiplication and the order relation defined by:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	d	a
c	a	e	c	c	e
d	a	b	d	d	b
e	a	e	a	c	a

$$\leq := \{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(c,c),(d,d),(e,e)\}.$$

We give the covering relation and the figure of S by

$$\preceq := \{(a,b),(a,c),(a,d),(a,e)\}$$



Then, we have:

$$[a]^*_{1,1} = (a]^*,\ [b]^*_{1,1} = (\{a,b\}]^*,\ [c]^*_{1,1} = (\{a,c\}]^*,\ [d]^*_{1,1} = (\{a,d\}]^*, [e]^*_{1,1} = (\{a,e\}]^*.$$

Therefore, by Proposition 1.4, S is $(1,1)$-regular(or regular).

4. Conclusion

The notion of an ideal is a main concept in some algebraic structures. For instance, ideals play a considerable role in ring theory, semiring theory, semigroup theory and their corresponding ordered structures. In this paper, we generalized the notion of (m,n)-ideals to (m,n)-high-ideals and investigated some properties of posemigroups using this new notion. Also, we extended the concept of 0-minimal to 0^*-minimal and get a connection between 0^*-minimal (m,n)-high-ideals and (m,n)-regular posemigroups. Another idea in this way is to study these notions for posemirings and get new characterization results concerning such structures.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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